LECTURES ON THE MORSE COMPLEX FOR INFINITE-DIMENSIONAL MANIFOLDS

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Introduction

These lectures consist of three parts. In the first one we review some results about the dynamics of differentiable flows with hyperbolic rest points, in a Banach space setting. In particular, we prove the local stable manifold theorem, the Grobman – Hartman linearization theorem, and we describe the global stable and unstable manifolds in the case of a flow admitting a Lyapunov function.

In the second part we study the Morse complex of gradient-like flows on Banach manifolds, assuming that all the rest points have finite Morse index. We introduce this chain complex as the cellular chain complex of a suitable cellular filtration of the underlying manifold M. In particular, the homology of the Morse complex is isomorphic to the singular homology of M (or to the singular homology of the pair $(\widehat{M}, A)$, in the relative case, in which we consider a gradient like flow on $\widehat{M}$, with a positively invariant open set A, and we consider the rest points in $M = \widehat{M} \setminus \overline{A}$ in the construction of the Morse complex). Then we describe the chain boundary operator in terms of the intersection numbers of the unstable and stable manifolds of pairs of rest points with index difference equal to 1. Finally, we specialize the analysis to the negative gradient flow of a Morse function on a Riemannian Hilbert manifold. In this case, we prove that the Morse – Smale transversality assumption holds for generic perturbations of the metric, and that the isomorphism class of the Morse complex does not depend on the metric. These results provide an alternative approach to infinite dimensional Morse theory, as developed by Palais and Smale in the sixties, see Palais (1963) and Smale (1964a; 1964b).

The third part is an exposition of recent results by the authors (see Abbondandolo and Majer, 2003b) about the Morse complex approach for gradient-like flows whose rest points have infinite Morse index and co-index. The framework is that of a Hilbert manifold M with a fixed infinite-dimensional and -codimensional subbundle $\mathcal{V}$ of the tangent bundle. When the gradient-like flow satisfies suitable compatibility conditions with respect to $\mathcal{V}$, each rest point x can be given a relative Morse index $m(x, \mathcal{V})$, and the unstable and stable manifolds of pairs of critical points x, y intersect in submanifolds of finite dimension $m(x, \mathcal{V}) - m(y, \mathcal{V})$. The study of the Hilbert Grassmannian, and in particular of the determinant bundle on the space of Fredholm pairs of subspaces of a Hilbert space, allow to prove that these intersections carry coherent orientations. Finally, suitable integrability assumptions on $\mathcal{V}$, together with compactness assumptions on the flow, imply that the above intersections have compact closure in M. These facts allow to define the Morse complex.

The first two parts contain fairly detailed proofs of all the statements, most of which — especially in the second part — are folklore results, for which we could not find appropriate reference in the literature. The style of the third part is different: proofs are only sketched, or given in a simplified framework. We refer to Abbondandolo and Majer (2003b) for a more complete presentation.

1. A few facts from hyperbolic dynamics

1.1. ADAPTED NORMS

Let E be a real Banach space. A bounded linear operator L on E is said *hyperbolic* if its spectrum does not meet the imaginary axis[1]: $\sigma(L) \cap i\mathbb{R} = \emptyset$. In this case, the decomposition of the spectrum of L into the disjoint closed subsets $\sigma^+(L) = \sigma(L) \cap \{\mathrm{Re}\, z > 0\}$ and $\sigma^-(L) = \sigma(L) \cap \{\mathrm{Re}\, z < 0\}$ induces the splitting $E = E^{\mathrm{u}} \oplus E^{\mathrm{s}}$ into L-invariant closed linear subspaces, such that $\sigma(L|_{E^{\mathrm{u}}}) = \sigma^+(L)$ and $\sigma(L|_{E^{\mathrm{s}}}) = \sigma^-(L)$, with projectors $P^{\mathrm{u}} = \chi_{\{\mathrm{Re}\, z > 0\}}(L)$, $P^{\mathrm{s}} = \chi_{\{\mathrm{Re}\, z < 0\}}(L)$. The spaces $E^{\mathrm{u}} = E^{\mathrm{u}}(L)$ and $E^{\mathrm{s}} = E^{\mathrm{s}}(L)$ are often called the *positive* (or *unstable*) and the *negative* (or *stable*) *eigenspaces of* L (although they may not consist of eigenvectors).

An L-*adapted norm* is an equivalent norm $\|\cdot\|$ on E such that:

$$\|\xi\| = \max\{\|P^{\mathrm{u}}\xi\|, \|P^{\mathrm{s}}\xi\|\}, \quad \forall \xi \in E, \tag{1}$$

and there is $\lambda > 0$ such that for every $t \geq 0$

$$\|e^{tL}\xi\| \leq e^{-\lambda t}\|\xi\| \;\forall \xi \in E^{\mathrm{s}}, \quad \|e^{-tL}\xi\| \leq e^{-\lambda t}\|\xi\| \;\forall \xi \in E^{\mathrm{u}}. \tag{2}$$

As a consequence, also

$$\|e^{-tL}\xi\| \geq e^{\lambda t}\|\xi\| \;\forall \xi \in E^{\mathrm{s}}, \quad \|e^{tL}\xi\| \geq e^{\lambda t}\|\xi\| \;\forall \xi \in E^{\mathrm{u}}, \tag{3}$$

[1] In the framework of discrete dynamical systems, a hyperbolic operator is a bounded operator whose spectrum does not meet the unit circle. In that context, an operator L satisfying $\sigma(L) \cap i\mathbb{R} = \emptyset$ should be called *infinitesimally hyperbolic*.

for every $t \geq 0$. Such an adapted norm exists. Actually, for every λ in the interval $]0, \min|\operatorname{Re}\sigma(L)|[$ there is a norm $\|\cdot\|$ satisfying (1), (2), and (3). The construction is based on the following lemma, applied to $L|_{E^s}$ and to $-L|_{E^u}$.

LEMMA 1.1. *Let L be a bounded operator on the Banach space* $(E, \|\cdot\|_0)$, *and let λ be a real number such that* $\lambda > \max \operatorname{Re}\sigma(L)$. *Then there exists a norm* $\|\cdot\|$ *on E equivalent to* $\|\cdot\|_0$ *such that* $\|e^{tL}\xi\| \leq e^{t\lambda}\|\xi\|$ *for every ξ in E and* $t \geq 0$.

Proof. Up to replacing L by $L - \lambda I$, we may assume that $\lambda = 0$, so $\alpha := \max \operatorname{Re}\sigma(L)$ is negative, and we must find an equivalent norm $\|\cdot\|$ for which $\|e^{tL}\xi\| \leq \|\xi\|$, for every $\xi \in E$ and $t \geq 0$. Still denoting by $\|\cdot\|_0$ the operator norm induced by $\|\cdot\|_0$, the spectral radius formula and the spectral mapping theorem imply

$$\lim_{t\to+\infty} \|e^{tL}\|_0^{1/t} = \max|\sigma(e^L)| = \max|e^{\sigma(L)}| = e^{\alpha} < 1.$$

Therefore, there exists $c_0 > 0$ such that $\|e^{tL}\|_0 \leq c_0 e^{\alpha t/2}$ for every $t \geq 0$, so

$$\|\xi\| := \int_0^{+\infty} \|e^{tL}\xi\|_0 \, dt \leq \frac{2c_0}{\alpha}\|\xi\|_0 \quad \forall \xi \in E,$$

defines a norm on E not finer than $\|\cdot\|_0$. On the other hand, by compactness $\|e^{-tL}\|_0 \leq c_1$ for every $t \in [0,1]$, so

$$\|\xi\| \geq \int_0^1 \|e^{tL}\xi\|_0 \, dt \geq \frac{1}{c_1}\|\xi\|_0 \quad \forall \xi \in E,$$

and the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_0$. Finally, for every $t \geq 0$ and $\xi \in E$,

$$\|e^{tL}\xi\| = \int_0^{+\infty} \|e^{(s+t)L}\xi\|_0 \, ds = \int_t^{+\infty} \|e^{sL}\xi\|_0 \, ds \leq \|\xi\|,$$

concluding the proof. □

EXERCISE 1.2. Find an adapted norm for the hyperbolic operator on $E = \mathbb{R}^2$ defined by the matrix

$$L = \begin{pmatrix} -1 & \mu \\ 0 & -1 \end{pmatrix},$$

where $\mu \in \mathbb{R}$, and draw the corresponding unit ball when μ is large.

EXERCISE 1.3. Prove that if L is a normal operator on a Hilbert space H, that is L commutes with its adjoint L^*, then the Hilbert norm is L-adapted.

1.2. LINEAR STABLE AND UNSTABLE SPACES OF AN ASYMPTOTICALLY HYPERBOLIC PATH

Let $A\colon [0,+\infty] \to \mathcal{L}(E)$ be a continuous path of bounded linear operators on the Banach space E, such that $A(+\infty)$ is hyperbolic. Let $X_A\colon [0,+\infty[\to \mathcal{L}(E)$ be the solution of the linear Cauchy problem

$$\begin{cases} X_A'(t) = A(t)X_A(t), \\ X_A(0) = I. \end{cases}$$

EXERCISE 1.4. Prove that $X_A(t)$ is an isomorphism for every t, and find a linear Cauchy problem solved by its inverse.

The linear subspace of E

$$W_A^{\mathrm{s}} = \{\xi \in E \mid \lim_{t\to+\infty} X_A(t)\xi = 0\}$$

is said the *linear stable space* of the asymptotically hyperbolic path A. Similarly, if $A\colon [-\infty,0] \to \mathcal{L}(E)$ is a continuous path of operators such that $A(-\infty)$ is hyperbolic, the linear unstable space of A is defined as

$$W_A^{\mathrm{u}} = \{\xi \in E \mid \lim_{t\to-\infty} X_A(t)\xi = 0\}.$$

EXERCISE 1.5. Prove that if $A(t) \equiv L$ is constant (and hyperbolic), then $W_A^{\mathrm{s}} = E^{\mathrm{s}}(L)$, the negative eigenspace of L, and $W_A^{\mathrm{u}} = E^{\mathrm{u}}(L)$, the positive eigenspace of L.

A consequence of the hyperbolicity of $A(+\infty)$ is that the linear subspaces W_A^{s} and W_A^{u} are closed and complemented in E, and they are isomorphic to $E^{\mathrm{s}}(A(+\infty))$ and to $E^{\mathrm{u}}(A(-\infty))$, respectively. Indeed, one can prove that if A is close enough to $A(+\infty)$ in the L^∞ norm, then W_A^{s} is the graph of a bounded operator from $E^{\mathrm{s}}(A(+\infty))$ to $E^{\mathrm{u}}(A(+\infty))$. The statement for a general asymptotically hyperbolic path A follows, because

$$W_A^{\mathrm{s}} = X_A(t)^{-1} W_{A(t+\cdot)}^{\mathrm{s}}.$$

See for instance Abbondandolo and Majer (2003c, Proposition 1.2) for a complete proof (the case of a Hilbert space is treated in that reference, but the proof in the Banach setting presents no difference).

Denote by $C_0^k([0,+\infty[,E)$ the Banach space of all C^k curves $u\colon [0,+\infty[\to E$ such that

$$\lim_{t\to+\infty} u^{(h)}(t) = 0 \quad \forall h \in \{0,1,\ldots,k\}.$$

PROPOSITION 1.6. *Let $A \in C^0([0,+\infty],\mathcal{L}(E))$ be a path of bounded linear operators on the Banach space E such that $L = A(+\infty)$ is hyperbolic.*

(i) *The bounded linear operator*

$$F_A^+ : C_0^1([0,+\infty[,E) \to C_0^0([0,+\infty[,E), \quad u \mapsto u' - Au,$$

is a left inverse. Moreover, F_A^+ admits a right inverse R_A^+ such that

$$W_A^{\mathrm{s}} + \{R_A^+ v(0) \mid v \in C_0^0([0,+\infty[,E), v(0) = 0\} = E. \tag{4}$$

(ii) *The evaluation map*

$$\ker F_A^+ \to E, \quad u \mapsto u(0),$$

is a right inverse.

Proof. We endow E with a Banach norm $\|\cdot\|$ adapted to L.

(i) Let us start by considering the case of the constant path $A(t) \equiv L$. By (2), the operator valued piecewise continuous function

$$G: \mathbb{R} \to \mathcal{L}(E), \quad G(t) = e^{tL}(\mathbb{1}_{\mathbb{R}^+}(t)P^{\mathrm{s}} - \mathbb{1}_{\mathbb{R}^-}(t)P^{\mathrm{u}}),$$

satisfies $\|G(t)\| \leq e^{-\lambda|t|}$, in particular it is integrable on $\mathbb{R}$. Let $v \in C_0^0([0,+\infty[,E)$. It is readily seen that the curve

$$u(t) = (G * v)(t) = \int_0^{+\infty} G(t-\tau)v(\tau)\, d\tau$$

is continuously differentiable and solves the equation

$$u'(t) - Lu(t) = v(t). \tag{5}$$

Moreover, the inequality

$$\|u(t)\| \leq \|G\|_{L^1(\mathbb{R},\mathcal{L}(E))}\|v\|_{\infty,[s,+\infty[} + \|G\|_{L^1(]t-s,t[,\mathcal{L}(E))}\|v\|_\infty, \tag{6}$$

shows that $u \in C_0^0([0,+\infty[,E)$, so by (5), $u \in C_0^1([0,+\infty[,E)$. We conclude that the operator

$$R_L^+ : C_0^0([0,+\infty[,E) \to C_0^1([0,+\infty[,E), \quad v \mapsto G * v,$$

is a right inverse of F_L^+. Indeed, such a linear map is continuous by (5) and (6) with $s = 0$. Let us check that the operator $v \mapsto R_L^+ v(0)$ maps $v \in C_c^\infty(]0,+\infty[,E)$ onto E^{u}; since E^{u} is a direct complement of $E^{\mathrm{s}} = W_L^{\mathrm{s}}$ this implies that the right inverse R_L^+ satisfies (4). Let φ be a smooth real function with $\operatorname{supp}\varphi \subset]0,+\infty[$ so small that the operator

$$U := \int_0^{+\infty} \varphi(\tau)e^{-\tau L}\, d\tau \in \mathcal{L}(E)$$

is an isomorphism. The operator U preserves the splitting $E = E^{\mathrm{u}} \oplus E^{\mathrm{s}}$. If $\xi \in E^{\mathrm{u}}$, setting $v = -\varphi U^{-1}\xi$, there holds

$$R_L^+ v(0) = \int_0^{+\infty} e^{-\tau L} P^{\mathrm{u}} \varphi(\tau) U^{-1}\xi \, d\tau = \left(\int_0^{+\infty} \varphi(\tau) e^{-\tau L} \, d\tau \right) U^{-1}\xi = \xi,$$

proving the claim.

Let us now consider the general case. Setting $A_s(t) = A(s+t)$, we have that

$$\lim_{s \to +\infty} F_{A_s}^+ = F_L^+$$

in the operator norm of $\mathcal{L}(C_0^1, C_0^0)$. Since the set of left inverses is open, by our previous case we deduce that $F_{A_s}^+$ has a right inverse $R_{A_s}^+$ for s large, such that $R_{A_s}^+ \to R_L^+$ in the operator norm for $s \to +\infty$. Since the space of surjective operators is open, $R_{A_s}^+$ satisfies (4) for s large.

Fix such a large s. We can now define a right inverse R_A^+ of F_A^+ by setting $R_A^+ v = u$, where u is the solution of the linear Cauchy problem

$$\begin{cases} u' - Au = v, \\ u(s) = R_{A_s}^+ v_s(0). \end{cases}$$

The continuity of R_A^+ is easily seen by the formula

$$(R_A^+ v)(t) = X_A(t) \left(X_A(s)^{-1} R_{A_s}^+ v_s(0) + \int_s^t X_A(\tau)^{-1} v(\tau) \, d\tau \right).$$

Finally, the fact that $R_{A_s}^+$ satisfies (4) implies that also R_A^+ satisfies (4). Indeed, let $\xi \in E$, and let $v \in C_0^0([0, +\infty[, E)$ with $v(0) = 0$ be such that

$$X_A(s)\xi \in R_{A_s}^+ v(0) + W_{A_s}^{\mathrm{s}}. \tag{7}$$

Since $v(0) = 0$, the curve

$$w(t) = \begin{cases} v(t-s) & \text{if } t \geq s, \\ 0 & \text{if } 0 \leq t \leq s, \end{cases}$$

belongs to $C_0^0([0, +\infty[, E)$. Since w vanishes on $[0, s]$, $R_A^+ w$ solves the equation $u' - Au = 0$ on $[0, s]$, so

$$R_{A_s}^+ v(0) = R_A^+ w(s) = X_A(s) R_A^+ w(0). \tag{8}$$

By (7) and (8),

$$\xi \in R_A^+ w(0) + X_A(s)^{-1} W_{A_s}^{\mathrm{s}} = R_A^+ w(0) + W_A^{\mathrm{s}},$$

concluding the proof of (i).

(ii) The kernel of F_A^+ is

$$\ker F_A^+ = \{X_A(t)\xi \mid \xi \in W_A^s\},$$

so if $Q \in \mathcal{L}(E)$ is a projector onto W_A^s, the linear map

$$E \to \ker F_A^+, \quad \xi \mapsto X_A(\cdot)Q\xi,$$

is a left inverse of the evaluation at 0,

$$\ker F_A^+ \to E, \quad u \mapsto u(0). \qquad \square$$

REMARK 1.7. If P is a projector onto W_A^s, it can be shown that

$$R_A^+ v(t) = \int_0^{+\infty} X_A(t)(\mathbb{1}_{\mathbb{R}^+}(t-\tau)P - \mathbb{1}_{\mathbb{R}^-}(t-\tau)(I-P))X_A(\tau)^{-1} v(\tau)\, d\tau$$

defines a right inverse of F_A^+. See Abbondandolo and Majer (2003c) for a more extensive discussion of the topics of this section.

We conclude this section by establishing some properties of the operator $d/dt - A(t)$ on the whole real line.

PROPOSITION 1.8. *Assume that $A \in C^0(\overline{\mathbb{R}}, \mathcal{L}(E))$ has hyperbolic asymptotic operators $A(-\infty)$ and $A(+\infty)$, both with finite-dimensional positive eigenspace. Then the bounded linear operator*

$$F_A \colon C_0^1(\mathbb{R}, E) \to C_0^0(\mathbb{R}, E), \quad u \mapsto u' - Au,$$

is Fredholm of index

$$\operatorname{ind} F_A = \dim E^u(A(-\infty)) - \dim E^u(A(+\infty)).$$

Moreover, $W_A^u + W_A^s$ is closed and

$$\ker F_A \cong W_A^u \cap W_A^s, \quad \operatorname{coker} F_A \cong E/(W_A^u + W_A^s). \tag{9}$$

Proof. Since $W_A^u \cong E^u(A(-\infty))$ and $W_A^s \cong E^s(A(+\infty))$, the first space is finite-dimensional and the second one is finite-codimensional, with

$$\dim W_A^u = \dim E^u(A(-\infty)), \quad \operatorname{codim} W_A^s = \dim E^u(A(+\infty)). \tag{10}$$

Therefore, $W_A^u + W_A^s$ is (closed and) finite-codimensional, and

$$\dim W_A^u \cap W_A^s - \operatorname{codim}(W_A^u + W_A^s) = \dim W_A^u - \operatorname{codim} W_A^s. \tag{11}$$

The kernel of F_A is the linear subspace

$$\ker F_A = \{X_A(t)\xi \mid \xi \in W_A^{\mathrm{u}} \cap W_A^{\mathrm{s}}\},$$

so

$$\dim \ker F_A = \dim W_A^{\mathrm{u}} \cap W_A^{\mathrm{s}}. \tag{12}$$

By Proposition 1.6(i), the operators

$$\begin{aligned} F_A^+ &: C_0^1([0,+\infty[, E) \to C_0^0([0,+\infty[, E), \quad u \mapsto u' - Au, \\ F_A^- &: C_0^1(]-\infty,0], E) \to C_0^0(]-\infty,0], E), \quad u \mapsto u' - Au, \end{aligned}$$

have right inverses R_A^+ and R_A^-. If v is an element of $C_0^0(\mathbb{R}, E)$, any solution of $u' - Au = v$ has the form

$$\begin{aligned} u(t) &= X_A(t)(u(0) - R_A^+ v(0)) + R_A^+ v(t), \quad \forall t \geq 0, \\ u(t) &= X_A(t)(u(0) - R_A^- v(0)) + R_A^- v(t), \quad \forall t \leq 0. \end{aligned}$$

Such a curve u belongs to $C_0^1(\mathbb{R}, E)$ if and only if $u(0) - R_A^+ v(0) \in W_A^{\mathrm{s}}$ and $u(0) - R_A^- v(0) \in W_A^{\mathrm{u}}$. Therefore, v belongs to the range of F_A if and only if the affine subspaces $R_A^+ v(0) + W_A^{\mathrm{s}}$ and $R_A^- v(0) + W_A^{\mathrm{u}}$ have nonempty intersection, that is if and only if $R_A^+ v(0) - R_A^- v(0)$ belongs to $W_A^{\mathrm{s}} + W_A^{\mathrm{u}}$. So the range of F_A is the linear subspace

$$\operatorname{ran} F_A = \{v \in C_0^0(\mathbb{R}, E) \bigm| R_A^+ v(0) - R_A^- v(0) \in W_A^{\mathrm{u}} + W_A^{\mathrm{s}}\}.$$

Such a linear subspace is closed. By the second assertion in Proposition 1.6(i), the operator

$$C_0^0(\mathbb{R}, E) \to \frac{E}{W_A^{\mathrm{u}} + W_A^{\mathrm{s}}}, \quad v \mapsto [R_A^+ v(0) - R_A^- v(0)],$$

is onto, so

$$\operatorname{codim} \operatorname{ran} F_A = \operatorname{codim}(W_A^{\mathrm{u}} + W_A^{\mathrm{s}}). \tag{13}$$

All the statements follow from (10)–(13). □

1.3. MORSE VECTOR FIELDS

Let M be a Banach manifold of class C^2, i.e., a paracompact Hausdorff topological space, locally homeomorphic to a Banach space E, endowed with an atlas whose transition maps are of class C^2. See Lang (1999) for foundational results on Banach manifolds. A C^1 tangent vector field X on M defines a local flow ϕ solving

$$\partial_t \phi(t,p) = X(\phi(t,p)), \; \phi(0,p) = p, \quad \forall p \in M, \; -\infty \leq t^-(p) < t < t^+(p) \leq +\infty,$$

where $]t^-(p), t^+(p)[$ denotes the maximal interval of existence of the above Cauchy problem. The functions t^- and t^+ are upper and lower semi-continuous, respectively. Denote by $\Omega(X)$ the subset of $\mathbb{R} \times M$ which lies strictly between the graph of t^- and the graph of t^+. Then $\Omega(X)$ is an open neighborhood of $\{0\} \times M$, and the map $\phi \colon \Omega(X) \to M$ is of class C^1. The vector field X is said *complete* (resp. *positively complete*, resp. *negatively complete*) if $\Omega(X) = \mathbb{R} \times M$ (resp. $\Omega(X) \supset [0, +\infty[\times M$, resp. $\Omega(X) \supset]-\infty, 0] \times M$).

Let A be a *positively invariant* subset of M: this means that if $p \in A$ then $\phi(t, p) \in A$ for every $t \in [0, t^+(p)[$. The vector field is said *positively complete with respect to* A if for every $p \in M$ such that $t^+(p) < +\infty$ there exists $t \in [0, t^+(p)[$ such that $\phi(t, p) \in A$. Similarly, one defines a negatively complete vector field with respect to a negatively invariant subset.

A *rest point* of X is a point $x \in M$ such that $X(x) = 0$. The set of rest points of X will be denoted by $\mathrm{rest}(X)$. The *Jacobian* of X at a rest point x is the bounded linear operator on $T_x M$ defined by $\nabla X(x)\xi = [X, Y](x)$, where $\xi \in T_x M$ and Y is a tangent vector field on M such that $Y(x) = \xi$. Indeed, the fact that $X(x) = 0$ implies that this definition does not depend on the choice of extension Y of ξ.

EXERCISE 1.9. Give an alternative definition of the Jacobian of a vector field at a rest point in terms of a local chart: if $\varphi \colon U \to E$ maps a neighborhood U of $x \in \mathrm{rest}(X)$ diffeomorphically onto an open subset of the Banach space E, define the operator $\nabla X(x)$ on $T_x M$ by

$$\varphi_*(\nabla X(x)\xi) = D(\varphi_* X)(\varphi(x))[\varphi_*\xi], \quad \forall \xi \in T_x M,$$

where, for $\eta \in T_p M$, $\varphi_*\eta$ is the vector in E defined by $\varphi_*\eta = D\varphi(p)[\eta]$. Show that such a definition does not depend on the choice of the chart φ.

A rest point x of X is said *hyperbolic* if the Jacobian of X at x is a hyperbolic operator. The corresponding splitting of the tangent space at x will be denoted by $T_x M = E^{\mathrm{u}}_x \oplus E^{\mathrm{s}}_x$. By the inverse mapping theorem, the hyperbolic rest points are isolated in $\mathrm{rest}(X)$. The *Morse index* $m(x) \in \mathbb{N} \cup \{+\infty\}$ of the hyperbolic rest point x is the dimension of the subspace E^{u}_x. The *Morse co-index* is the dimension of E^{s}_x. If all the rest points of X are hyperbolic, the vector field X is said a *Morse vector field*.

1.4. LOCAL DYNAMICS NEAR A HYPERBOLIC REST POINT

Let U be an open neighborhood of 0 in the Banach space E, and let $X \in C^1(U, E)$ be a vector field having 0 as a hyperbolic rest point. Denote by $\phi \colon \Omega(X) \to U$ the local flow of X. Let $L := \nabla X(0) = DX(0)$, with splitting $E = E^{\mathrm{u}} \oplus E^{\mathrm{s}}$ and projectors P^{u}, P^{s}, and let us endow E with an L-adapted norm $\|\cdot\|$. If $V \subset E$ is a closed linear subspace, $V(r)$ will denote the closed ball in V of radius r centered in 0, and $\partial V(r)$ will be the relative boundary of $V(r)$ in V. Consider the cones

$$C^{\mathrm{u}} = \{\xi \in E \mid \|P^{\mathrm{s}}\xi\| \le \|P^{\mathrm{u}}\xi\|\} \supset E^{\mathrm{u}}, \quad C^{\mathrm{s}} = \{\xi \in E \mid \|P^{\mathrm{u}}\xi\| \le \|P^{\mathrm{s}}\xi\|\} \supset E^{\mathrm{s}}.$$

We recall that if $A \subset B \subset U$ the set A is said *positively* (*negatively*) *invariant with respect to* B if for every $\xi \in A$ and for every $t > 0$, $\phi([0,t] \times \{\xi\}) \subset B$ implies $\phi([0,t] \times \{\xi\}) \subset A$ (resp. for every $\xi \in A$ and for every $t < 0$, $\phi([t,0] \times \{\xi\}) \subset B$ implies $\phi([t,0] \times \{\xi\}) \subset A$).

LEMMA 1.10. *For every* $r > 0$ *small enough there holds*:

(i) *the set* $C^{\mathrm{u}} \cap E(r)$ *is positively invariant with respect to* $E(r)$;

(ii) *the set* $C^{\mathrm{s}} \cap E(r)$ *is negatively invariant with respect to* $E(r)$;

(iii) *if* ξ *belongs to the set* $C^{\mathrm{u}} \cap \partial E(r) = \partial E^{\mathrm{u}}(r) \times E^{\mathrm{s}}(r)$ *then* $\|P^{\mathrm{u}}\phi(t,\xi)\| > r$ *for every* $t \in]0,1]$, *and* $\|P^{\mathrm{u}}\phi(t,\xi)\| < r$ *for every* $t \in [-1,0[$;

(iv) *if* ξ *belongs to the set* $C^{\mathrm{s}} \cap \partial E(r) = E^{\mathrm{u}}(r) \times \partial E^{\mathrm{s}}(r)$ *then* $\|P^{\mathrm{s}}\phi(t,\xi)\| < r$ *for every* $t \in]0,1]$, *and* $\|P^{\mathrm{s}}\phi(t,\xi)\| > r$ *for every* $t \in [-1,0[$.

Proof. Since $t^+(0) = +\infty$ and $t^-(0) = -\infty$, we have $t^+(\xi) > 1$ and $t^-(\xi) < -1$ for $\|\xi\|$ small enough. A first order expansion of $\phi(t,\cdot)$ at 0 yields to

$$\phi(t,\xi) = e^{tL}\xi + o(\xi)t \quad \text{for } \xi \to 0,$$

uniformly in $t \in [-1,1]$. Therefore, if $r > 0$ is small enough, for every $\xi \in C^{\mathrm{s}} \cap E(r)$ and $t \in [0,1]$, (2) implies

$$\begin{aligned} \|P^{\mathrm{s}}\phi(t,\xi)\| = \|P^{\mathrm{s}}e^{tL}\xi\| + o(\xi)t &= \|e^{tL}P^{\mathrm{s}}\xi\| + o(P^{\mathrm{s}}\xi)t \\ &\le e^{-\lambda t}\|P^{\mathrm{s}}\xi\| + o(P^{\mathrm{s}}\xi)t \le e^{-\lambda t/2}\|P^{\mathrm{s}}\xi\|, \end{aligned}$$

and similarly, for every $\xi \in C^{\mathrm{u}} \cap E(r)$ and $t \in [0,1]$, (3) implies

$$\|P^{\mathrm{u}}\phi(t,\xi)\| \ge e^{\lambda t/2}\|P^{\mathrm{u}}\xi\|.$$

All the statements follow from the above inequalities and from the analogous inequalities holding for $t \in [-1,0]$. □

REMARK 1.11. In the language of Conley theory, $E(r)$ is an isolating neighborhood for the invariant set $\{0\}$, and $\partial E^{\mathrm{u}}(r) \times E^{\mathrm{s}}(r)$ is its exit set.

1.5. LOCAL STABLE AND UNSTABLE MANIFOLDS

Given $r > 0$, the *local unstable manifold* and the *local stable manifold* of 0 are the sets

$$W^{\mathrm{u}}_{\mathrm{loc},r}(0) = \Big\{\xi \in E(r) \;\Big|\; t^-(\xi) = -\infty,\ \phi(]-\infty, 0] \times \{\xi\}) \subset E(r),\ \lim_{t\to-\infty} \phi(t,\xi) = 0\Big\},$$
$$W^{\mathrm{s}}_{\mathrm{loc},r}(0) = \Big\{\xi \in E(r) \;\Big|\; t^+(\xi) = +\infty,\ \phi([0, +\infty[\times \{\xi\}) \subset E(r),\ \lim_{t\to+\infty} \phi(t,\xi) = 0\Big\}.$$

When r is small, these sets are actually graphs of regular maps.

THEOREM 1.12 (Local (un)stable manifold theorem). *Assume that* 0 *is a hyperbolic rest point of the* C^k *vector field* $X: U \to E$, $k \geq 1$. *For any* $r > 0$ *small enough,* $W^{\mathrm{s}}_{\mathrm{loc},r}(0)$ *is the graph of a* C^k *map* $\sigma^{\mathrm{s}}: E^{\mathrm{s}}(r) \to E^{\mathrm{u}}(r)$ *such that* $\sigma^{\mathrm{s}}(0) = 0$ *and* $D\sigma^{\mathrm{s}}(0) = 0$. *Similarly,* $W^{\mathrm{u}}_{\mathrm{loc},r}(0)$ *is the graph of a* C^k *map* $\sigma^{\mathrm{u}}: E^{\mathrm{u}}(r) \to E^{\mathrm{s}}(r)$ *such that* $\sigma^{\mathrm{u}}(0) = 0$, $D\sigma^{\mathrm{u}}(0) = 0$.

See Shub (1987, Chapter 5) for a proof based on the graph transform method. Here we will present a proof based on the study of the orbit space and on Proposition 1.6.

Proof. We shall prove the conclusion for the local stable manifold, the case of the unstable one following by considering the vector field $-X$. The map

$$\Phi: C^1_0([0, +\infty[, U) \to C^0_0([0, +\infty[, E), \quad u \mapsto u' - X \circ u,$$

is of class C^k, and its differential at $u \in C^1_0([0, +\infty[, U)$ is

$$D\Phi(u): C^1_0([0, +\infty[, E) \to C^0_0([0, +\infty[, E), \quad v \mapsto v' - DX(u)v.$$

Since $DX(u(t))$ converges to $L = DX(0)$ for $t \to +\infty$, statement (i) of Proposition 1.6 implies that $D\Phi(u)$ is a left inverse, so Φ is a C^k submersion. In particular, its set of zeros $\Phi^{-1}(\{0\})$ is a C^k submanifold of $C^1_0([0, +\infty[, U)$. The set of zeros is nonempty, because it contains the curve 0. Actually

$$T_0\Phi^{-1}(\{0\}) = \ker D\Phi(0) = \ker(v \mapsto v' - Lv) = \{e^{tL}\xi \mid \xi \in E^{\mathrm{s}}\}. \tag{14}$$

By statement (ii) of Proposition 1.6, the evaluation map $\mathrm{ev}_0: u \mapsto u(0)$ subordinates an immersion

$$\mathrm{ev}_0: \Phi^{-1}(\{0\}) \to U,$$

which is injective by the uniqueness of the solution of Cauchy problems. Therefore,

$$W^{\mathrm{s}}(0) := \mathrm{ev}_0(\Phi^{-1}(\{0\})) = \Big\{\xi \in U \;\Big|\; t^+(\xi) = +\infty,\ \lim_{t\to+\infty} \phi(t,\xi) = 0\Big\}$$

is the image of an injective C^k immersion. The point 0 belongs to $W^s(0)$, and by (14),

$$T_0 W^s(0) = D\,\mathrm{ev}_0(0) T_0 \Phi^{-1}(\{0\}) = \mathrm{ev}_0(T_0 \Phi^{-1}(\{0\})) = E^s.$$

By the implicit function theorem, if r is small enough the path-connected component of $W^s(0) \cap E(r)$ containing 0 — call it Z_r — is the graph of a C^k map

$$\sigma^s \colon E^s(r) \to E^u(r)$$

such that $\sigma^s(0) = 0$ and $D\sigma^s(0) = 0$.

We claim that if r is so small that the conclusions of Lemma 1.10 hold, and that the Lipschitz norm of σ^s is less than 1, then $Z_r = W^s_{\mathrm{loc},r}(0)$, which concludes the proof. Indeed, by definition $W^s_{\mathrm{loc},r}(0) \subset Z_r$, a path connecting $\xi \in W^s_{\mathrm{loc},r}(0)$ to 0 within $W^s(0) \cap E(r)$ being provided by the orbit of ξ. On the other hand, notice that by definition Z_r is positively invariant with respect to $E(r)$. So if there exists $\xi \in Z_r \setminus W^s_{\mathrm{loc},r}(0)$, by Lemma 1.10 there is some $t > 0$ for which $\phi(t, \xi) \in (\partial E^u(r) \times E^s(r))$ (the latter is the exit set of $E(r)$) and $\phi(t, \xi) \in Z_r$ (Z_r is positively invariant with respect to $E(r)$). Therefore $Z_r \cap (\partial E^u(r) \times E^s(r))$ is nonempty, contradicting the fact that Z_r is the graph of a map whose Lipschitz constant is less than 1, taking value 0 at 0. □

1.6. THE GROBMAN – HARTMAN LINEARIZATION THEOREM

The Grobman – Hartman theorem says that up to a change of variables, the dynamics near a hyperbolic point is the dynamics given by a linear vector field. We will deduce this fact from the analogous statement for discrete dynamical systems. The proof is adapted from Shub (1987). Let us start with a result about the existence, uniqueness, and Hölder regularity of a semi-conjugacy between two perturbations of a linear operator.

PROPOSITION 1.13. *Let $E = E^u \oplus E^s$ be an invariant splitting for the bounded invertible operator T. Let P^u and P^s be the corresponding projectors, and assume that there exists $\mu < 1$ such that*

$$\max\{\|P^s T P^s\|, \|P^u T^{-1} P^u\|\} \leq \mu.$$

Let φ and ψ be Lipschitz continuous maps from E to E such that:

(i) $\|\varphi - \psi\|_\infty < +\infty$;

(ii) $\mathrm{lip}\,\varphi < 1 - \mu$;

(iii) $\mathrm{lip}\,\psi < 1/\|T^{-1}\|$.

Then there exists a unique bounded map $g\colon E \to E$ *such that*

$$(T+\varphi)\circ(I+g) = (I+g)\circ(T+\psi). \tag{15}$$

Moreover,

$$\|g\|_\infty \leq \frac{\|\varphi-\psi\|_\infty}{1-(\mu+\operatorname{lip}\varphi)},$$

and setting

$$\theta := \max\left\{\|T\| + \operatorname{lip}\psi, \frac{\|T^{-1}\|}{1-\|T^{-1}\|\operatorname{lip}\psi}\right\},$$

g *is* α*-Hölder continuous for every*

$$\alpha < \frac{-\log(\mu+\operatorname{lip}\varphi)}{\log\theta}.$$

Notice that if $E^{\mathrm{u}} \neq (0)$, then $\|T\| \geq \|P^{\mathrm{u}}TP^{\mathrm{u}}\| \geq 1/\mu$, while if $E^{\mathrm{s}} \neq (0)$, then $\|T^{-1}\| \geq \|P^{\mathrm{s}}T^{-1}P^{\mathrm{s}}\| \geq 1/\mu$. Therefore $\log\theta \geq -\log\mu$, so the quantity $-\log(\mu+\operatorname{lip}\varphi)/\log\theta$ appearing in the above proposition does not exceed 1. In general, the map g is not locally Lipschitz, even when φ and ψ are smooth.

Proof. For an E-valued map f, we denote by f_{u} and f_{s} its components with respect to the splitting $E = E^{\mathrm{u}} \oplus E^{\mathrm{s}}$, that is $f_{\mathrm{u}} := P^{\mathrm{u}}f$, $f_{\mathrm{s}} := P^{\mathrm{s}}f$. By applying the projectors P^{u} and P^{s}, (15) is equivalent to

$$\begin{cases} (T_{\mathrm{u}}+\varphi_{\mathrm{u}})\circ(I+g) = (P^{\mathrm{u}}+g_{\mathrm{u}})\circ(T+\psi), \\ (T_{\mathrm{s}}+\varphi_{\mathrm{s}})\circ(I+g) = (P^{\mathrm{s}}+g_{\mathrm{s}})\circ(T+\psi). \end{cases} \tag{16}$$

Since $\operatorname{lip} T^{-1}\psi \leq \|T^{-1}\|\operatorname{lip}\psi < 1$, the map $T+\psi = T(I+T^{-1}\psi)$ is a homeomorphism of E onto E. Actually, its inverse is Lipschitz continuous with

$$\operatorname{lip}(T+\psi)^{-1} = \operatorname{lip}((I+T^{-1}\psi)^{-1}T^{-1}) \leq \frac{\|T^{-1}\|}{1-\|T^{-1}\|\operatorname{lip}\psi}. \tag{17}$$

By a simple algebraic manipulation, (16) is equivalent to the fixed point problem $F(g) = g$, where

$$\begin{aligned} F(g)_{\mathrm{u}} &= T_{\mathrm{u}}^{-1}(g_{\mathrm{u}}\circ(T+\psi) - \varphi_{\mathrm{u}}\circ(I+g) + \psi_{\mathrm{u}}), \\ F(g)_{\mathrm{s}} &= (T_{\mathrm{s}}g_{\mathrm{s}} + \varphi_{\mathrm{s}}\circ(I+g) - \psi_{\mathrm{s}})\circ(T+\psi)^{-1}. \end{aligned}$$

Since

$$\begin{aligned} \|F(g)_{\mathrm{u}}\|_\infty &\leq \mu((1+\operatorname{lip}\varphi)\|g\|_\infty + \|\varphi-\psi\|_\infty) \\ &\leq (\mu+\operatorname{lip}\varphi)\|g\|_\infty + \mu\|\varphi-\psi\|_\infty, \qquad (18) \\ \|F(g)_{\mathrm{s}}\|_\infty &\leq (\|T_{\mathrm{s}}\| + \operatorname{lip}\varphi)\|g\|_\infty + \|\varphi-\psi\|_\infty \\ &\leq (\mu+\operatorname{lip}\varphi)\|g\|_\infty + \|\varphi-\psi\|_\infty, \qquad (19) \end{aligned}$$

F maps $B(E,E)$, the Banach space of bounded maps from E to E, into itself. Actually, (18) and (19) imply that if $\|g\|_\infty \leq R$, with

$$R := \frac{\|\varphi - \psi\|_\infty}{1 - (\mu + \operatorname{lip}\varphi)},$$

then $\|F(g)\|_\infty \leq R$. Moreover, the maps $F_{\mathrm{u}} = P^{\mathrm{u}} F \colon B(E,E) \to B(E,E^{\mathrm{u}})$ and $F_{\mathrm{s}} = P^{\mathrm{s}} F \colon B(E,E) \to B(E,E^{\mathrm{s}})$ are Lipschitz with

$$\begin{aligned}\operatorname{lip} F_{\mathrm{u}} &\leq \|T_{\mathrm{u}}\|^{-1}(1 + \operatorname{lip}\varphi) \leq \mu(1 + \operatorname{lip}\varphi) < 1,\\ \operatorname{lip} F_{\mathrm{s}} &\leq \|T_{\mathrm{s}}\| + \operatorname{lip}\varphi \leq \mu + \operatorname{lip}\varphi < 1,\end{aligned}$$

so $F \colon B(E,E) \to B(E,E)$ is a contraction, proving that there exists a unique $g \in B(E,E)$ satisfying (15). Since F maps the closed R-ball of $C^0 \cap B(E,E)$ into itself, the fixed point g is continuous and bounded by R.

If $h \in B(E,E)$ has modulus of continuity[2] ω, then $F(h)_{\mathrm{u}}$ has modulus of continuity

$$t \mapsto \mu\omega((\|T\| + \operatorname{lip}\psi)t) + \mu \operatorname{lip}\varphi\,\omega(t) + \mu(\operatorname{lip}\varphi + \operatorname{lip}\psi)t, \tag{20}$$

while by (17), $F(h)_{\mathrm{s}}$ has modulus of continuity

$$t \mapsto (\mu + \operatorname{lip}\varphi)\omega(\sigma t) + (\operatorname{lip}\psi + \operatorname{lip}\varphi)\sigma t, \tag{21}$$

where $\sigma := \|T^{-1}\|/(1 - \|T^{-1}\| \operatorname{lip}\psi)$. Comparing (20) and (21), we find that setting

$$a := (\operatorname{lip}\varphi + \operatorname{lip}\psi)\sigma,$$

the function

$$t \mapsto (\mu + \operatorname{lip}\varphi)\omega(\theta t) + at$$

is a modulus of continuity for $F(h)$. If moreover $\|h\|_\infty \leq R$, we have that $\|F(h)\|_\infty \leq R$, so $F(h)$ has modulus of continuity

$$t \mapsto \min\{(\mu + \operatorname{lip}\varphi)\omega(\theta t) + at, 2R\}.$$

Therefore, if a modulus of continuity ω satisfies

$$\min\{(\mu + \operatorname{lip}\varphi)\omega(\theta t) + at, 2R\} \leq \omega(t) \quad \forall t \in [0, +\infty[, \tag{22}$$

we deduce that the nonempty closed subset of $B(E,E)$

$$\{h \in B(E,E) \mid \|h\|_\infty \leq R, h \text{ has modulus of continuity } \omega\}$$

[2] Here, moduli of continuity are always assumed to be nondecreasing.

is F-invariant, hence the fixed point g has modulus of continuity ω. A function of the form $\omega(t) = ct^\alpha$ satisfies (22) if

$$(\mu + \operatorname{lip}\varphi)\theta^\alpha < 1,$$

and c is large enough. The conclusion follows. □

If we symmetrize the assumptions of the above proposition, a standard argument involving uniqueness yields to the following global version of the Grobman – Hartman theorem for discrete dynamical systems.

COROLLARY 1.14. *Let T be an invertible bounded operator on E satisfying the same assumptions of Proposition* 1.13. *Let φ and ψ be Lipschitz continuous maps from E to E such that*

(i) $\|\varphi - \psi\|_\infty < +\infty$;

(ii) $\operatorname{lip}\varphi < \min\{1-\mu, 1/\|T^{-1}\|\}$, $\operatorname{lip}\psi < \min\{1-\mu, 1/\|T^{-1}\|\}$.

Then there exists a unique bounded map $g \colon E \to E$ such that

$$(T + \varphi) \circ (I + g) = (I + g) \circ (T + \psi).$$

Moreover, g is Hölder continuous, and $I + g$ is homeomorphism of E onto E.

Proof. Applying Proposition 1.13 to the pair (φ, ψ) and to the pair (ψ, φ), we find Hölder continuous bounded maps $g \colon E \to E$ and $h \colon E \to E$ such that

$$\begin{aligned}(T + \varphi) \circ (I + g) &= (I + g) \circ (T + \psi),\\ (T + \psi) \circ (I + h) &= (I + h) \circ (T + \varphi).\end{aligned}$$

It follows that $(I + g) \circ (I + h)$, which is of the form $I + k$ with $k \in B(E, E)$, satisfies

$$(T + \varphi) \circ (I + k) = (I + k) \circ (T + \varphi).$$

By the uniqueness statement of Proposition 1.13 applied to the pair (φ, φ), k must be the zero map, that is $(I + g) \circ (I + h) = I$. Similarly, $(I + h) \circ (I + g) = I$, so $I + g$ is a homeomorphism of E onto E with inverse $I + h$. □

Now we can derive a global version of the Grobman – Hartman theorem for flows.

THEOREM 1.15. *Let L be a hyperbolic operator on E. Let $\|\cdot\|$ be an L-adapted norm on E, satisfying* (1) *and* (2) *for some positive λ. Let $B_1 \colon E \to E$ and $B_2 \colon E \to E$ be Lipschitz continuous maps such that*

(i) $\|B_1 - B_2\|_\infty < +\infty$;

(ii) $\operatorname{lip} B_1 < \lambda$, $\operatorname{lip} B_2 < \lambda$.

Then the flows $\phi_1, \phi_2 \colon \mathbb{R} \times E \to E$ *of the vector fields* $X_1(\xi) = L\xi + B_1(\xi)$ *and* $X_2(\xi) = L\xi + B_2(\xi)$ *are conjugated. More precisely, there is a unique bounded map* $g \colon E \to E$ *such that*

$$\phi_1(t, (I+g)(\xi)) = (I+g)(\phi_2(t,\xi) \quad \forall (t,\xi) \in \mathbb{R} \times E,$$

and $I+g$ *is a homeomorphism of* E *onto* E. *Moreover,* g *is* α*-Hölder continuous for every*

$$\alpha < \frac{\lambda - \operatorname{lip} B_1}{\|L\| + \operatorname{lip} B_2}.$$

Proof. A c-Lipschitz vector field X produces a globally defined flow ϕ, with $\operatorname{lip} \phi(t,\cdot) \leq e^{c|t|}$. If two c-Lipschitz vector fields X_1, X_2 have bounded distance, then

$$\|\phi_1(t,\cdot) - \phi_2(t,\cdot)\|_\infty \leq \|X_1 - X_2\|_\infty |t| e^{c|t|}.$$

Let $\psi_i(t,\xi) = \phi_i(t,\xi) - e^{tL}\xi$, for $i = 1,2$. By our initial considerations, for every t the maps $\psi_1(t,\cdot)$ and $\psi_2(t,\cdot)$ are Lipschitz and have bounded distance. The maps ψ_i satisfy

$$\psi_i(t,\xi) = \int_0^t e^{(t-s)L} B_i(e^{sL}\xi + \psi_i(s,\xi))\, ds \quad \forall (t,\xi) \in \mathbb{R} \times E. \tag{23}$$

By (23),

$$\begin{aligned}
\operatorname{lip} \psi_i(t,\cdot) &\leq \operatorname{lip} B_i \left| \int_0^t \|e^{(t-s)L}\| (\|e^{sL}\| + \operatorname{lip} \psi_i(s,\cdot))\, ds \right| \\
&\leq |t| \operatorname{lip} B_i \sup_{|s|\leq|t|} \|e^{sL}\| \left(\sup_{|s|\leq|t|} \|e^{sL}\| + \sup_{|s|\leq|t|} \operatorname{lip} \psi_i(s,\cdot) \right) \\
&= |t| \operatorname{lip} B_i \Big(1 + \sup_{|s|\leq|t|} \operatorname{lip} \psi_i(s,\cdot) \Big) (1 + o(1)),
\end{aligned}$$

for $t \to 0$. Taking the supremum for all $|t| \leq \tau$, we obtain

$$\sup_{|t|\leq\tau} \operatorname{lip} \psi_i(t,\cdot) \leq \operatorname{lip} B_i |\tau| (1 + o(1)) \quad \text{for } \tau \to 0. \tag{24}$$

Since $\operatorname{lip} B_i < \lambda$, the last inequality implies that there exists $\tau > 0$ such that

$$\operatorname{lip} \psi_i(t,\cdot) < 1 - e^{-\lambda|t|}, \ \operatorname{lip} \psi_i(t,\cdot) < 1/\|e^{-tL}\|, \quad \forall\, 0 < |t| \leq \tau, \ i = 1,2.$$

By Corollary 1.14 applied to $T = e^{tL}$, $\mu = e^{-\lambda|t|}$, $\varphi = \psi_1(t,\cdot)$, and $\psi = \psi_2(t,\cdot)$, for every $0 < |t| \leq \tau$ there exists a unique $g_t \in B(E,E)$ such that

$$\phi_1(t, (I + g_t)(\xi)) = (I + g_t)(\phi_2(t,\xi)), \tag{25}$$

and $I+g_t$ is a homeomorphism of E onto E. If $n \in \mathbb{Z}\setminus\{0\}$ and $|nt| \leq \tau$, (25) implies

$$\phi_1(nt, (I+g_t)(\cdot)) = (I+g_t)(\phi_2(nt,\cdot)),$$

so by uniqueness $g_t = g_{nt}$. If p, q are rational numbers in $[-\tau,\tau]\setminus\{0\}$, they have a common sub-multiple, so $g_p = g_q$. Therefore $g_p = g$ for every rational $p \in [-\tau,\tau]\setminus\{0\}$. By the continuity of ϕ_1 and ϕ_2 with respect of t,

$$\phi_1(t, (I+g)(\cdot)) = (I+g)(\phi_2(t,\cdot))$$

holds for every $|t| \leq \tau$, hence by taking iterates, for every $t \in \mathbb{R}$.

There remains to estimate the Hölder exponent of g. Let $0 < t \leq \tau$. By Proposition 1.13 g_t is α-Hölder for every

$$\alpha < \frac{-\log(e^{-\lambda t} + \operatorname{lip}\psi_1(t,\cdot))}{\log\theta(t)},$$

where

$$\theta(t) = \max\left\{\|e^{tL}\| + \operatorname{lip}\psi_2(t,\cdot), \frac{\|e^{-tL}\|}{1-\|e^{-tL}\|\operatorname{lip}\psi_2(t,\cdot)}\right\}.$$

Since $g_t = g$, g is α-Hölder for every

$$\alpha < \beta := \limsup_{t\to 0^+} \frac{-\log(e^{-\lambda t} + \operatorname{lip}\psi_1(t,\cdot))}{\log\theta(t)}.$$

By (24),

$$-\log(e^{-\lambda t} + \operatorname{lip}\psi_1(t,\cdot)) \geq -\log(e^{-\lambda t} + \operatorname{lip} B_1 t + o(t)) = (\lambda - \operatorname{lip} B_1)t + o(t) \quad (26)$$

for $t \to 0^+$. Since

$$\|e^{\pm tL}\| \leq e^{t\|L\|} = 1 + \|L\|t + o(t),$$

for $t \to 0^+$, by (24) there holds

$$\begin{aligned} \|e^{tL}\| + \operatorname{lip}\psi_2(t,\cdot) &\leq 1 + (\|L\| + \operatorname{lip} B_2)t + o(t), \\ \frac{\|e^{-tL}\|}{1-\|e^{-tL}\|\operatorname{lip}\psi_2(t,\cdot)} &\leq \frac{1+\|L\|t + o(t)}{1 - \operatorname{lip} B_2 t + o(t)} = 1 + (\|L\| + \operatorname{lip} B_2)t + o(t), \end{aligned}$$

for $t \to 0^+$. Therefore

$$\log\theta(t) \leq \log(1 + (\|L\| + \operatorname{lip} B_2)t + o(t)) = (\|L\| + \operatorname{lip} B_2)t + o(t), \quad (27)$$

for $t \to 0^+$. The inequalities (26) and (27) imply that

$$\beta = \limsup_{t\to 0^+} \frac{-\log(e^{-\lambda t} + \operatorname{lip}\psi_1(t,\cdot))}{\log\theta(t)} \geq \lim_{t\to 0^+} \frac{(\lambda - \operatorname{lip} B_1)t + o(t)}{(\|L\| + \operatorname{lip} B_2)t + o(t)} = \frac{\lambda - \operatorname{lip} B_1}{\|L\| + \operatorname{lip} B_2},$$

concluding the proof. □

It is then straightforward to deduce the following local linearization result.

COROLLARY 1.16. *Assume that* 0 *is a hyperbolic rest point of the* C^1 *vector field* $X\colon U \to E$, *and let* $L = DX(0)$. *If* $r > 0$ *is small enough, the local flow* ϕ *restricted to* $E(r)$ *is conjugated to the linear flow* $(t,\xi) \mapsto e^{tL}\xi$ *by a bi-Hölder continuous homeomorphism. More precisely, there exists a bi-Hölder continuous homeomorphism* $h\colon E(r) \to h(E(r)) \subset E$ *such that*

$$h(\phi(t,\xi)) = e^{tL}h(\xi) \quad \forall (t,\xi) \in \Omega(X|_{E(r)}).$$

We conclude the discussion about the local dynamics at a rest point with the following proposition.

PROPOSITION 1.17. *For every* $r > 0$ *small enough there holds*: *for every sequence* $(\xi_n) \subset E$ *converging to 0 and for every sequence* $(t_n) \subset [0,+\infty[$ *such that* $\phi([0,t_n] \times \{\xi_n\}) \subset E(r)$ *and* $\phi(t_n,\xi_n) \in \partial E(r)$, *there holds*

$$\operatorname{dist}(\phi(t_n,\xi_n), W^{\mathrm{u}}_{\mathrm{loc},r}(0) \cap \partial E(r)) \to 0.$$

Proof. If the vector field is linear, $X(\xi) = L\xi$, the conclusion is immediate: indeed in this case $W^{\mathrm{u}}_{\mathrm{loc},r}(0) = E^{\mathrm{u}}(r)$, and for any $(\xi_n) \subset E$ converging to 0 and any $(t_n) \subset [0,+\infty[$, by (2) we have

$$\limsup_{n\to\infty} \operatorname{dist}(e^{t_nL}\xi_n, E^{\mathrm{u}}) = \limsup_{n\to\infty} \|P^{\mathrm{s}}e^{t_nL}\xi_n\| \leq \limsup_{n\to\infty} e^{-t_n\lambda}\|P^{\mathrm{s}}\xi_n\| = 0.$$

By the Grobman – Hartman theorem, if $r_0 > 0$ is small enough the local flow ϕ restricted to $E(r_0)$ is conjugated to its linearization $(t,\xi) \mapsto e^{tL}\xi$, by a bi-uniformly continuous homeomorphism. By the local (un)stable manifold theorem, we may also assume that r_0 is so small that $W^{\mathrm{u}}_{\mathrm{loc},r_0}(0)$ is the graph of a uniformly continuous map $\sigma^{\mathrm{u}}\colon E^{\mathrm{u}}(r_0) \to E^{\mathrm{s}}(r_0)$.

Let $r < r_0$ and set $\eta_n := \phi(t_n,\xi_n) \in \partial E(r)$, with $\xi_n \to 0$ and $t_n \geq 0$. By Lemma 1.10, $\|P^{\mathrm{u}}\eta_n\| = r$. By the linear case and by the uniform continuity of the conjugacy, there exists $(\eta'_n) \subset W^{\mathrm{u}}_{\mathrm{loc},r_0}(0)$ such that $\|\eta_n - \eta'_n\|$ is infinitesimal. Setting $\eta''_n = (P^{\mathrm{u}}\eta_n, \sigma^{\mathrm{u}}(P^{\mathrm{u}}\eta_n)) \in W^{\mathrm{u}}_{\mathrm{loc},r}(0) \cap \partial E(r)$, by the uniform continuity of σ^{u} we have

$$\begin{aligned}
&\operatorname{dist}(\eta_n, W^{\mathrm{u}}_{\mathrm{loc},r}(0) \cap \partial E(r)) \\
&\qquad \leq \|\eta_n - \eta''_n\| \\
&\qquad \leq \|\eta_n - \eta'_n\| + \|P^{\mathrm{u}}\eta'_n - P^{\mathrm{u}}\eta''_n\| + \|P^{\mathrm{s}}\eta'_n - P^{\mathrm{s}}\eta''_n\| \\
&\qquad = \|\eta_n - \eta'_n\| + \|P^{\mathrm{u}}\eta'_n - P^{\mathrm{u}}\eta_n\| + \|\sigma^{\mathrm{u}}(P^{\mathrm{u}}\eta'_n) - \sigma^{\mathrm{u}}(P^{\mathrm{u}}\eta_n)\| \to 0,
\end{aligned}$$

concluding the proof. □

1.7. GLOBAL STABLE AND UNSTABLE MANIFOLDS

Let us assume that X is a C^1 tangent vector field on the Banach manifold M, and that x is a hyperbolic rest point of X. We shall identify a neighborhood of x with a neighborhood of 0 in the Banach space $E_x = T_x M$, identifying x with 0. We shall consider a $\nabla X(x)$-adapted norm on E_x, and we will use the notation introduced in the above sections: for instance, $E_x(r) \subset M$ will denote the closed r ball centered in x.

The *unstable* and the *stable manifolds* of the rest point x are the subsets of M

$$\begin{aligned} W^{\mathrm{u}}(x) &= \left\{ p \in M \;\middle|\; t^-(p) = -\infty \text{ and } \lim_{t \to -\infty} \phi(t,p) = x \right\}, \\ W^{\mathrm{s}}(x) &= \left\{ p \in M \;\middle|\; t^+(p) = +\infty \text{ and } \lim_{t \to +\infty} \phi(t,p) = x \right\}. \end{aligned}$$

THEOREM 1.18. *Let $x \in M$ be a hyperbolic rest point of the C^k vector field X, $k \geq 1$, on the Banach manifold M. Then $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$ are the images of injective C^k immersions of manifolds which are homeomorphic to E^{u}_x and E^{s}_x, respectively.*

Proof. By Theorem 1.12, if r is small enough the local unstable manifold $W^{\mathrm{u}}_{\mathrm{loc},r}(x)$ is the graph of a C^k map $\sigma^{\mathrm{u}}\colon E^{\mathrm{u}}(r) \to E^{\mathrm{s}}(r)$. Since

$$W^{\mathrm{u}}(x) = \{\phi(t,p) \mid p \in W^{\mathrm{u}}_{\mathrm{loc},r}(x),\ 0 \leq t < t^+(p)\},$$

the set $W^{\mathrm{u}}(x)$ inherits the structure of a C^k manifold from that of $W^{\mathrm{u}}_{\mathrm{loc},r}(x)$ by the maps $\{\phi(t,\cdot)\}$, and the inclusion of $W^{\mathrm{u}}(x)$ into M is a C^k injective immersion.

If $\theta\colon E^{\mathrm{u}}(r) \to W^{\mathrm{u}}_{\mathrm{loc},r}(x)$ is the C^k diffeomorphism $\theta(\xi) = \xi + \sigma^{\mathrm{u}}(\xi)$, the map

$$A := \{\xi \in E^{\mathrm{u}}_x \mid \log\|\xi\| < t^+(\theta(r\xi/\|\xi\|))\} \to W^{\mathrm{u}}(x), \quad \xi \mapsto \phi(\log\|\xi\|, \theta(r\xi/\|\xi\|)),$$

is a homeomorphism from a star-shaped open subset of E^{u}_x - thus homeomorphic to E^{u}_x itself - onto $W^{\mathrm{u}}(x)$. The analogous results for $W^{\mathrm{s}}(x)$ follow by considering the vector field $-X$. □

REMARK 1.19. If M is a Hilbert manifold, then the regularity of the norm implies that $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$ are actually images of C^k immersions of E^{u}_x and E^{s}_x, respectively.

In general $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$ need not be embedded submanifolds: actually, they need not be locally closed.

A *Lyapunov function* for X is a C^1 function $f\colon M \to \mathbb{R}$ such that $Df(p)[X(p)] < 0$ for every $p \in M \setminus \mathrm{rest}(X)$. In this case, of course $\mathrm{crit}(f) \subset \mathrm{rest}(X)$. If X is a Morse vector field, the two sets actually coincide. Indeed, we can assume that $M = E$ is a Banach space, so if $x \in \mathrm{rest}(X)$ and $t \in \mathbb{R}$ we have

$$Df(x+tv)[X(x+tv)] = Df(x)[X(x+tv)]+o(t) = tDf(x)[DX(x)v]+o(t) \quad \text{for } t \to 0.$$

The principal part of the right-hand side is an odd function of t. Since the above quantity has to be negative for every $t \neq 0$, such a principal part has to be identically zero. Since $DX(x)$ is an isomorphism, we deduce that $Df(x) = 0$.

THEOREM 1.20. *Assume that X admits a Lyapunov function f, and that for every $r_0 > 0$ small enough there holds*

$$\sup\{f(p) \mid p \in W^{\mathrm{u}}_{\mathrm{loc},r_0}(x) \cap \partial E_x(r_0)\} < \inf\{f(p) \mid p \in W^{\mathrm{s}}_{\mathrm{loc},r_0}(x) \cap \partial E_x(r_0)\}. \quad (28)$$

Then if $r > 0$ is small enough:

(i) *for every $p \in M$, the closed set $I = \{t \in]t^-(p), t^+(p)[\mid \phi(t,p) \in E_x(r)\}$ is an interval, and its interior is $\{t \in]t^-(p), t^+(p)[\mid \phi(t,p) \in \mathring{E}_x(r)\}$;*

(ii) *if I is upper bounded, then $\phi(\max I, p) \in \partial E^{\mathrm{u}}_x(r) \cap E^{\mathrm{s}}(r)$; conversely, if $\phi(t,p) \in \partial E^{\mathrm{u}}_x(r) \cap E^{\mathrm{s}}(r)$, then $t = \max I$;*

(iii) *if I is lower bounded, then $\phi(\min I, p) \in E^{\mathrm{u}}_x(r) \cap \partial E^{\mathrm{s}}(r)$; conversely, if $\phi(t,p) \in E^{\mathrm{u}}_x(r) \cap \partial E^{\mathrm{s}}(r)$, then $t = \min I$;*

(iv) *$W^{\mathrm{u}}(x) \cap E_x(r) = W^{\mathrm{u}}_{\mathrm{loc},r}(x)$, and $W^{\mathrm{s}}(x) \cap E_x(r) = W^{\mathrm{s}}_{\mathrm{loc},r}(x)$;*

(v) *$W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$ are submanifolds of M.*

Proof. Let r_0 be so small that (28) and the conclusions of Lemma 1.10, Theorem 1.12, and Proposition 1.17 hold. Since f is of class C^1, up to choosing a smaller r_0 we may also assume that f is uniformly continuous on $E_x(r_0)$.

We shall prove the first assertion in (i) arguing by contradiction. In fact, assume that there exist an infinitesimal sequence of positive numbers (ρ_n), $\rho_n < r_0$, a sequence of points $(\xi_n) \in \partial E_x(\rho_n)$, and a sequence of positive numbers (t_n) such that $\phi(t_n, \xi_n) \in \partial E_x(\rho_n)$ and $\phi(]0, t_n[\times \{\xi_n\}) \cap E_x(\rho_n) = \varnothing$. Lemma 1.10(iv) implies that (at least for n large) $\xi_n \in \partial E^{\mathrm{u}}_x(\rho_n) \times E^{\mathrm{s}}_x(\rho_n) \subset C^{\mathrm{u}} \cap E_x(r_0)$. By Lemma 1.10(i), $C^{\mathrm{u}} \cap E_x(r_0)$ is positively invariant with respect to $E_x(r_0)$, and by (iii) if $t > 0$ and $\phi([0,t] \times \{\xi_n\}) \subset C^{\mathrm{u}} \cap E_x(r_0)$ then $\phi(t, \xi_n) \notin E_x(\rho_n)$. Therefore, there exists $a_n \in]0, t_n[$ such that $\phi([0, a_n] \times \{\xi_n\}) \subset E_x(r_0)$ and $\phi(a_n, \xi_n) \in \partial E_x(r_0)$. Similarly, there exists $b_n \in [a_n, t_n[$ such that $\phi(b_n, \xi_n) \in \partial E_x(r_0)$ and $\phi([b_, t_n] \times \{\xi_n\}) \subset E_x(r_0)$.

Since $\xi_n \to 0$ and $\phi(t_n, \xi_n) \to 0$, Proposition 1.17 implies that

$$\mathrm{dist}(\phi(a_n, \xi_n), W^{\mathrm{u}}_{\mathrm{loc},r_0}(x) \cap \partial E_x(r_0)) \to 0, \quad \mathrm{dist}(\phi(b_n, \xi_n), W^{\mathrm{s}}_{\mathrm{loc},r_0}(x) \cap \partial E_x(r_0)) \to 0.$$

Therefore, by (28), taking into account the fact that f is uniformly continuous on $E_x(r_0)$,

$$\limsup_{n\to\infty} f(\phi(a_n, \xi_n)) \leq \sup_{W^{\mathrm{u}}_{\mathrm{loc},r_0}(x) \cap \partial E_x(r_0)} f < \inf_{W^{\mathrm{s}}_{\mathrm{loc},r_0}(x) \cap \partial E_x(r_0)} f \leq \liminf_{n\to\infty} f(\phi(b_n, \xi_n)),$$

a contradiction because $a_n \leq b_n$ implies $f(\phi(a_n, \xi_n)) \geq f(\phi(b_n, \xi_n))$.

The second statement in (i), and statements (ii), (iii) are immediate consequences of Lemma 1.10. The inclusions

$$W^{\mathrm{u}}_{\mathrm{loc},r}(x) \subset W^{\mathrm{u}}(x) \cap E_x(r), \quad W^{\mathrm{s}}_{\mathrm{loc},r}(x) \subset W^{\mathrm{s}}(x) \cap E_x(r),$$

are obvious. The opposite inclusions follow from statement (i). Then $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$ are submanifold of M, because

$$\begin{aligned} W^{\mathrm{u}}(x) &= \{\phi(t,p) \mid p \in W^{\mathrm{u}}(x) \cap E(r),\ 0 \leq t < t^+(p)\}, \\ W^{\mathrm{s}}(x) &= \{\phi(t,p) \mid p \in W^{\mathrm{s}}(x) \cap E(r),\ t^-(p) < t \leq 0\}, \end{aligned}$$

and because $\phi(t,\cdot)$ is a diffeomorphism. □

REMARK 1.21. The weak inequality always holds in (28). The strict inequality holds if either:

(i) x has finite Morse index;

(ii) M is a Hilbert manifold, f is twice differentiable at x and the second differential of f at x satisfies $D^2 f(x)[\xi,\xi] \leq -\lambda\|\xi\|^2$ for every $\xi \in E^{\mathrm{u}}_x$, for some positive constant λ.

Indeed, in the first case $W^{\mathrm{u}}_{\mathrm{loc},r_0}(x)$ is a compact set, so

$$\sup_{W^{\mathrm{u}}_{\mathrm{loc},r_0}(x)\cap\partial E_x(r_0)} f = \max_{W^{\mathrm{u}}_{\mathrm{loc},r_0}(x)\cap\partial E_x(r_0)} f < f(x) \leq \inf_{W^{\mathrm{s}}_{\mathrm{loc},r_0}(x)\cap\partial E_x(r_0)} f.$$

In the second case, a second order expansion of f at x yields to the same conclusion.

2. The Morse complex in the case of finite Morse indices

2.1. THE PALAIS – SMALE CONDITION

Assume that f is a Lyapunov function for the vector field X on the Banach manifold M. A *Palais – Smale* (PS) *sequence at level c* is a sequence $(p_n) \subset M$ such that $(f(p_n))$ converges to c and $(Df(p_n)[X(p_n)])$ is infinitesimal. We shall say that (X, f) satisfies the *Palais – Smale* (PS) *condition at level c* if every Palais – Smale sequence at level c is compact.

If (X, f) satisfies the (PS) condition at every $c \in [a,b]$, then $\mathrm{rest}(X) \cap f^{-1}([a,b])$ is compact, so if X is also Morse this set is finite.

REMARK 2.1. Assume that $J_n = \phi([0, t_n] \times \{p_n\})$, $t_n > 0$, is contained in a strip $\{a \leq f \leq b\}$, and that

$$\lim_{n\to\infty} \frac{f(p_n) - f(\phi(t_n, p_n))}{t_n} = 0. \tag{29}$$

Then there is a (PS) sequence $q_n \in J_n$. Indeed, by the mean value theorem there is $s_n \in]0, t_n[$ such that

$$Df(\phi(s_n, p_n))[X(\phi(s_n, p_n))] = \frac{f(p_n) - f(\phi(t_n, p_n))}{t_n},$$

and by (29), $q_n = \phi(s_n, p_n)$ is a (PS) sequence.

Actually, the above observation could be used to give a weaker formulation of the (PS) condition, which does not require f to be differentiable, and could be used to study flows in the continuous category.

2.2. THE MORSE – SMALE CONDITION

We recall that two closed linear subspaces V_1, V_2 of a Banach space E are said *transverse* if $V_1 + V_2 = E$ and $V_1 \cap V_2$ is complemented in E. Two C^1 submanifolds M_1 and M_2 of the Banach manifold M are said *transverse* if for every $p \in M_1 \cap M_2$ the closed linear subspaces T_pM_1 and T_pM_2 are transverse in T_pM.

Let X be a Morse vector field having only rest points with finite Morse index and admitting a Lyapunov function. We will say that X satisfies the *Morse – Smale condition up to order* $k \in \mathbb{N}$ if for every pair of rest points x, y satisfying $m(x) - m(y) \leq k$, the submanifolds $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(y)$ are transverse. In this case, the implicit function theorem implies that $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ — if nonempty — is a submanifold of dimension $m(x) - m(y)$.

Notice that the presence of a Lyapunov function implies that $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(x) = \{x\}$, and such an intersection is always transverse. Notice also that the fact that $\phi(t, \cdot)$ is a diffeomorphism implies that if $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ meet transversally at some $p \in M$, they meet transversally at every point of the orbit of p.

2.3. THE ASSUMPTIONS

Let M be an open subset of the Banach manifold $\widehat{M}$, and let $\widehat{X}$ be a C^1 vector field on $\widehat{M}$ (possibly, $M = \widehat{M}$). Denote by A the open subset $\widehat{M} \setminus \overline{M}$, and denote by X the restriction of $\widehat{X}$ to M.

We shall construct the Morse complex for X on M under the following assumptions:

(A1) A is positively invariant with respect to the flow of $\widehat{X}$, and $\widehat{X}$ is positively complete with respect to A;

(A2) X is a Morse vector field on M;

(A3) every rest point of X has finite Morse index;

(A4) X admits a Lyapunov function $f \in C^1(M) \cap C^0(\overline{M})$;

(A5) f is bounded below on M;

(A6) (X, f) satisfies the (PS) condition at every level $c \in f(\overline{M})$;

(A7) X satisfies the Morse – Smale condition up to order 0.

The local flow of $\widehat{X}$ will be denoted by ϕ. By (A1), the local flow of X is just the restriction of ϕ to

$$\Omega(X) = \{(t, p) \in \Omega(\widehat{X}) \mid p \in M,\ \phi(t, p) \in M\}.$$

In most applications f is actually defined on the whole $\widehat{M}$ and A is a sublevel of f.

Notice that (A6) and the fact that $f \in C^0(\overline{M})$ imply that there are no rest points on the boundary of M: such a rest point would be the limit of a (PS) sequence in M, which does not converge in M.

Notice also that (A7) means asking that $W^u(x)$ does not meet $W^s(y)$ whenever $x \neq y$ are rest points with $m(x) \le m(y)$.

2.4. FORWARD COMPACTNESS

The (PS) condition plays a crucial role in the following compactness result.

PROPOSITION 2.2. *Assume* (A1) – (A7). *Then*

(i) *for every $p \in M$, $\phi(t, p)$ either converges to a rest point of X for $t \to +\infty$ or eventually enters A;*

(ii) *if $(p_n) \subset M$ converges to $p \in \widehat{M}$, $(t_n) \subset [0, +\infty[$, and $(\phi(t_n, p_n)) \subset M$, then the sequence $(\phi(t_n, p_n))$ is compact in $\widehat{M}$.*

Proof. (i) Let $p \in M$. Assume that $\phi(t, p)$ never enters A: by (A1) this implies that $t^+(p) = +\infty$. By Remark 2.1, with $p_n = p$, $t_n \to +\infty$, $a = \inf f$, $b = f(p)$, and by (PS) we can find a sequence $s_n \to +\infty$ such that $\phi(s_n, p)$ converges to a rest point $x \in M$. The function $t \mapsto f(\phi(t, p))$ converges for $t \to +\infty$, being monotone, therefore

$$\lim_{t \to +\infty} f(\phi(t, p)) = \lim_{n \to \infty} f(\phi(s_n, p)) = f(x).$$

Assume by contradiction that $\phi(t, p)$ does not converge to x for $t \to +\infty$. Then we can find $r > 0$ (as small as we like), two sequences $a_n \le b_n \le a_{n+1}$, $a_n \to +\infty$, such that $\phi(a_n, p) \in \partial E_x(r)$, $\phi(b_n, p) \in \partial E_x(2r)$, $\phi([a_n, b_n] \times \{p\}) \subset \overline{E_x(2r) \setminus E_x(r)}$.

Choosing r so small that X is bounded on $E_x(2r) \subset M$, one has that $b_n - a_n$ is bounded away from 0. Since

$$\lim_{n\to\infty} f(\phi(a_n, p)) = \lim_{n\to\infty} f(\phi(b_n, p)) = \lim_{t\to+\infty} f(\phi(t, p)) = f(x),$$

we have

$$\lim_{n\to\infty} \frac{f(\phi(b_n, p)) - f(\phi(a_n, p))}{b_n - a_n} = 0,$$

so by Remark 2.1 there is a (PS) sequence in $\overline{E_x(2r) \setminus E_x(r)}$, converging by (PS) to a rest point. Since r is arbitrarily small, x is not isolated in rest(X), contradicting (A2).

(ii) If (t_n) is bounded, then

$$\limsup_{n\to\infty} t_n < t^+(p). \tag{30}$$

Indeed, if by contradiction $t^+(p) \leq \limsup_{n\to\infty} t_n$, $t^+(p)$ is finite, so by (A1) there exists $s \in [0, t^+(p)[$ such that $\phi(s, p) \in A$. Then $\phi(s, p_n)$ eventually belongs to A, so $s > t_n$ for n large, and $\limsup_{n\to\infty} t_n \leq s < t^+(p)$, a contradiction.

When (t_n) is bounded, the continuity of ϕ and (30) imply that $(\phi(t_n, p_n))$ is compact in $\widehat{M}$, so we may assume that $t_n \to +\infty$.

By Remark 2.1 and (PS) there exists a sequence $a_n \in [0, t_n]$ such that, up to a subsequence, $\phi(a_n, p_n)$ converges to a rest point $x \in M$, with $\inf f \leq f(x) \leq f(p)$. Since there are finitely many rest points in this strip, we may assume that $f(x)$ is minimal, that is for no sequence $a_n' \in [0, t_n]$, $\phi(a_n', p_n)$ has a subsequence converging to a rest point $y \in M$ with $f(y) < f(x)$.

If $\phi(t_n, p_n)$ converges to x then there is nothing to prove, otherwise up to a subsequence we can find $r > 0$ (as small as we like) and $b_n \in [a_n, t_n]$ such that $\phi(b_n, p_n) \in \partial E_x(r)$ and $\phi([a_n, b_n] \times \{p_n\}) \subset E_x(r)$. By Proposition 1.17, the sequence $(\phi(b_n, p_n))$ is compact, since its distance from the compact set $W^{\mathrm{u}}_{\mathrm{loc},r}(x) \cap \partial E_x(r)$ tends to 0 (here we are using the fact that x has finite Morse index). So a subsequence of $(\phi(b_n, p_n))$ converges to a point q with

$$f(q) \leq \max_{W^{\mathrm{u}}_{\mathrm{loc},r}(x)\cap\partial E_x(r)} f < f(x).$$

The sequence $t_n - b_n$ is bounded: otherwise by Remark 2.1 and (PS) there would exist $c_n \in [b_n, t_n]$ such that a subsequence of $(\phi(c_n, p_n))$ converges to a rest point y with $f(y) \leq f(q) < f(x)$, contradicting the minimality of $f(x)$. Therefore $\phi(t_n, p_n) = \phi(t_n - b_n, \phi(b_n, p_n))$ is compact in $\widehat{M}$. □

The above result has the following immediate consequence.

COROLLARY 2.3. *For every $x \in$ rest(X), $W^{\mathrm{u}}(x) \cap M$ has compact closure in $\widehat{M}$.*

Another consequence is the following convergence result for forward orbits: if $(p_n) \subset M$ converges to $p \in M$, up to a subsequence the forward orbit of p_n converges to a "broken orbit" consisting of $h+1$ flow lines, $h \geq 0$, matching at h rest points $x_h, \dots, x_1$. The first of these flow lines is the forward orbit of p, the last one either converges to a rest point x_0, which is also the common limit of $\phi(t, p_n)$ for $t \to +\infty$, or eventually enters A, together with all the orbits of p_n. More precisely, the situation is described by the following corollary.

COROLLARY 2.4. *Assume that $(p_n) \subset M$ converges to some $p \in M$. Then there exists a subsequence (p_{k_n}) such that one of the following two alternatives holds*:

(a) $t^+(p_{k_n}) = +\infty$, *and there exists $x_0 \in \operatorname{rest}(X)$ such that $\phi(t, p_{k_n})$ converges to x_0 for $t \to +\infty$, for every $n \in \mathbb{N}$*;

(b) *for every $n \in \mathbb{N}$, $\phi(t, p_{k_n})$ eventually enters A.*

Moreover, there exist $h \in \mathbb{N}$, a set $\{x_j\}_{1 \leq j \leq h} \subset \operatorname{rest}(X)$, with $f(x_1) < \cdots < f(x_h)$, sequences of real numbers $t_0^n > t_1^n > \cdots > t_h^n = 0$, and points $q_0, q_1, \dots, q_h = p$ in M such that:

(i) $q_j \in W^{\mathrm{s}}(x_j) \cap W^{\mathrm{u}}(x_{j+1})$ *for every* $1 \leq j \leq h-1$;

(ii) $q_h = p \in W^{\mathrm{s}}(x_h)$, *unless case* (b) *holds and $h = 0$, in which case $\phi(t, q_h) = \phi(t, p)$ eventually enters A*;

(iii) $q_0 \in W^{\mathrm{u}}(x_1)$ *if $h \geq 1$; in case* (a) $q_0 \in W^{\mathrm{s}}(x_0)$, *in case* (b) $\phi(t, q_0)$ *eventually enters A*;

(iv) $\lim_{n\to\infty} \phi(t_n^j, p_{k_n}) = q_j$ *for every* $0 \leq j \leq h$.

The proof is an easy application of Proposition 2.2, together with an induction argument. Details are left to the reader.

2.5. CONSEQUENCES OF COMPACTNESS AND TRANSVERSALITY

Given a subset $B \subset \widehat{M}$, we will denote by $\phi([0, +\infty[\times B)$ its forward evolution, although this set should more properly be indicated by

$$\phi(([0, +\infty[\times B) \cap \Omega(\widehat{X})).$$

The Morse – Smale condition up to order zero, assumption (A7), has the following consequence.

LEMMA 2.5. *Assume* (A1) – (A7). *Let x, y be distinct rest points of X, with $m(x) \leq m(y)$. Then there exists $r > 0$ such that*

$$\phi([0, +\infty[\times E_x(r)) \cap E_y(r) = \varnothing.$$

Proof. Assume the contrary: there exist a sequence $(p_n) \subset M$ converging to x and a sequence $(t_n) \subset [0, +\infty[$ such that $(\phi(t_n, p_n))$ converges to y. By Corollary 2.4, a subsequence of the sequence of forward orbits of p_n converges to a “broken orbit” passing through x and y. In particular, there are pairwise distinct rest points $z_1 = x, z_2, \dots, z_k = y$, $k \geq 2$, such that $W^{\mathrm{u}}(z_i) \cap W^{\mathrm{s}}(z_{i+1}) \neq \varnothing$ for every $1 \leq i \leq k-1$. The Morse – Smale condition up to order zero implies that $m(x) = m(z_1) > \cdots > m(z_k) = m(y)$, a contradiction. □

In particular, the closure of the unstable manifold of a rest point x of index k does not contain rest points of index greater than or equal to k, other than x itself. Let us state a stronger assumption, which will be later removed:

(A8) every rest point y does not belong to the closure of the union of the unstable manifolds of rest points $x \neq y$ with $m(x) \leq m(y)$:

$$y \notin \overline{\bigcup_{\substack{x \in \mathrm{rest}(X) \setminus \{y\} \\ m(x) \leq m(y)}} W^{\mathrm{u}}(x)}.$$

Since the closure of a finite union is the union of the closures, by Lemma 2.5 condition (A8) is implied by the Morse – Smale condition up to order zero (A7) when X has finitely many rest points of index k, for every $k \in \mathbb{N}$. In general it is strictly more restrictive.

Assumption (A8) implies the following result.

PROPOSITION 2.6. *Assume* (A1) – (A8). *Then there exists a positive function* $\rho \colon \mathrm{rest}(X) \to]0, +\infty[$ *such that*

$$\phi\Big([0, +\infty[\times E_x(\rho(x))\Big) \cap E_y(\rho(y)) = \varnothing$$

for all pairs of rest points $x \neq y$ *with* $m(x) \leq m(y)$.

Proof. By (A8) there exists a function $\sigma \colon \mathrm{rest}(X) \to]0, +\infty[$ such that

$$E_y(\sigma(y)) \cap \bigcup_{\substack{x \in \mathrm{rest}(X) \setminus \{y\} \\ m(x) \leq m(y)}} W^{\mathrm{u}}(x) = \varnothing \quad \forall y \in \mathrm{rest}(X). \tag{31}$$

Let us prove that for every $x \in \mathrm{rest}(X)$ there is a positive number $\theta(x)$ such that

$$\phi\Big([0, +\infty[\times E_x(\theta(x))\Big) \cap E_y(\sigma(y)) = \varnothing \quad \forall y \in \mathrm{rest}(X) \setminus \{x\},\ m(y) \geq m(x). \tag{32}$$

Then the function $\rho(x) \colon \mathrm{rest}(X) \to]0, +\infty[$, $x \mapsto \min\{\sigma(x), \theta(x)\}$, will satisfy the requirements.

We argue by contradiction, assuming that there exists $x \in \mathrm{rest}(X)$ for which (32) does not hold, no matter how small $\theta(x)$ is. Since there are finitely many rest points in $\{p \in M \mid f(p) \leq f(x)\}$, we can find a sequence $(p_n) \subset M$ converging to x and a sequence $(t_n) \subset [0, +\infty[$ such that $\phi(t_n, p_n) \in E_y(\sigma(y))$, for some $y \in \operatorname{rest}(X) \setminus \{x\}$ with $m(y) \geq m(x)$. By Corollary 2.4, a subsequence of the sequence of the forward orbits of p_n converges to a "broken orbit" starting from x and passing through $E_y(\sigma(y))$. In particular, there are rest points $z_1 = x, \dots, z_k \neq y$, $k \geq 1$, such that $W^{\mathrm{u}}(z_i) \cap W^{\mathrm{s}}(z_{i+1}) \neq \emptyset$ for $1 \leq i \leq k-1$, and

$$W^{\mathrm{u}}(z_k) \cap E_y(\sigma(y)) \neq \emptyset. \tag{33}$$

By the Morse – Smale condition up to order 0, $m(z_k) \leq m(x) \leq m(y)$, and since $z_k \neq y$, (33) contradicts (31). □

2.6. CELLULAR FILTRATIONS

Cellular filtrations are a useful tool to compute the singular homology of a topological space. See Dold (1980, Section V.1) for a more extensive discussion and for the proof of the results stated in this section.

Let T be a topological space. A sequence $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ of subsets of T is said a *cellular filtration of T* if:

(i) $F_n \subset F_{n+1}$ for every $n \in \mathbb{Z}$;

(ii) every singular simplex in T is a simplex in F_n for some n;

(iii) the k-th singular homology group $H_k(F_n, F_{n-1})$ vanishes for every $k \neq n$.

Notice that (ii) is fulfilled when T is the union of the family $\{F_n\}$ and each F_n is open. The space F_{-1} may be empty. The spaces F_n for $n \leq -2$ will be actually irrelevant in the construction. Singular homology is always meant to have integer coefficients.

If $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a cellular filtration of T, we denote by $W_k\mathcal{F}$ the Abelian group

$$W_k\mathcal{F} := H_k(F_k, F_{k-1}).$$

The homomorphism $\partial_k \colon W_k\mathcal{F} \to W_{k-1}\mathcal{F}$ is given by the composition

$$H_k(F_k, F_{k-1}) \to H_{k-1}(F_{k-1}) \to H_{k-1}(F_{k-1}, F_{k-2}),$$

where the first map is the boundary homomorphism of the pair (F_k, F_{k-1}), and the second map is induced by the inclusion. It is readily seen that $\partial_k \partial_{k+1} = 0$, so $W_*\mathcal{F}$ is a chain complex of Abelian groups, said the *cellular complex of the filtration* $\mathcal{F}$.

A *cellular map* $g \colon (T, \mathcal{F}) \to (T', \mathcal{F}')$ is a continuous map from T to T' mapping each F_n into F'_n. Such a map induces homomorphisms

$$W_k g \colon W_k\mathcal{F} \to W_k\mathcal{F}', \quad W_k g = g_* \colon H_k(F_k, F_{k-1}) \to H_k(F'_k, F'_{k-1}),$$

which are readily seen to form a chain map $W_* g: W_* \mathcal{F} \to W_* \mathcal{F}'$. This makes W a functor from the category of cellular filtrations and cellular maps to the category of chain complexes of Abelian groups and chain maps.

THEOREM 2.7. *If $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ is a cellular filtration of the topological space T, then there is an isomorphism*

$$H_k(\{W_*\mathcal{F}, \partial_*\}) \cong H_k(T, F_{-1}).$$

Such isomorphisms form a natural transformation between the functor HW and the singular homology functor H, in the sense that if $g: (T, \mathcal{F}) \to (T', \mathcal{F}')$ is a cellular map, then the diagram

$$\begin{tikzcd} H_k(\{W_*\mathcal{F}, \partial_*\}) \arrow[r, "\cong"] \arrow[d, "H_k W_k g"'] & H_k(T, F_{-1}) \arrow[d, "g_*"] \\ H_k(\{W_*\mathcal{F}', \partial_*\}) \arrow[r, "\cong"] & H_k(T', F'_{-1}) \end{tikzcd}$$

commutes.

A *cellular homotopy* h between two cellular maps $g_0, g_1: (T, \mathcal{F}) \to (T', \mathcal{F}')$ is cellular map $h: ([0,1] \times T, \widehat{\mathcal{F}}) \to (T', \mathcal{F}')$, $\widehat{\mathcal{F}}$ being the cellular filtration $\{[0,1] \times F_n\}_{n\in\mathbb{Z}}$, such that $h(0,\cdot) = g_0$ and $h(1,\cdot) = g_1$. If there is a cellular homotopy between g and g', the homotopy invariance of singular homology implies that $W_* g = W_* g'$.

A cellular map $g: (T, \mathcal{F}) \to (T', \mathcal{F}')$ is said a *cellular homotopy equivalence* if there are a cellular map $g': (T', \mathcal{F}') \to (T, \mathcal{F})$, said a *cellular homotopy inverse of* g, and cellular homotopies h between $g' \circ g$ and $\mathrm{id}_{(T,\mathcal{F})}$ and h' between $g \circ g'$ and $\mathrm{id}_{(T',\mathcal{F}')}$. By functoriality and homotopy invariance, if g is a cellular homotopy equivalence then $W_* g$ is an isomorphism.

2.7. THE MORSE COMPLEX

Denote by $\mathrm{rest}_k(X)$ the set of rest points of X of Morse index k, and let $C_k(X)$ be the free Abelian group generated by the elements of $\mathrm{rest}_k(X)$.

Let $\rho: \mathrm{rest}(X) \to]0, +\infty[$ be a function satisfying

$$\phi\Big([0, +\infty[\times E_x(\rho(x))\Big) \cap E_y(\rho(y)) = \emptyset, \quad \forall x \neq y \in \mathrm{rest}(X),\ m(x) \leq m(y), \quad (34)$$

whose existence is established by Proposition 2.6. Consider the subsets of $\widehat{M}$

$$M_k = M_k(\rho) := A \cup \bigcup_{\substack{x \in \mathrm{rest}(X) \\ m(x) \leq k}} \phi([0, +\infty[\times \mathring{E}_x(\rho(x)))\ \forall k \geq 0, \quad M_k = A\ \forall k < 0,$$

and $M_\infty = M_\infty(\rho) := \bigcup_{k\in\mathbb{Z}} M_k$. Each M_k is open and positively invariant.

We shall denote by D^k the closed unit ball of $\mathbb{R}^k$, and by ω_k the generator of $H_k(D^k, \partial D^k)$ corresponding to the standard orientation of $\mathbb{R}^k$. Here is the main result of this second part.

THEOREM 2.8. *Assume* (A1) – (A8)*. Let* $\rho\colon \mathrm{rest}(X) \to]0, +\infty[$ *be a function satisfying* (34)*, and let* M_k *be the sets defined above. Then*:

(i) *The inclusion* $(M_\infty, A) \hookrightarrow (\widehat{M}, A)$ *is a homotopy equivalence.*

(ii) $\mathcal{M} = \mathcal{M}(\rho) := \{M_k\}_{k\in\mathbb{Z}}$ *is a cellular filtration of* M_∞*, with*

$$W_k\mathcal{M} = H_k(M_k, M_{k-1}) \cong C_k(X), \quad \forall k \in \mathbb{N}.$$

More precisely, the choice of an orientation of each unstable manifold $W^{\mathrm{u}}(x)$ *determines an isomorphism*

$$\Theta_k(\rho)\colon C_k(X) \cong W_k\mathcal{M}(\rho), \; x \mapsto \theta^x_*(\omega_k), \quad \forall x \in \mathrm{rest}_k(X),$$

where $\theta^x\colon (D^k, \partial D^k) \to (M_k, M_{k-1})$ *is a map of the form* $\theta^x(\xi) = \phi(t(\xi), w(\xi))$*, with* w *an orientation preserving embedding of* D^k *onto an open neighborhood of* x *in* $W^{\mathrm{u}}(x)$*, and* $0 \leq t < t^+$ *so large that* $\phi(t(\xi), w(\xi)) \in M_{k-1}$ *for every* $\xi \in \partial D^k$*.*

(iii) *If* $\rho' \leq \rho$*, then the inclusion* $j = j_{\rho'\rho}\colon M_\infty(\rho') \hookrightarrow M_\infty(\rho)$ *is a cellular homotopy equivalence with respect to the cellular filtrations* $\{M_k(\rho')\}_{k\in\mathbb{Z}}$ *and* $\{M_k(\rho)\}_{k\in\mathbb{Z}}$*. Moreover, the diagram*

$$\begin{tikzcd} & W_k\mathcal{M}(\rho') \arrow[d, "W_k j"] \\ C_k(X) \arrow[ur, "\Theta_k(\rho')"] \arrow[r, "\Theta_k(\rho)"'] & W_k\mathcal{M}(\rho) \end{tikzcd} \tag{35}$$

commutes.

By (iii), the isomorphism class of the cellular chain complex $W_k\mathcal{M}(\rho)$ does not depend on the choice of the function ρ satisfying (34). In order to fix a standard representative, we can define

$$W_*(X) := \lim_{\rho\downarrow 0} W_*\mathcal{M}(\rho),$$

the limit of the direct system of chain complexes $\{W_*\mathcal{M}(\rho), W_* j_{\rho'\rho}\}$. The chain complex $W_*(X)$ is said the *Morse complex of* X. By Theorem 2.7, the homology of such a chain complex is isomorphic to the singular homology of (M_∞, A), which by statement (i) of the theorem above is isomorphic to the singular homology of $(\widehat{M}, A)$:

$$H_k W_*(X) \cong H_k(\widehat{M}, A) \quad \forall k \in \mathbb{N}.$$

In particular when A is the empty set (so that X is a positively complete Morse vector field on M admitting a Lyapunov function which is bounded below), the homology of the Morse complex is isomorphic to the singular homology of M.

By (ii) and by the commutativity of diagram (35), a choice of an orientation of each unstable manifold allows to identify the groups $C_k(X)$ and $W_k(X)$, by the isomorphism

$$\Theta_k = \lim_{\rho \downarrow 0} \Theta_k(\rho) \colon C_k(X) \cong W_k(X).$$

EXERCISE 2.9. Deduce the so called *strong Morse relations*: there exists a formal series Q with coefficients in $\mathbb{N} \cup \{+\infty\}$ such that

$$\sum_{k=0}^{\infty} |\operatorname{rest}_k(X)|\, t^k = \sum_{k=0}^{\infty} \beta_k(\widehat{M}, A) t^k + (1+t) Q(t), \tag{36}$$

where $\beta_k(\widehat{M}, A) = \operatorname{rank} H_k(\widehat{M}, A) \in \mathbb{N} \cup \{+\infty\}$ is the kth Betti number of $(\widehat{M}, A)$.

Before proving Theorem 2.8, we recall the semi-continuity properties of the *entrance time function* into a subset $C \subset \widehat{M}$:

$$t_C(p) := \inf\{t \in [0, t^+(p)[\;\big|\; \phi(t, p) \in C\} \in [0, +\infty].$$

LEMMA 2.10. *If C is open, t_C is upper semi-continuous. If C is closed, t_C is lower semi-continuous.*

Proof. Assume that C is open. It $t_C(p) < t$, there exists $s \in [t_C(p), t[$ such that $\phi(s, p) \in C$. By continuity, $\phi(s, q) \in C$ for every q in a neighborhood of p, so $t_C(q) \le s < t$ in such a neighborhood.

Assume that C is closed. If $t_C(p) > t$, choosing $t' \in]t, t_C(p)[$ we have that $\phi(s, p)$ belongs to the open set $\widehat{M} \setminus C$ for every $s \in [0, t']$. By continuity and compactness, $\phi(s, q) \in \widehat{M} \setminus C$ for every $s \in [0, t']$ and every q in a neighborhood of p. Therefore, $t_C(q) \ge t' > t$ in such a neighborhood. □

Proof of Theorem 2.8. (i) By Proposition 2.2(i), the orbit of every $p \in \widehat{M}$ either converges to some rest point $x \in M$ for $t \to +\infty$, or eventually enters A. Since M_∞ is a neighborhood of $\operatorname{rest}(X)$ and contains A, for every $p \in \widehat{M}$ the entrance time of $\phi(\cdot, p)$ in M_∞,

$$t_{M_\infty}(p) := \inf\{t \in [0, t^+(p)[\;\big|\; \phi(t, p) \in M_\infty\}$$

is finite, and less than $t^+(p)$. Since M_∞ is open, by Lemma 2.10 the function t_{M_∞} is upper semi-continuous.

On the other hand, the function t^+ is lower semi-continuous. A simple argument with partitions of unity (also known as Dowker theorem; see Dugundji, 1978, VIII.4.3) shows that on a paracompact topological space we can always find a continuous function between an upper semi-continuous function and a lower semi-continuous one. So we can find a continuous function $s : \widehat{M} \to \mathbb{R}$ such that $t_{M_\infty} < s < t^+$. Then the continuous map

$$r\colon (\widehat{M}, A) \to (M_\infty, A), \quad r(p) = \phi(s(p), p),$$

is a homotopical inverse of the inclusion $i\colon (M_\infty, A) \hookrightarrow (\widehat{M}, A)$, the homotopies $\mathrm{id}_{(\widehat{M},A)} \sim i \circ r$ and $\mathrm{id}_{(M_\infty,A)} \sim r \circ i$ being the map

$$([0,1] \times \widehat{M}, [0,1] \times A) \to (\widehat{M}, A), \quad (\lambda, p) \mapsto \phi(\lambda s(p), p),$$

and its restriction to $([0,1] \times M_\infty, [0,1] \times A)$ into (M_∞, A).

(ii) Let us prove that $\mathcal{M}$ is a cellular filtration. Since $\mathcal{M}$ is an open covering of M_∞, we just need to compute the singular homology of (M_k, M_{k-1}). Since M_k is the union of the open sets M_{k-1} and

$$U_k := \bigcup_{x \in \mathrm{rest}_k(X)} \phi\Big([0, +\infty[\times \mathring{E}_x(\rho(x))\Big),$$

by excision the singular homology of (M_k, M_{k-1}) is isomorphic to the singular homology of $(U_k, U_k \cap M_{k-1})$. Condition (34) implies that the open sets $U(x) := \phi\Big([0, +\infty[\times \mathring{E}_x(\rho(x) bigr)\Big)$, $x \in \mathrm{rest}_k(X)$, are pairwise disjoint, so

$$H_*(M_k, M_{k-1}) \cong H_*(U_k, U_k \cap M_{k-1}) \cong \bigoplus_{x \in \mathrm{rest}_k(X)} H_*(U(x), U(x) \cap M_{k-1}).$$

We shall prove that $(U(x), U(x) \cap M_{k-1})$ is homotopically equivalent to a k-dimensional disc modulo its boundary, so that

$$H_j(U(x), U(x) \cap M_{k-1}) = \begin{cases} 0 & \text{if } j \neq k, \\ \mathbb{Z} & \text{if } j = k, \end{cases}$$

proving that $\mathcal{M}$ is a cellular filtration.

Set for simplicity $\rho = \rho(x)$. By Lemma 1.10(iii), $E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho) \subset U(x)$. Let $p \in U(x) \setminus E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho)$. By Proposition 2.2(ii), the orbit of p either eventually enters A, or converges to a rest point y for $t \to +\infty$. In the latter case, $y \neq x$ because of Theorem 1.20(i), so by (34) $m(y) \leq k - 1$. In both cases, the orbit of p eventually enters M_{k-1}. The upper semi-continuous function

$$\tilde{a} : U(x) \to \mathbb{R}, \quad p \mapsto \begin{cases} 0 & \text{if } p \in \mathring{E}_x(\rho), \\ t_{M_{k-1}}(p) & \text{if } p \in U(x) \setminus \mathring{E}_x(\rho), \end{cases}$$

is strictly less than the lower semi-continuous function t^+, so we can find a continuous function $a\colon U(x) \to [0, +\infty[$ such that $\tilde{a} < a < t^+$, so that

$$\phi(a(p), p) \in M_{k-1} \quad \forall p \in U(x) \setminus \mathring{E}_x(\rho). \tag{37}$$

Then we can define the continuous map

$$\alpha\colon (E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho), \partial E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho)) \to (U(x), U(x) \cap M_{k-1}), \quad p \mapsto \phi(a(p), p).$$

By Theorem 1.20(i), for every $p \in U(x)$ there holds

$$b(p) := \sup\{t \in]t^-(p), 0] \bigm| \phi(t, p) \in \mathring{E}_x(\rho)\} = \max\{t \in]t^-(p), 0] \bigm| \phi(t, p) \in E_x(\rho)\},$$

so by Lemma 2.10 the function $b\colon U(x) \to]-\infty, 0]$ is both lower and upper semi-continuous, hence continuous. The map $p \mapsto \phi(b(p), p)$ is the identity on $E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho)$, and maps all the other points of $U(x)$ into $\partial E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho)$. Since by (34) $E_x(\rho) \cap M_{k-1} = \emptyset$, the continuous map

$$\beta\colon (U(x), U(x) \cap M_{k-1}) \to (E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho), \partial E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho)), \quad p \mapsto \phi(b(p), p),$$

is well-defined.

It is easy to check that α and β are homotopy inverses. Indeed,

$$(\lambda, p) \mapsto \beta(\phi(\lambda a(p), p))$$

is a homotopy between $\beta \circ \alpha$ and the identity map on $(E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho), \partial E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho))$. On the other hand, by (37),

$$(\lambda, p) \mapsto \phi(a(\phi(\lambda b(p), p)), \phi(\lambda b(p), p))$$

is a homotopy between $\alpha \circ \beta$ and the map

$$(U(x), U(x) \cap M_{k-1}) \to (U(x), U(x) \cap M_{k-1}), \quad p \mapsto \phi(a(p), p),$$

which is clearly homotopy equivalent to the identity on $(U(x), U(x) \cap M_{k-1})$.

We conclude that $(U(x), U(x) \cap M_{k-1})$ is homotopy equivalent to $(E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho), \partial E^{\mathrm{u}}_x(\rho) \times \mathring{E}^{\mathrm{s}}_x(\rho))$, which is homotopy equivalent to $(E^{\mathrm{u}}_x(\rho), \partial E^{\mathrm{u}}_x(\rho))$, a k-dimensional disc modulo its boundary. The latter pair is homeomorphic to $(W^{\mathrm{u}}_{\mathrm{loc},\rho}(x), \partial W^{\mathrm{u}}_{\mathrm{loc},\rho}(x))$, and the statement about the form of the isomorphism Θ_k easily follows.

(iii) Since $M_k(\rho') \subset M_k(\rho)$ for every k, j is a cellular map; we will construct a cellular homotopy inverse of j of the form

$$\gamma(p) = \phi(c(p), p), \tag{38}$$

with c a suitable positive continuous function.

Given $p \in M_\infty(\rho)$, set

$$\kappa(p) := \min\{k \in \mathbb{N} \mid p \in \overline{M_k(\rho)}\},$$

and

$$\tilde{c}(p) := t_{M_{\kappa(p)}(\rho')}(p),$$

the entrance time of p into the open set $M_{\kappa(p)}(\rho')$. By (34), every point in $\overline{M_k(\rho)}$ either eventually enters A or converges to a rest point x with $m(x) \leq k$; in both cases, p eventually enters $M_k(\rho')$. Therefore, $\tilde{c} < t^+$.

Since $\{M_h(\rho')\}_{h\in\mathbb{Z}}$ is a filtration, $t_{M_h(\rho')}$ is nonincreasing in h, so

$$\tilde{c}(p) = \min\{t_{M_h(\rho')}(p) \mid 0 \leq h \leq \kappa(p)\} = \min\{t_{M_h(\rho')}(p)\chi_h(p) \mid h \in \mathbb{N}\},$$

where $\chi_h(p) = 1$ if $h \leq \kappa(p)$, i.e., $p \notin \overline{M_{h-1}(\rho)}$, and $\chi_h(p) = +\infty$ otherwise; hence the positive function χ_h is upper semi-continuous. Since also $t_{M_h(\rho')}$ is upper semi-continuous and nonnegative, so is the function $\tilde{c}$.

Let $c\colon M_\infty(\rho) \to \mathbb{R}$ be a continuous function such that $\tilde{c} < c < t^+$, and let $\gamma\colon M_\infty(\rho) \to M_\infty(\rho')$ be the map defined in (38). By construction, γ maps $M_k(\rho)$ into $M_k(\rho')$, so it is a cellular map. The cellular homotopies $\mathrm{id}_{M_\infty(\rho)} \sim j \circ \gamma$ and $\mathrm{id}_{M_\infty(\rho')} \sim \gamma \circ j$ are given by the cellular map $(\lambda, p) \mapsto \phi(\lambda c(p), p)$ on the respective domains.

If $\theta^x(\rho)\colon (D^k, \partial D^k) \to (M_k(\rho), M_{k-1}(\rho))$ and $\theta^x(\rho')\colon (D^k, \partial D^k) \to (M_k(\rho'), M_{k-1}(\rho'))$ are the continuous maps appearing in (ii), then $j \circ \theta^x(\rho')$ is homotopic to $\theta^x(\rho)$, so the diagram (35) commutes. □

2.8. REPRESENTATION OF ∂_* IN TERMS OF INTERSECTION NUMBERS

Let us strengthen the Morse – Smale assumption (A7) by requiring:

(A7′) X satisfies the Morse – Smale condition up to order 1.

In this case, the boundary operator ∂_k of the Morse complex of X can be expressed in terms of intersection numbers of unstable and stable manifolds of rest points of index difference 1.

First of all notice that if $m(x) - m(y) = 1$, the assumption (A7′) implies that $W^u(x) \cap W^s(y)$ is a flow-invariant 1-dimensional manifold, that is a discrete set of flow lines. We claim that $W^u(x) \cap W^s(y)$ is compact: otherwise Corollary 2.4 would imply the existence of a “broken orbit” from $z_0 = x$ to $z_h = y$, with intermediate rest points $z_1, \dots, z_{h-1}$, for some $h \geq 2$. By the Morse – Smale condition (up to order 0) $m(z_0) > m(z_1) > \cdots > m(z_h)$, a contradiction because $m(z_0) - m(z_h) = 1$. Therefore $W^u(x) \cap W^s(y)$ consists of finitely many flow lines.

Let us fix an orientation of each unstable manifold $W^u(x)$. As we have seen in Section 2.7, this choice determines a preferred isomorphism $\Theta_k\colon C_k(X) \cong W_k(X)$.

Moreover, it determines an orientation of each transverse intersection $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$. Indeed, the orientation of each unstable manifold determines a co-orientation of each stable manifold (that is an orientation of its normal bundle), and the transverse intersection of a finite-dimensional oriented submanifold with a finite-codimensional co-oriented submanifold carries a canonical orientation: if $p \in W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ and $V \subset T_p W^{\mathrm{u}}(x)$ is a linear complement of $T_p(W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y))$ in $T_p W^{\mathrm{u}}(x)$, by transversality V is also a complement of $T_p W^{\mathrm{s}}(x)$ in $T_p M$, so it is oriented, and the orientation of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is the one for which

$$T_p W^{\mathrm{u}}(x) = T_p(W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)) \oplus V$$

is an oriented sum.

In particular, if $m(x) - m(y) = 1$ each connected component W of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is an oriented line, and we can define $\epsilon(W)$ to be $+1$ if ϕ is orientation preserving on W, -1 otherwise. Then we can define the integer

$$n(x, y) = \sum_{\substack{W \text{ connected component} \\ \text{of } W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)}} \epsilon(W), \quad \forall x, y \in \operatorname{rest}(X),\ m(x) - m(y) = 1.$$

Assume that conditions (A1)–(A6), (A7′), and (A8) hold. Then we have the following fact.

THEOREM 2.11. *In terms of the preferred isomorphism $\Theta_k \colon C_k(X) \cong W_k(X)$, the boundary operator of the Morse complex of X has the form*

$$\partial_k x = \sum_{y \in \operatorname{rest}_{k-1}(X)} n(x, y) y, \quad \forall x \in \operatorname{rest}_k(X) \subset C_k(X). \tag{39}$$

Before proving this result, we recall that if σ_n denotes the generator of $H_n(S^n)$ corresponding to the standard orientation of $\partial D^{n+1} = S^n$, that is the one for which $\mathbb{R}^{n+1} = \mathbb{R}\zeta \oplus T_\zeta S^n$ is an oriented sum, for every ζ in S^n, we have that the boundary homomorphism $H_{n+1}(D^{n+1}, \partial D^{n+1}) \to H_n(\partial D^{n+1})$ maps ω_{n+1} into σ_n.

EXERCISE 2.12. Let $A_1, \dots, A_h$ be pairwise disjoint closed n-discs in S^n, with maps

$$a^i \colon (D^n, \partial D^n) \to \left(S^n, S^n \setminus \bigcup_{i=1}^h \mathring{A}_i\right),$$

mapping D^n homeomorphically onto A_i, preserving the standard orientations. Let $j \colon S^n \to (S^n, S^n \setminus \bigcup_{i=1}^h \mathring{A}_i)$ be the inclusion. Then

$$j_*(\sigma_n) = \sum_{i=1}^h a^i_*(\omega_n).$$

Proof of Theorem 2.11. Notice first of all that by (A2) and (A6), for every $x \in \mathrm{rest}(X)$ there are finitely many rest points $y \in \mathrm{rest}(X)$ with $f(y) < f(x)$, so the sum appearing in (39) is finite.

Let $\rho\colon \mathrm{rest}(X) \to \,]0, +\infty[$ be a function satisfying (34), and let $M_k = M_k(\rho)$, for $k \in \mathbb{N} \cup \{\infty\}$. Let us fix a rest point x of Morse index k.

By the naturality of the boundary homomorphism of pairs in singular homology, we have the commutative diagram

$$\begin{array}{ccc} H_k(D^k, \partial D^k) & \xrightarrow{\theta^x_*} & H_k(M_k, M_{k-1}) \\ \downarrow & & \downarrow \\ H_{k-1}(\partial D^k) & \xrightarrow{\alpha_*} & H_{k-1}(M_{k-1}) \end{array}$$

where $\alpha\colon \partial D^k \to M_{k-1}$ is the restriction of θ^x. The cellular boundary homomorphism ∂_k of the cellular filtration $\{M_k\}_{k\in\mathbb{Z}}$ is the composition of the right vertical arrow with the homomorphism induced by the inclusion $i\colon M_{k-1} \hookrightarrow (M_{k-1}, M_{k-2})$. On the other hand, the left vertical arrow is an isomorphism mapping ω_k into σ_{k-1}. Therefore, ∂_k maps the generator $\theta^x_*(\omega_k)$ of $H_k(M_k, M_{k-1})$ into $i_*\alpha_*(\sigma_{k-1}) \in H_{k-1}(M_{k-1}, M_{k-2})$, and we must express the latter element in terms of the generators $\theta^y_*(\omega_{k-1})$ of $H_{k-1}(M_{k-1}, M_{k-2})$, for $y \in \mathrm{rest}_{k-1}(X)$.

By the Morse – Smale condition up to order 1,

$$\alpha^{-1}\Big(\bigcup_{y\in\mathrm{rest}_{k-1}(X)} W^{\mathrm{s}}(y)\Big) = \{\zeta_1, \dots, \zeta_h\}$$

is a finite subset of ∂D^k, and α maps all the other points into points which either belong to stable manifolds of rest points of index less than $k-1$, or eventually enter A; so the orbit of any point in $\alpha(\partial D^k \setminus \{\zeta_1, \dots, \zeta_h\})$ eventually enters M_{k-2}. Choose $r > 0$ so small that the closed r-balls $\overline{B}_r(\zeta_i) \subset \partial D^k$ centered in ζ_i are pairwise disjoint $(k-1)$-discs. Let $b\colon \partial D^k \to \mathbb{R}$ be a continuous function such that

$$\chi t_{M_{k-2}} \circ \alpha < b < t^+ \circ \alpha,$$

where χ is the characteristic function of the open set $\partial D^k \setminus \bigcup_{i=1}^h \overline{B_r}(\zeta_i)$, and $t_{M_{k-2}}$ is the entrance time function into M_{k-2}. Then α is homotopic to the map

$$\beta\colon \partial D^k \to M_{k-1}, \quad \zeta \mapsto \phi(b(\zeta), \alpha(\zeta)),$$

so

$$\theta^x_*(\omega_k) = i_*\alpha_*(\sigma_{k-1}) = i_*\beta_*(\sigma_{k-1}).$$

Denote by

$$\gamma_i\colon (D^{k-1}, \partial D^{k-1}) \to (M_{k-1}, M_{k-2})$$

the composition of $i \circ \beta$ with an orientation preserving homeomorphism

$$(D^{k-1}, \partial D^{k-1}) \to (\overline{B}_\rho(\zeta_i), \partial B_\rho(\zeta_i)).$$

Then the result of Exercise 2.12 shows that:

$$\theta^x_*(\omega_k) = i_* \beta_*(\sigma_{k-1}) = \sum_{i=1}^{h} \gamma_{i*}(\omega_{k-1}). \tag{40}$$

Fix some $i \in \{1, \dots, h\}$, let y be the rest point of index $k-1$ toward which the orbit of $\alpha(\zeta_i)$, i.e., of $\beta(\zeta_i)$, converges for $t \to +\infty$, and let W_i be the connected component of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ consisting of such an orbit.

We claim that γ_i is homotopic to either θ^y, in the case $\epsilon(W_i) = 1$, or to $\theta^y \circ \mu$, where μ is an orientation reversing automorphism of $(D^{k-1}, \partial D^{k-1})$, in the case $\epsilon(W_i) = -1$. Therefore

$$\gamma_{i*}(\omega_{k-1}) = \epsilon(W_i)\theta^y_*(\omega_{k-1}),$$

and (40) allows to conclude.

Let us proof the claim. Up to a small perturbation, we may assume that γ_i is a C^1 embedding of a closed $(k-1)$-disc, meeting $W^{\mathrm{s}}(y)$ transversally at the single point $p = \gamma_i(0)$. The diffeomorphism γ_i induces an orientation of $T_p\gamma_i(D^{k-1})$, the one for which

$$T_p W^{\mathrm{u}}(x) = \mathbb{R}X(p) \oplus T_p\gamma_i(D^{k-1})$$

is an oriented sum. The differential of the flow $D_2\phi(t, \cdot)$ at p maps the tangent space of $\gamma_i(D^{k-1})$ at p onto a subspace of $T_{\phi(t,p)}M$ which converges to $T_y W^{\mathrm{u}}(y)$ for $t \to +\infty$ (see for instance Abbondandolo and Majer, 2003c, Theorem 2.1(iii)). A first consequence is that the orientation of $T_p\gamma_i(D^{k-1})$ defined above is $\epsilon(W_i)$ times the orientation obtained by seeing $T_p\gamma_i(D^{k-1})$ as a complement of $T_p W^{\mathrm{s}}(y)$ in T_pM. A second consequence is that, by the evolution of graphs of Lipschitz maps from $E^{\mathrm{u}}_y(r)$ to $E^{\mathrm{s}}_y(r)$ near the hyperbolic rest point y (see Shub, 1987, or Abbondandolo and Majer, 2001, Proposition A.3 and Addendum A.5), if $r > 0$ is small and $t \geq 0$ is large then $\phi(\{t\} \times \gamma_i(D^{k-1})) \cap E_y(r)$ is the graph of a map[3] $\tau : E^{\mathrm{u}}_y(r) \to E^{\mathrm{s}}_y(r)$. Let $K \subset D^{k-1}$ be the closed neighborhood of 0 such that

$$\phi(\{t\} \times \gamma_i(K)) = \operatorname{graph} \tau.$$

Since K is a closed $(k-1)$-disc, it is a deformation retract of D^{k-1}. Since the local unstable manifold $W^{\mathrm{u}}_{\mathrm{loc},r}(y)$ is also the graph of a map $\sigma^{\mathrm{u}} : E^{\mathrm{u}}_y(r) \to E^{\mathrm{s}}_y(r)$, it is now easy to combine the above maps to construct a homotopy between γ_i and an embedding of $(D^{k-1}, \partial D^{k-1})$ into $(W^{\mathrm{u}}(y), W^{\mathrm{u}}(y) \cap M_{k-2})$, which is orientation

[3] This statement is part of the content of the so called λ-lemma, in the particular case of a gradient-like flow. See Palis (1968) and Palis and de Melo (1982).

preserving, hence homotopic to θ^y, if $\epsilon(W_i) = 1$, orientation reversing, hence homotopic to $\theta^y \circ \mu$, if $\epsilon(W_i) = -1$. $\square$

2.9. HOW TO REMOVE THE ASSUMPTION (A8)

If we drop assumption (A8), there need not exist a function ρ satisfying (34), and it becomes more difficult to associate a cellular filtration to X. Nevertheless, we can make the graded group $C_*(X)$ into a chain complex by taking a direct limit of the Morse complexes on sublevels $\{f < a\}$, for $a \uparrow \sup f$. On these domains indeed, there are finitely many rest points and condition (A7) guarantees condition (A8). Not being forced to assume (A8) is a positive fact, in that assumption (A7) can be more easily achieved by generic perturbations, as we shall see in Section 2.12.

If the supremum of f on $\overline{M}$ is attained, by (A2) and (A6) X has finitely many rest points, so (A8) is implied by (A7). Thus, we can assume that $\sup f$ is not attained.

For $a < \sup f$, let $W_*(X)^a$ be the Morse complex associated to $\widehat{M}^a := A \cup \{f < a\}$, and if $a < b < \sup f$, let

$$w_{ab} \colon W_*(X)^a \to W_*(X)^b$$

be the chain map induced by the inclusion $\widehat{M}^a \hookrightarrow \widehat{M}^b$. The *Morse complex of X* is defined to be the chain complex

$$W_*(X) := \lim_{a \uparrow \sup f} W_*(X)^a,$$

the limit of the direct system $\{W_*(X), w_{ab}\}$. Notice that if (A8) holds, so that $W_*(X)$ is the chain complex defined in Section 2.7, the family of chain complexes $\{W_*(X)^a\}_{a<\sup f}$ is identified with an increasing and exhausting family of sub-complexes of $W_*(X)$, so this definition of the Morse complex agrees with the previous one.

Since the homology of a direct limit of chain complexes is the direct limit of the homologies (see Dold, 1980, VIII.5.20),

$$H_k W_*(X) = \lim_{a \uparrow \sup f} H_k W_*(X)^a.$$

Similarly, the singular homology of an increasing union of open subsets is the limit of the singular homologies (see Dold, 1980, VIII.5.22), so

$$H_k(\widehat{M}, A) = \lim_{a \uparrow \sup f} H_k(\widehat{M}^a, A).$$

We conclude that the homology of the Morse complex of X is isomorphic to the singular homology of $(\widehat{M}, A)$,

$$H_k(W_*(X)) \cong H_k(\widehat{M}, A) \quad \forall k \in \mathbb{N}.$$

Finally, having fixed an orientation for each unstable manifold, we have the isomorphisms

$$\Theta_k^a \colon C_k(X)^a \cong W_k(X)^a,$$

$C_k(X)^a$ being the subgroup of $C_k(X)$ generated by the rest points x with $f(x) < a$, and the limit of this direct system defines an isomorphism

$$\Theta_k \colon C_k(X) \cong W_k(X).$$

REMARK 2.13. Since the boundary of $x \in \mathrm{rest}(X)$ in $C_*(X)$ and in $C_*(X)^a$ coincide when $f(x) < a < \sup f$, the formula for the boundary homomorphism under the Morse – Smale condition up to order 1 (Theorem 2.11) holds also without assuming (A8).

2.10. MORSE FUNCTIONS ON HILBERT MANIFOLDS

A particular but important case is the following situation: f is a C^2 Morse function on a smooth Hilbert manifold N, endowed with a C^1 Riemannian metric g, $-\infty < a < b \leq +\infty$, $\widehat{M} = \{p \in N \mid f(p) < b\}$, $M = \{p \in N \mid a < f(p) < b\}$, and $\widehat{X} = -\nabla f$, the negative gradient of f with respect to the metric g. Let us see what the assumptions (A1) – (A8) look like in this situation.

In this case, of course, $\mathrm{rest}(X) = \mathrm{crit}(f) \cap \{a < f < b\}$, the set of critical points of f with values between a and b. Condition (A2) is equivalent to saying that f is a Morse function on M, and in condition (A3) the Morse index is the standard Morse index of a critical point of $f|_M$. The set of critical points of f with index k will be denoted by $\mathrm{crit}_k(f)$.

Condition (A4) is automatically fulfilled, f itself being a Lyapunov function for $-\nabla f$, and so is condition (A5).

In the case of a gradient flow the (PS) condition can be restated in the more familiar way: *the pair* (f, g) *satisfies the* (PS) *condition at level* $c \in \mathbb{R}$ if every sequence $(p_n) \subset \widehat{M}$ such that $f(p_n) \to c$ and $\|df(p_n)\| \to 0$ is compact (here the norm $\|\cdot\|$ on T^*M is induced by the Riemannian structure g). The assumption (A6) is equivalent to: (f, g) satisfies the (PS) condition at level c for every $c \in [a, b[$, and a is a regular value for f.

As we shall see in Section 2.12, the Morse – Smale condition required in (A7) can be always achieved by perturbing the metric g.

Finally, assumption (A1) is automatically fulfilled when $(\widehat{M}, g)$ is complete. Indeed, the following fact holds.

PROPOSITION 2.14. *Let* $f \in C^2(\widehat{M}, \mathbb{R})$ *and* $a \in \mathbb{R}$ *be such that the strip* $\{a \leq f \leq c\}$ *is complete* (*with respect to the geodesic distance* d *on* $\widehat{M}$ *induced by the Riemannian metric* g), *for every* $c < \sup f$. *Then the vector field* $-\nabla f$ *is positively complete with respect to* $\{f < a\}$.

Proof. Let $p \in \widehat{M}$ and consider the curve $u\colon [0, t^+(p)[\to \widehat{M}$, $u(t) = \phi(t, p)$. If $f(p) = \sup f$, then p is a critical point of f, so $t^+(p) = +\infty$. If $\inf f \circ u < a$ then $u(t)$ eventually enters $\{f < a\}$.

Therefore we can assume that $f(p) < \sup f$ and $u([0, t^+(p)[) \subset \{a \leq f \leq f(p)\}$, and we must prove that $t^+(p) = +\infty$. Let $0 \leq s < t$. Then

$$f(u(t)) - f(u(s)) = \int_s^t Df(u(\tau))[-\nabla f(u(\tau))]\, d\tau = -\int_s^t g\Big(\nabla f(u(\tau)), \nabla f(u(\tau))\Big)\, d\tau,$$

and the Cauchy – Schwarz inequality implies that

$$\begin{aligned} d(u(s), u(t)) &\leq \int_s^t \sqrt{g(u'(\tau), u'(\tau))}\, d\tau \\ &= \int_s^t \sqrt{g\Big(\nabla f(u(\tau)), \nabla f(u(\tau))\Big)}\, d\tau \\ &\leq \sqrt{t-s}\Big(\int_s^t g\Big(\nabla f(u(\tau)), \nabla f(u(\tau))\Big)\, d\tau\Big)^{1/2} \\ &= \sqrt{t-s}\sqrt{f(u(s)) - f(u(t))} \leq \sqrt{t-s}\sqrt{f(p) - \inf f \circ u}. \end{aligned}$$

The above estimate shows that u is uniformly continuous. If by contradiction $t^+(p) < +\infty$, by the completeness of the strip $\{a \leq f \leq f(p)\}$ we deduce that $u(t)$ converges for $t \to t^+(p)$. But then the solution u of $u' = -\nabla f(u)$, $u(0) = p$, can be extended to a right neighborhood of $t^+(p)$, contradicting the maximality of $t^+(p)$. □

We summarize the above discussion into the following proposition.

PROPOSITION 2.15. *Let f be a C^2 function on the smooth Hilbert manifold N, endowed with a C^1 Riemannian metric g, and let $-\infty < a < b \leq +\infty$. Assume that*

(B1) *a is a regular value of f*;

(B2) *f is a Morse function on $\{a < f < b\}$, and it has only critical points of finite Morse index in such a strip*;

(B3) *for every $c < b$, the strip $\{a \leq f \leq c\}$ is complete*;

(B4) *f satisfies the* (PS) *condition at every level $c \in [a, b[$.*

Then, setting $\widehat{M} = \{f < b\}$, $\widehat{X} = -\nabla f|_{\widehat{M}}$ and $M = \{a < f < b\}$, the conditions (A1) – (A6) *are fulfilled.*

Notice that only (B3) and (B4) involve the metric. Moreover, if (B3) and (B4) hold for some metric, they hold also for every uniformly equivalent metric.

Under the assumptions (B1) – (B4), the free Abelian group generated by the critical points of f of index k in $\{a < f < b\}$ will be denoted by $C_k(f)_a^b$. The lower index will be omitted when $a < \inf f$, the upper index will be omitted when $b = +\infty$.

If $-\nabla f$ satisfies also the Morse – Smale condition up to order 0 on $\{a < f < b\}$, the boundary operator of the Morse complex of $-\nabla f$ on $\{a < f < b\}$ will be denoted by

$$\partial_k(f,g)_a^b \colon C_k(f)_a^b \to C_{k-1}(f)_a^b.$$

Its homology is isomorphic to the singular homology of $(\{f < b\}, \{f < a\})$:

$$H_k(\{C_*(f)_a^b, \partial_*(f,g)_a^b\}) \cong H_k(\{f < b\}, \{f < a\}).$$

2.11. BASIC RESULTS IN TRANSVERSALITY THEORY

In the following lemma we single out a useful family of linear mappings whose kernel is complemented.

LEMMA 2.16. *Let E, F, G be Banach spaces, and assume that $A \in \mathcal{L}(E, G)$ has complemented kernel and finite-codimensional range. Then for every $B \in \mathcal{L}(F, G)$ the kernel of the operator $C \in \mathcal{L}(E \times F, G)$, $C(e, f) = Ae - Bf$, is complemented in $E \times F$.*

Proof. Let $E_0 := \ker A$, E_1 be a closed complement of E_0 in E, and P_0, P_1 be the associated projectors. Let $G_1 := \operatorname{ran} A$, G_0 be a (finite-dimensional) complement of G_1 in G, and Q_0, Q_1 be the associated projectors. Then A induces an isomorphism from E_1 onto G_1, whose inverse will be denoted by $T \in \mathcal{L}(G_1, E_1)$.

The equation $C(e, f) = 0$ is equivalent to $AP_1e = Bf$, which is equivalent to the system

$$\begin{cases} AP_1e = Q_1Bf, \\ Q_0Bf = 0, \end{cases}$$

again equivalent to

$$\begin{cases} P_1e = TQ_1Bf, \\ Q_0Bf = 0. \end{cases} \tag{41}$$

Since Q_0B has finite rank, its kernel — say F_0 — has a (finite-dimensional) complement F_1. By (41), the kernel of C is

$$\ker C = \{(e_0 + TQ_1Bf_0, f_0) \in E \times F \mid e_0 \in E_0,\ f_0 \in F_0\},$$

and the closed linear subspace $E_1 \times F_1$ is a complement of $\ker C$. □

Let us recall some definitions and basic facts about transversality in a Banach setting. A classical reference for these topics is Abraham and Robbin (1967). If

$\varphi: M \to N$ is a C^k map between Banach manifolds, $k \geq 1$, a point $q \in N$ is said a *regular value for* φ if for every $p \in \varphi^{-1}(\{q\})$ the differential $D\varphi(p): T_pM \to T_qN$ is a left inverse, i.e., if it is onto and its kernel is complemented. In this case, $\varphi^{-1}(\{q\})$ is a submanifold of class C^k.

A C^1 map $\varphi: M \to N$ between Banach manifolds is said a *Fredholm map* if its differential at every point is a Fredholm operator. When the index of the differential is constant (for instance when M is connected), this integer is said the *Fredholm index of* φ.

PROPOSITION 2.17. *Let M, N, O be Banach manifolds, and let $\varphi \in C^1(M,N)$, $\psi \in C^1(M,O)$ be maps with regular values $p \in N$ and $q \in O$. Then*:

(i) *p is a regular value for $\varphi|_{\psi^{-1}(\{q\})}$ if and only if q is a regular value for $\psi|_{\varphi^{-1}(\{p\})}$*;

(ii) *$\varphi|_{\psi^{-1}(\{q\})}$ is a Fredholm map if and only if $\psi|_{\varphi^{-1}(\{p\})}$ is a Fredholm map, in which case the indices coincide.*

This proposition is a consequence of the following linear statements.

PROPOSITION 2.18. *Let E, F, G be Banach spaces, and let $A \in \mathcal{L}(E,F)$, $B \in \mathcal{L}(E,G)$ be left inverses. Then*:

(i) *$A|_{\ker B}$ is a left inverse if and only if $B|_{\ker A}$ is a left inverse*;

(ii) *$A|_{\ker B}$ is Fredholm if and only if $B|_{\ker A}$ is Fredholm, in which case the indices coincide.*

Proof. Let $R \in \mathcal{L}(F,E)$ and $S \in \mathcal{L}(G,E)$ be right inverses of A and B, respectively.

(i) If $R_0 \in \mathcal{L}(F, \ker B)$ is a right inverse of $A|_{\ker B}$, i.e., a right inverse of A with range in $\ker B$, the map $S_0 := (I_E - R_0 A)S$ is a right inverse of B, being a perturbation of S by an operator with range in $\ker B$, and it takes value in $\ker A$ because

$$AS_0 = AS - AR_0AS = AS - I_F AS = 0.$$

Therefore, S_0 is a right inverse of $B|_{\ker A}$.

(ii) The kernels of $A|_{\ker B}$ and $B|_{\ker A}$ coincide:

$$\ker A|_{\ker B} = \ker B|_{\ker A} = \ker A \cap \ker B.$$

Moreover, since $R: F \to RF$ is an isomorphism and since $I - RA$ is a projector onto $\ker A$,

$$\operatorname{coker} A|_{\ker B} = \frac{F}{A \ker B} \overset{R}{\cong} \frac{RF}{RA \ker B} \cong \frac{\ker A + RF}{\ker A + RA \ker B} = \frac{E}{\ker A + \ker B}.$$

We conclude that the assertions in (ii) are equivalent, each of them being equivalent to the fact that the pair of subspaces $(\ker A, \ker B)$ is Fredholm, i.e., $\ker A \cap$

$\ker B$ is finite-dimensional, and $\ker A + \ker B$ is finite-codimensional.[4] The index of $A|_{\ker B}$ and of $B|_{\ker A}$ equals the index of $(\ker A, \ker B)$,

$$\mathrm{ind}(\ker A, \ker B) = \dim \ker A \cap \ker B - \mathrm{codim}(\ker A + \ker B). \qquad \square$$

We recall that a subspace T' of a topological space T is said *residual* if it contains a countable intersection of open and dense subspaces of T. Baire theorem guarantees that a residual subspace of a complete metric space is dense.

The following Sard – Smale Theorem, combined with Proposition 2.17, is the basic tool to deal with transversality questions.

THEOREM 2.19. *Let M, N be C^h Banach manifolds, $h \geq 1$, with M Lindelöf. Let $\varphi\colon M \to N$ be a C^h Fredholm map of index m. If $h >$ $\max\{0, m\}$ then the set of regular values of φ is residual in N.*

The proof can be found in Smale (1965).

2.12. GENERICITY OF THE MORSE – SMALE CONDITION

Let f be a C^{h+1} real function, $h \geq 1$, on the smooth Hilbert manifold N, endowed with a Riemannian metric g of class C^h. Let $-\infty < a < b \leq +\infty$, and assume (B1) – (B4). The aim of this section is to show that it is possible to perturb the metric g obtaining a uniformly equivalent metric such that the associated negative gradient of f has the Morse – Smale property up to order h. We shall assume N to be infinite-dimensional and second countable (in particular, it is modeled on a separable Hilbert space).

A well-known theorem by Eells and Elworthy (1970) implies that every infinite-dimensional Hilbert manifold can be smoothly embedded as an open subset of a Hilbert space. So we may assume that N is an open subset of the separable Hilbert space $(H, \langle\cdot,\cdot\rangle)$.[5]

Denote by $\mathrm{Sym}(H)$ the Banach space of self-adjoint bounded linear operators on H. The metric g can be represented by a C^h map $G\colon N \to \mathrm{Sym}(H)$ taking values in the cone of positive operators, such that

$$g(p)[\xi,\eta] = \langle G(p)\xi, \eta\rangle \quad \forall p \in N,\ \forall \xi, \eta \in T_pN = H.$$

We shall always denote by a lower case letter a symmetric bilinear form, and by the corresponding upper case letter the associated self-adjoint operator. The

[4] See also Section 3.2.

[5] Viewing N as an open subset of a Hilbert space is useful to simplify the notation (some spaces of maps are Banach spaces and not Banach manifolds, some sections of Banach bundles are just maps between Banach spaces, and so on) but it is by no means necessary. Therefore the results of this section hold also for a finite-dimensional N which is not diffeomorphic to an open subset of $\mathbb{R}^n$.

gradient of f with respect to the metric g is $\nabla_g f(p) = G(p)^{-1}\nabla f(p)$, where ∇f denotes the gradient of f with respect to the Hilbert inner product $\langle\cdot,\cdot\rangle$.

The Morse – Smale property will be achieved by rank 2 perturbations of G. In order to describe the space of such perturbations, let $\theta\colon N \to [0,+\infty[$ be a continuous function such that

$$\theta(p) \leq \frac{1}{\|G(p)^{-1}\|} \quad \forall p \in N. \tag{42}$$

The vector space

$$\mathcal{K} := \{K \in C^h_b(N, \mathrm{Sym}(H)) \,\big|\, \operatorname{rank} K(p) \leq 2 \;\forall p \in N, \\ \exists c \geq 0 \text{ such that } \|K(p)\| \leq c\,\theta(p) \;\forall p \in N\}$$

is a Banach space with the norm

$$\|K\|_{\mathcal{K}} := \|K\|_{C^h} + \sup_{\theta(p)\neq 0} \frac{\|K(p)\|}{\theta(p)}.$$

As usual, the symbol C^h_b denotes the space of maps whose differentials up to the hth order are continuous and bounded. Notice that the maps $K \in \mathcal{K}$ vanish on the set of zeroes of θ. By (42), for every $p \in N$

$$\|G(p)^{-1}K(p)\| \leq \|G(p)^{-1}\|\|K\|_{\mathcal{K}}\theta(p) \leq \|K\|_{\mathcal{K}},$$

so if $\|K\|_{\mathcal{K}} < 1$, $G + K = G(I + G^{-1}K)$ is positive, and defines a metric $g + k$ which is uniformly equivalent to g. Denote by $\mathcal{K}_1$ the open unit ball of $\mathcal{K}$. The main result of this section is the following theorem.

THEOREM 2.20. *Let f be a C^{h+1} function, $h \geq 1$, on the smooth second countable Hilbert manifold $N \subset H$, endowed with a Riemannian metric g of class C^h. Let $-\infty < a < b \leq +\infty$, and assume* (B2). *Assume that the continuous function $\theta\colon N \to [0,+\infty[$ satisfies* (42), *that its set of zeroes is the closure of an open set, and that it has the following property*: *if x, y are critical points in $\{a < f < b\}$ with $m(x) - m(y) \leq h$, such that $W^u(x)$ and $W^s(y)$* (*with respect to $-\nabla_g f$*) *have a nontransverse intersection at p, then $\theta > 0$ somewhere on the orbit of p.*

Then for every K in a residual subspace of $\mathcal{K}_1$, the metric $g + k$ associated to $G + K$ is such that the vector field $-\nabla_{g+k} f$ satisfies the Morse – Smale property up to order h.

Notice that high regularity of f and g is needed if we want to achieve the Morse – Smale property up to a high order. This phenomenon is determined by the regularity versus Fredholm index assumption required by the Sard – Smale Theorem 2.19. In a finite-dimensional setting this problem does not occur because there C^h functions can always be C^h-approximated by smooth ones, while such an approximation may not be possible on an infinite-dimensional Hilbert space (see for instance Nemirovskiĭ and Semenov, 1973 and Lasry and Lions, 1986). Notice that C^2 regularity of f is enough to get the Morse – Smale property up to order 1, which is just what we need in order to have the Morse complex and to represent it by intersection numbers.

The possibility of having a function θ which vanishes on some regions where the intersections are already transversal and which can be very small elsewhere will be useful in Section 2.13.

Let us set up the proof of Theorem 2.20. Fix two critical points $x \neq y$ in $\{a < f < b\}$ with $m(x) - m(y) \leq h$, and consider the space of curves

$$\mathcal{C} = \mathcal{C}(x, y) := \left\{ u \in C^1(\mathbb{R}, N) \;\middle|\; \lim_{t\to-\infty} u(t) = x,\ \lim_{t\to+\infty} u(t) = y,\ \lim_{t\to\pm\infty} u'(t) = 0 \right\}.$$

The space $\mathcal{C}$ is a smooth Banach manifold, being an open subset of an affine Banach space modeled on $C^1_0(\mathbb{R}, H)$ (the spaces C^h_0 are defined in Section 1.2). Therefore, $T_u\mathcal{C} = C^1_0(\mathbb{R}, H)$. The map

$$\Psi \colon \mathcal{C} \times \mathcal{K}_1 \to C^0_0(\mathbb{R}, H), \quad (u, K) \mapsto u' + \nabla_{g+k} f(u) = u' + (G + K)^{-1}(u)\nabla f(u),$$

is of class C^h, and its zeroes are the pairs (u, K) such that u is a negative gradient flow line of f with respect to the metric $g+k$, going from x to y. Set $\mathcal{Z} := \Psi^{-1}(\{0\})$. The following two lemmas describe some properties of the differential of Ψ with respect to the first, respectively the second variable.

LEMMA 2.21. *Let* $(u, K) \in \mathcal{Z}$. *Then*:

(i) *the operator* $D_1\Psi(u, K) \colon T_u\mathcal{C} \to C^0_0(\mathbb{R}, H)$ *is Fredholm of index* $m(x) - m(y)$;

(ii) *the operator* $D_1\Psi(u, K)$ *is onto if and only if the unstable manifold of* x *and the stable manifold of* y *with respect to the vector field* $-\nabla_{g+k} f$ *intersect transversally at* $u(t)$ *for some (hence all)* $t \in \mathbb{R}$;

(iii) *if* $w \in C^0_0(\mathbb{R}, H)$ *and* $a < b$ *are real numbers, then there exists* $v \in T_u\mathcal{C}$ *such that*
$$D_1\Psi(u, K)[v](t) = w(t) \quad \forall t \in\,]-\infty, a] \cup [b, +\infty[.$$

Proof. The differential of Ψ with respect to the first variable is of the form

$$D_1\Psi(u, K) \colon C^1_0(\mathbb{R}, H) \to C^0_0(\mathbb{R}, H), \quad v \mapsto v' - Av,$$

where $A \colon \mathbb{R} \to \mathcal{L}(H)$ is defined by

$$A(t) := -(G + K)^{-1}(u(t))D^2 f(u(t)) - D(G + K)^{-1})(u(t))\nabla f(u(t)).$$

Since $u(t)$ converges to x, resp. to y, for $t \to -\infty$, resp. $t \to +\infty$, $A(t)$ converges in norm to the operators

$$\begin{aligned} A(-\infty) &= -(G+K)^{-1}(x) D^2 f(x) = -\nabla^2_{g+k} f(x), \\ A(+\infty) &= -(G+K)^{-1}(y) D^2 f(y) = -\nabla^2_{g+k} f(y), \end{aligned}$$

which are hyperbolic, and have positive eigenspaces of dimension $m(x)$ and $m(y)$, respectively. Then (i) follows from Proposition 1.8. Claim (ii) follows from the second identity in (9) , and from the identities

$$T_{u(0)} W^{\mathrm{u}}(x) = W^{\mathrm{u}}_A, \quad T_{u(0)} W^{\mathrm{s}}(y) = W^{\mathrm{s}}_A.$$

As for claim (iii), up to a translation we may assume that $a < 0 < b$. Then the conclusion follows from Proposition 1.6(i), applied to $A|_{[0,+\infty]}$ and to $A|_{[-\infty,0]}(-\cdot)$. $\square$

LEMMA 2.22. *Let $(u, K) \in \mathcal{Z}$, and let $a < b$ be real numbers such that $\theta(u(t)) \neq 0$ for every $t \in [a, b]$. Let $w \in C^h(\mathbb{R}, H)$ be a curve with support in $[a, b]$. Then there exists $J \in \mathcal{K}$ such that $D_2\Psi(u, K)[J] = w$.*

Proof. The differential of Ψ with respect to the second variable is

$$D_2\Psi(u, K)[J] = -(G+K)^{-1}(u) J(u) (G+K)^{-1}(u) \nabla f(u).$$

Since $(u, K) \in \mathcal{Z}$, the curve u is a flow line of the vector field $-\nabla_{g+k} f$ going from x to y. In particular, u is a C^{h+1} embedding of $\mathbb{R}$ into N, and $\nabla f_{g+k} \circ u$ never vanishes.

It is easy to find a C^h curve $J_0 \colon \mathbb{R} \to \mathrm{Sym}(H)$ with support in $[a, b]$ such that for every $t \in \mathbb{R}$ the symmetric operator $J(t)$ has rank not exceeding 2, and maps the nonzero vector $(G+K)^{-1}(u(t))\nabla f(u(t)) = \nabla_{g+k} f(u(t))$ into the vector $-(G+K)(u(t))w(t)$. Indeed, one may write an explicit formula for J_0 by noticing that if $\xi \neq 0$ and η are two elements of H, the bounded linear operator on H

$$\zeta \mapsto \frac{\langle \xi, \zeta \rangle}{|\xi|^2} \eta + \frac{\langle \eta, \zeta \rangle}{|\xi|^2} \xi - \frac{\langle \xi, \eta \rangle \langle \xi, \zeta \rangle}{|\xi|^4} \xi,$$

is self-adjoint, has rank not exceeding 2, vanishes when $\eta = 0$, maps ξ into η, and depends smoothly on $(\xi, \eta) \in (H \setminus \{0\}) \times H$.

Since u is a C^{h+1} embedding, given $\delta > 0$ we can find an open neighborhood U of $u(]a-\delta, b+\delta[)$ and a C^{h+1} submersion $\tau \colon U \to]a-\delta, b+\delta[$ such that $\tau(u(t)) = t$ for every $t \in]a-\delta, b+\delta[$. Since θ is positive on $u([a, b])$, up to choosing a smaller δ and a smaller U we may assume that $\inf_U \theta > 0$, and also that τ has bounded derivatives up to order $h+1$. If $\psi \in C^\infty_b(H, \mathbb{R})$ is a cut-off function with support in U and taking value 1 on $u([a, b])$, the C^h map $J \colon N \to \mathrm{Sym}(H)$, $J(p) = \psi(p) J_0(\tau(p))$, belongs to $\mathcal{K}$ and has the required property. $\square$

The following lemma is the key point in the proof of Theorem 2.20.

LEMMA 2.23. *Let $(u,K) \in \mathcal{Z}$. Then the differential $D\Psi(u,K)\colon T_uC \times \mathcal{K} \to C^0_0(\mathbb{R},H)$ is a left inverse.*

Proof. We must prove that the operator

$$D\Psi(u,K)[(v,J)] = D_1\Psi(u,K)[v] + D_2\Psi(u,K)[J]$$

is onto and that its kernel is complemented in $T_uC \times \mathcal{K}$. By Lemma 2.21(i), the operator $D_1\Psi(u,K)$ is Fredholm, so Lemma 2.16 implies that $\ker D\Psi(u,K)$ is complemented in $T_uC \times \mathcal{K}$. Moreover, the range of $D\Psi(u,K)$ contains the range of $D_1\Psi(u,K)$, in particular it has finite codimension.

If $\theta \circ u(t) = 0$ for every $t \in \mathbb{R}$, also $K \circ u$ vanishes identically, so $\nabla_{g+k} f \circ u = \nabla_g f \circ u$, and (recalling that the set of zeroes of θ is the closure of an open set) the tangent spaces of the unstable and stable manifolds of x and y along u are the same for $-\nabla_{g+k} f$ and for $-\nabla_g f$. Therefore, the assumption of Theorem 2.20 guarantees that these manifolds meet transversally along u. By Lemma 2.21(ii), $D_1\Psi(u,K)$ is onto, and so is $D\Psi(u,K)$.

If $\theta \circ u$ is not identically zero, we can find real numbers $a < b$ such that $\theta(u(t)) \neq 0$ for every $t \in [a,b]$. Let $w \in C^0_0(\mathbb{R},H)$ and let $\epsilon > 0$. By Lemma 2.21(iii), there exists $v \in T_uC$ such that

$$D_1\Psi(u,K)[v](t) = w(t) \quad \forall t \in]-\infty,a] \cup [b,+\infty[.$$

The curve $w - D_1\Psi(u,K)[v]$ is continuous and has support in $[a,b]$, and we can find a C^h curve $z : \mathbb{R} \to H$ with support in $[a,b]$ such that

$$\|z - (w - D_1\Psi(u,K)[v])\|_\infty < \epsilon.$$

Since z has support in $[a,b]$, where $\theta \circ u$ does not vanish, by Lemma 2.22 there exists $J \in \mathcal{K}$ such that $D_2\Psi(u,K)[J] = z$. Hence

$$\|D\Psi(u,K)[(v,J)] - w\|_\infty = \|D_1\Psi(u,K)[v] + z - w\|_\infty < \epsilon.$$

Therefore, $D\Psi(u,K)$ has dense and finite-codimensional range, so it is onto. □

In particular, $\mathcal{Z}$ is a C^h submanifold of $C \times \mathcal{K}_1$. Let π be the restriction to $\mathcal{Z}$ of the projection onto the second factor in the product $C \times \mathcal{K}_1$.

LEMMA 2.24. *The map $\pi\colon \mathcal{Z} \to \mathcal{K}_1$ is Fredholm of index $m(x) - m(y)$.*

Proof. Everything follows from Proposition 2.17(ii), applied to $M = C \times \mathcal{K}_1$, $N = \mathcal{K}_1$, $O = C^0_0(\mathbb{R},H)$, $\varphi\colon C \times \mathcal{K}_1 \to \mathcal{K}_1$ projection onto the second factor, $\psi = \Psi$, together with Lemma 2.21(i) and Proposition 2.23. □

Denote by $\mathcal{H}(x,y)$ the set of regular values of π.

LEMMA 2.25. *The set $\mathcal{H}(x,y)$ is residual in $\mathcal{K}_1$.*

Proof. Notice that $\mathbb{R}$ acts freely on the submanifold $\mathcal{Z}$ by $(t,(u,K)) \mapsto (u(t+\cdot),K)$, and the map π is invariant with respect to this action. Therefore, the quotient space $\widetilde{\mathcal{Z}} = \mathcal{Z}/\mathbb{R}$ is still a C^h manifold, and the induced map $\widetilde{\pi}\colon \widetilde{\mathcal{Z}} \to \mathcal{K}_1$ is of class C^h and Fredholm index $m(x) - m(y) - 1 \leq h - 1$, by Lemma 2.24. Moreover, K is a regular value for π if and only if it is a regular value for $\widetilde{\pi}$.

Since N, and thus H, is assumed to be second countable, $\mathcal{C} \times \mathcal{K}_1$ is second countable, and so are $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$. Since the level of differentiability of $\widetilde{\pi}$ is strictly greater than its Fredholm index, the Sard – Smale Theorem 2.19 implies that the set of regular values of $\widetilde{\pi}$ — and thus of π — is residual in $\mathcal{K}_1$. □

Proof of Theorem 2.20. By Proposition 2.17(i), $\mathcal{H}(x,y)$ is also the set of $K \in \mathcal{K}_1$ for which the map $\Psi(\cdot,K) : \mathcal{C}(x,y) \to C^0_0(\mathbb{R},H)$ has 0 as a regular value. By Lemma 2.21 (ii), $\mathcal{H}(x,y)$ is also the set of $K \in \mathcal{K}_1$ such that the unstable manifold of x and the stable manifold of y with respect to $-\nabla_{g+k} f$ meet transversally. By Lemma 2.25, the countable intersection

$$\bigcap_{\substack{x,y \in \mathrm{crit}(f) \cap \{a < f < b\} \\ x \neq y,\ m(x)-m(y) \leq h}} \mathcal{H}(x,y)$$

is the required residual subset of $\mathcal{K}_1$. □

2.13. INVARIANCE OF THE MORSE COMPLEX

Let $f \in C^2(M)$ be a Morse function on the smooth second countable Hilbert manifold M, with critical points of finite index. Assume that f is bounded below and that M admits a complete Riemannian metric g such that (f,g) satisfies the Palais – Smale condition. We know from the previous section that by perturbing g we can achieve also the Morse – Smale property up to order 1. In general, different Morse – Smale metrics will produce different Morse complexes: the groups $C_k(f)$ are the same, but the boundary operators ∂_k may vary. Of course the homology of the Morse complex does not vary, being isomorphic to the singular homology of M, but we can say more: varying the metric we obtain isomorphic chain complexes. This fact was observed by Cornea and Ranicki (2003) (together with other interesting rigidity results) for finite-dimensional manifolds, and for some cases of Floer theory. The proof we give here in our infinite-dimensional situation uses an idea from Abbondandolo and Majer (2001) (see also Poźniak, 1991).

THEOREM 2.26. *Let $f \in C^2(M)$ be a Morse function, bounded below, having only critical points of finite Morse index. Let g_0 and g_1 be complete Riemannian* *metrics on M, such that both (f, g_0) and (f, g_1) satisfy* (PS) *and the Morse – Smale property up to order* 1. *Then there is a chain complex isomorphism*

$$\Phi \colon \{C_*(f), \partial_*(f, g_0)\} \cong \{C_*(f), \partial_*(f, g_1)\}$$

of the form

$$\Phi x = x + \sum_{\substack{y \in \mathrm{crit}_k(f) \\ f(y) < f(x)}} n(x, y) y, \quad \forall x \in \mathrm{crit}_k(f), \ k \in \mathbb{N}, \tag{43}$$

for suitable integers $n(x, y)$.

The following lemma will be needed in the proof:

LEMMA 2.27. *Let a be a nondegenerate continuous symmetric bilinear form on the real Hilbert space $(H, \langle \cdot, \cdot \rangle)$, with either finite Morse index or finite Morse co-index. Let $t_0 \geq 0$, and let $t \mapsto \langle \cdot, \cdot \rangle_t$, $t \in \mathbb{R}$, be a continuous path of inner products on H — equivalent to $\langle \cdot, \cdot \rangle$ — constant for $t \geq t_0$ and for $t \leq -t_0$. Let $A(t)$ be the $\langle \cdot, \cdot \rangle_t$-self-adjoint bounded operator on H representing a with respect to the inner product $\langle \cdot, \cdot \rangle_t$: $a(\xi, \eta) = \langle A(t)\xi, \eta \rangle_t$ for every $\xi, \eta \in H$. Then the linear stable and unstable spaces of the path A* (*see Section* 1.2) *satisfy*

$$H = W_A^{\mathrm{s}} \oplus W_A^{\mathrm{u}}.$$

Proof. The path A is continuous and it is constant for $t \geq t_0$ and for $t \leq -t_0$. Let us assume that a has finite Morse index, the other case being easily reducible to this one. The linear stable space W_A^{s} has dimension $m(a)$, the Morse index of a, while the linear unstable space W_A^{u} is closed and has codimension $m(a)$. Therefore, it is enough to prove that $W_A^{\mathrm{s}} \cap W_A^{\mathrm{u}} = (0)$.

Let $u_0 \in W_A^{\mathrm{s}} \cap W_A^{\mathrm{u}}$, and let $u \colon \mathbb{R} \to H$ be the solution of the linear Cauchy problem

$$\begin{cases} u'(t) = A(t) u(t), \\ u(0) = u_0. \end{cases}$$

Since A is constant for $t \geq t_0$ and for $t \leq -t_0$,

$$u(t) = e^{(t - t_0) A(t_0)} u(t_0) \ \forall t \geq t_0, \quad u(t) = e^{(t + t_0) A(-t_0)} u(-t_0) \ \forall t \leq -t_0.$$

Since $u(t) \to 0$ for $|t| \to 0$, we deduce that $u(t_0)$ belongs to the negative eigenspace of $A(t_0)$, and $u(-t_0)$ belongs to the positive eigenspace of $A(-t_0)$. Since both $A(t_0)$ and $A(-t_0)$ represent the symmetric form a, we have

$$a(u(t_0), u(t_0)) \leq 0, \quad a(u(-t_0), u(-t_0)) \geq 0. \tag{44}$$

On the other hand, since $A(t)$ represent a for every t, the inequality

$$\frac{1}{2} \frac{d}{dt} a(u(t), u(t)) = a(u(t), u'(t)) = a(u(t), A(t) u(t)) = \langle A(t) u(t), A(t) u(t) \rangle_t \geq 0$$

is compatible with (44) if and only if $u(t) = 0$ for every $t \in [-t_0, t_0]$ (hence for every $t \in \mathbb{R}$), proving that $u_0 = 0$. Therefore $W^{\mathrm{s}}_A \cap W^{\mathrm{u}}_A = (0)$. □

Proof of Theorem 2.26. We introduce the smooth Morse function

$$\varphi \colon \mathbb{R} \to \mathbb{R}, \quad \varphi(s) = 2s^3 - 3s^2 + 1,$$

which has two critical points, namely a local maximum at 0, with $\varphi(0) = 1$, and a local minimum at 1, with $\varphi(1) = 0$. Moreover $\varphi'(s)$ diverges for $|s| \to +\infty$.

On the manifold $\widetilde{M} = \mathbb{R} \times M$ consider the C^2 function

$$\tilde{f} \colon \widetilde{M} \to \mathbb{R}, \quad \tilde{f}(s,p) = \varphi(s) + f(p).$$

It is a Morse function, with critical points of finite Morse index, and

$$\mathrm{crit}_k(\tilde{f}) = (\{0\} \times \mathrm{crit}_{k-1}(f)) \cup (\{1\} \times \mathrm{crit}_k(f)),$$

for every $k \in \mathbb{N}$. Therefore

$$C_k(\tilde{f}) \cong C_{k-1}(f) \oplus C_k(f), \quad \forall k \in \mathbb{N}, \tag{45}$$

the first group in the sum corresponding to the critical points in $\{0\} \times M$, the second one to critical points in $\{1\} \times M$.

If $\chi \colon \mathbb{R} \to [0,1]$ is a smooth cut-off function such that $\chi(s) = 1$ for $s \leq 1/3$ and $\chi(s) = 0$ for $s \geq \frac{2}{3}$, we can consider the complete Riemannian metric on $\widetilde{M}$

$$\tilde{g}(s,p)[(\sigma,\xi),(\sigma',\xi')] = \sigma\sigma' + \chi(s) g_0(p)[\xi,\xi'] + (1-\chi(s)) g_1(p)[\xi,\xi'],$$

for every $(\sigma,\xi),(\sigma',\xi') \in T_{(s,p)}\widetilde{M} = \mathbb{R} \oplus T_pM$.

Let $((s_n,p_n))$ be a (PS) sequence for $(\tilde{f},\tilde{g})$. Since $\|\nabla_{\tilde{g}} \tilde{f}(s,p)\|_{\tilde{g}} \geq |\varphi'(s)|$, we can find a subsequence of (s_n) which converges either to 0 or to 1. Since (f,g_0) and (f,g_1) satisfy (PS) and $\tilde{g}(s,p)|_{(0)\oplus TM}$ is just g_0 for s close to 0 and g_1 for s close to 1, we conclude that $(\tilde{f},\tilde{g})$ satisfies (PS).

Let us examine the negative gradient flow of $\tilde{f}$ with respect to the metric $\tilde{g}$.

(i) The hypersurfaces $\{0\} \times M$ and $\{1\} \times M$ are flow-invariant, and the restriction of the flow to $\{i\} \times M$ is nothing else but the negative gradient flow of f with respect to the metric g_i, for $i = 0,1$.

Moreover the invariant set $\{0\} \times M$ is a repeller, while $\{1\} \times M$ is an attractor. Therefore:

(ii) The only flow lines going from a critical point in $\{i\} \times M$ to a critical point in the same hypersurface are those which are fully contained in $\{i\} \times M$, for $i = 0,1$.

(iii) There are no flow lines going from a critical point in $\{1\} \times M$ to a critical point in $\{0\} \times M$.

If we view f as a function on $\widetilde{M}$, we have

$$\begin{aligned} Df(s,p)[-\nabla_{\tilde{g}}\tilde{f}(s,p)] &= Df(s,p)[(-\varphi'(s), -\nabla_{\tilde{g}(s,\cdot)} f(p)] \\ &= -\|\nabla_{\tilde{g}(s,\cdot)} f(p)\|^2_{\tilde{g}(s,\cdot)}. \end{aligned} \tag{46}$$

This implies that f is almost a Lyapunov function for the vector field $-\nabla_{\tilde{g}}\tilde{f}$:

(iv) f decreases strictly on all the nonconstant orbits, apart from those of the form

$$t \mapsto (s(t), x), \quad \text{with } x \in \operatorname{crit}(f),\ s'(t) = -\varphi'(s(t)).$$

In particular, up to time shifts there is exactly one flow line going from $(0,x)$ to $(1,x)$, for $x \in \operatorname{crit}(f)$, namely the orbit

$$t \mapsto (\bar{s}(t), x), \quad \text{with } \begin{cases} \bar{s}'(t) = -\varphi'(\bar{s}(t)), \\ \bar{s}(0) = \frac{1}{2}. \end{cases} \tag{47}$$

We claim that the intersection $W^{\mathrm{u}}((0,x)) \cap W^{\mathrm{s}}((1,x)) =]0,1[\times \{x\}$ is transverse. Indeed, by linearizing along the flow line $(\bar{s}(t), x)$, we easily see that

$$T_{(1/2,x)}W^{\mathrm{u}}((0,x)) = \mathbb{R} \oplus W^{\mathrm{u}}_A, \quad T_{(1/2,x)}W^{\mathrm{s}}((1,x)) = \mathbb{R} \oplus W^{\mathrm{s}}_A,$$

where the bounded linear operator $A(t) : T_xM \to T_xM$ is minus the Hessian of f at the critical point x with respect to the inner product

$$\tilde{g}(\bar{s}(t), x)|_{(0)\oplus T_xM} = \chi(\bar{s}(t))g_0 + (1 - \chi(\bar{s}(t)))g_1.$$

Then $A(t)$ represents the second differential of f at x with respect to the above inner product, so by Lemma 2.27, $T_xM = W^{\mathrm{s}}_A \oplus W^{\mathrm{u}}_A$. Therefore

$$T_{(1/2,x)}W^{\mathrm{u}}((0,x)) \oplus T_{(1/2,x)}W^{\mathrm{s}}((1,x)) = \mathbb{R} \oplus T_xM = T_{(1/2,x)}\widetilde{M},$$

proving transversality.

The vector field $-\nabla_{\tilde{g}}\tilde{f}$ need not satisfy the Morse – Smale condition up to order 1, but the only points where transversality can fail are the intersections of the unstable manifold of a critical point $(0,x)$ with the stable manifold of a critical point $(1,y)$, with $x \neq y$ critical points of f. We can perturb the metric $\tilde{g}$ in order to achieve the Morse – Smale property up to order 1 without loosing the nice features (i) – (iv) of the vector field $-\nabla_{\tilde{g}}\tilde{f}$. More precisely, by Theorem 2.20, taking into account (46), we can find a complete metric g on $\widetilde{M}$ such that

(a) $(\tilde{f}, g)$ satisfies (PS);

(b) g coincides with $\tilde{g}$ on the sets $]-\infty, \frac{1}{3}] \times M$, $[\frac{2}{3}, +\infty[\times M$, and $\mathbb{R} \times U$, where $U \subset M$ is a neighborhood of $\operatorname{crit}(f)$;

(c) $Df(s,p)[-\nabla_g\tilde{f}(s,p)] < 0$ if $p \notin \operatorname{crit}(f)$;

(d) $(\tilde{f}, g)$ satisfies the Morse – Smale property up to order 1.

Indeed, the function θ appearing in the statement of Theorem 2.20 can be chosen to vanish on the regions indicated in (b), where the intersections are already transverse, and to be so small that the metrics belonging to the space of perturbations satisfy (c). By property (b), the flow of $-\nabla_g \tilde{f}$ still satisfies (i), (ii), (iii). By (c), it satisfies also (iv).

We can now consider the Morse complex of $(\tilde{f}, g)$ relative to the sublevel $\{\tilde{f} < \inf f - 1\}$. Notice that this sublevel contains no critical points. The boundary operator $\partial_k(\tilde{f}, g)$ can be described by using Theorem 2.11 and Remark 2.13. To this purpose, it is convenient to choose the orientations of the unstable manifolds in the following way: since for every $x \in \operatorname{crit}(f)$ there is a privileged isomorphism

$$T_x W^{\mathrm{u}}(x; -\nabla_{g_0} f) \cong T_x W^{\mathrm{u}}(x; -\nabla_{g_1} f),$$

namely the restriction to the first space of the projection onto the first factor in the splitting

$$T_x M = T_x W^{\mathrm{u}}(x; -\nabla_{g_1} f) \oplus T_x W^{\mathrm{s}}(x; -\nabla_{g_1} f),$$

we can endow these two spaces with orientations which are compatible with this isomorphism. Then

$$T_{(0,x)} W^{\mathrm{u}}((0,x); -\nabla_g \tilde{f}) = \mathbb{R} \oplus T_x W^{\mathrm{u}}(x; -\nabla_{g_0} f)$$

and

$$T_{(1,x)} W^{\mathrm{u}}((1,x); -\nabla_g \tilde{f}) = (0) \oplus T_x W^{\mathrm{u}}(x; -\nabla_{g_1} f)$$

can be given the product orientations by the the standard orientations of $\mathbb{R}$ and (0). In this way, we have chosen an orientation for the unstable manifold of each critical point of $\tilde{f}$. With this choice the transverse intersection

$$W^{\mathrm{u}}((0,x)) \cap W^{\mathrm{s}}((1,x)) =]0,1[\times \{x\} \tag{48}$$

is given the orientation corresponding to the vector $\partial/\partial s$, which agrees with the direction of the flow.

By (i), (ii), (iii), and (45) the boundary operator

$$\partial_k(\tilde{f}, g) \colon C_{k-1}(f) \oplus C_k(f) \to C_{k-2}(f) \oplus C_{k-1}(f)$$

can be written as

$$\partial_k(\tilde{f}, g) = \begin{pmatrix} \partial_{k-1}(f, g_0) & 0 \\ \Phi_{k-1} & \partial_k(f, g_1) \end{pmatrix},$$

for some homomorphism

$$\Phi_k \colon C_k(f) \to C_k(f).$$

The fact that $\partial_*(\tilde{f}, g)$ is a boundary, i.e., $\partial_k(\tilde{f}, g)\, \partial_{k+1}(\tilde{f}, g) = 0$, implies that

$$\Phi_{k-1} \partial_k(f, g_0) = \partial_k(f, g_1) \Phi_k,$$

that is $(\Phi_k)_{k\in\mathbb{N}}$ is a chain homomorphism from the Morse complex of (f, g_0) to the Morse complex of (f, g_1).

By (iv), the intersection

$$W^{\mathrm{u}}((0,x); -\nabla_g \tilde{f}) \cap W^{\mathrm{s}}((1,y); -\nabla_g \tilde{f})$$

can be nonempty only if $f(y) < f(x)$ or $x = y$, and in the latter case it consists of the single orbit (48).

Together with the previous discussion on orientations, this fact implies that Φ has the form (43). So if we order the critical points of f with Morse index k by increasing value of f, we see that the homomorphism Φ_k is represented by an upper-triangular matrix, with 1 on the diagonal entries. A homomorphism of this form must be an isomorphism: this is well known when $C_k(f)$ has finite rank, because in this case Φ_k is represented by a finite matrix with determinant 1, an invertible element of $\mathbb{Z}$, but it remains true if the rank of $C_k(f)$ is infinite. Indeed if $x_1, x_2, \dots$, are the critical points of index k ordered by increasing value of f, the inverse of Φ_k is defined inductively by

$$\Phi_k^{-1} x_1 = x_1, \quad \Phi_k^{-1} x_h = x_h - \sum_{i=1}^{h-1} n(x_h, x_i) \Phi_k^{-1} x_i, \ \forall h \geq 2. \qquad \square$$

EXERCISE 2.28. Generalize this result to the case of a strip $\{a < f < b\}$.

EXERCISE 2.29. When f satisfies the condition (A8), it is possible to obtain the same conclusion of Theorem 2.26 by looking directly at the two cellular filtrations induced by the two negative gradient flows. Prove this fact. Then use the limit arguments of Section 2.9 to prove Theorem 2.26 under the hypothesis that (f, g_0) and (f, g_1) satisfy the Morse – Smale condition only up to order 0.

3. The Morse complex in the case of infinite Morse indices

3.1. THE PROGRAM

In this part we will consider a gradient-like C^1 vector field X on a Hilbert manifold M, whose rest points have infinite Morse index and co-index. In this case, the stable and the unstable manifolds of rest points are infinite-dimensional, and the flow of X does not produce a meaningful cellular filtration of M. Indeed, the infinite-dimensional Hilbert ball is retractable onto its boundary, so the rest points of X are homotopically invisible.

However, we may hope that in some cases the unstable and the stable manifolds of pairs of rest points have finite-dimensional intersections. If this holds, we could use the formula for the boundary operator of Theorem 2.11 not as a description, but rather as the definition of the Morse complex. Our program is to follow this idea.

Of course this program cannot be pursued in full generality. A first reason is that in general the unstable and stable manifolds may not have finite-dimensional intersections. A deeper reason is that the setting of gradient-like flows for a Morse function with critical points of infinite Morse index and co-index has too little rigidity. For instance, the following result was proved in Abbondandolo and Majer (2004). A sketch of the proof will be presented at the end of Section 3.3.

THEOREM 3.1. *Let $f\colon M \to \mathbb{R}$ be a smooth Morse function on a separable Hilbert manifold, whose critical points have infinite Morse index and co-index. Let $a\colon \mathrm{crit}(f) \to \mathbb{Z}$ be an arbitrary function. Then there exists a Riemannian metric g on M such that the corresponding negative gradient flow of f has the following property: for every pair of critical points x, y, the intersection $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is transverse and — if nonempty — has dimension $a(x) - a(y)$.*

Moreover, the metric g can be chosen to be uniformly equivalent to any given metric g_0 on M. Finally, if (x_i, y_i), $i = 1, \dots, k$, are pairs of critical points such that x_i and y_i can be connected by a path $u_i\colon [0,1] \to M$ such that $Df(u_i(t))[u_i'(t)]$ is negative for every $t \in\,]0,1[$, the metric g can be chosen in such a way that $W^{\mathrm{u}}(x_i) \cap W^{\mathrm{s}}(y_i)$ is not empty.

Therefore the situation is drastically less rigid than the case of finite Morse indices, where the Morse index of a critical point does not involve the metric, and where we have seen that the isomorphism class of the Morse complex does not depend on the metric, and that its homology does not even depend on f.

Let us examine another example of the lack of rigidity determined by infinite Morse indices and co-indices. We have seen that when the Morse indices are finite, the transverse intersection $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is always orientable, and each of its components has the same dimension $m(x) - m(y)$. On the other hand, if Z is any separable Hilbert manifold (finite-dimensional or not, possibly with components of different dimension), there exists a smooth gradient-like flow on the Hilbert space H with exactly two rest points x and y, such that the intersection $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is transverse and diffeomorphic to $Z \times \mathbb{R}$ (see Abbondandolo and Majer, 2003b, Section 4).

These phenomena suggest that a Morse theory for functions $f : M \to \mathbb{R}$ with critical points of infinite Morse index and co-index requires more structure than just the pair (M, f). Our choice will be to consider a subbundle $\mathcal{V}$ of TM, suitably compatible with the gradient-like flow.

3.2. FREDHOLM PAIRS AND COMPACT PERTURBATIONS OF LINEAR SUBSPACES

Before proceeding, we need to review some facts about the *Hilbert Grassmannian* $\mathrm{Gr}(H)$, the set of all closed linear subspaces of the separable Hilbert space H. See Abbondandolo and Majer (2003a) for a more complete presentation. If $V \in \mathrm{Gr}(H)$, we shall denote by P_V the orthogonal projection onto V. The set $\mathrm{Gr}(H)$ is a complete metric space with the distance $\mathrm{dist}(V, W) := \|P_V - P_W\|$. The connected components of $\mathrm{Gr}(H)$ are the subspaces

$$\mathrm{Gr}_{n,m}(H) = \{V \in \mathrm{Gr}(H) \mid \dim V = n, \operatorname{codim} V = m\}, \qquad \text{where } n, m \in \mathbb{N} \cup \{\infty\},\ n + m = \infty.$$

A pair $(V, W) \in \mathrm{Gr}(H)\times\mathrm{Gr}(H)$ is a *Fredholm pair* if $V\cap W$ is finite-dimensional and $V + W$ is finite-codimensional. In this case, the number $\mathrm{ind}(V, W) := \dim V \cap W - \operatorname{codim}(V + W)$ is said the *Fredholm index of* (V, W). The space of Fredholm pairs, denoted by $\mathrm{Fp}(H)$, is an open subset of $\mathrm{Gr}(H) \times \mathrm{Gr}(H)$, and the Fredholm index is a continuous (i.e., locally constant) function on it. See for instance Kato (1980, IV §4).

Let $W \in \mathrm{Gr}(H)$. A closed linear subspace V is a *compact perturbation of* W if the operator $P_V - P_W$ is compact. In this case, the pair $(V, W^\perp)$ is Fredholm, and its index is said the *relative dimension of* V *with respect to* W, denoted by

$$\dim(V, W) := \mathrm{ind}(V, W^\perp) = \dim V \cap W^\perp - \dim V^\perp \cap W.$$

If (V, W) is a Fredholm pair and Z is a compact perturbation of V, then (Z, W) is still a Fredholm pair, and its index is

$$\mathrm{ind}(Z, W) = \mathrm{ind}(V, W) + \dim(Z, V). \tag{49}$$

3.3. FINITE-DIMENSIONAL INTERSECTIONS

Let M be a smooth Hilbert manifold, and let X be a C^1 Morse vector field on M, with local flow $\phi\colon \Omega(X) \to M$. We shall always assume that X has a Lyapunov function f. In view of Remark 1.21(ii), we shall assume that $f \in C^2(M)$ and that it is a *nondegenerate Lyapunov function*, meaning that for every $x \in \mathrm{rest}(X)$ the quadratic form $\xi \mapsto D^2 f(x)[\xi, \xi]$ is coercive on $E^{\mathrm{s}}(\nabla X(x))$, while $\xi \mapsto -D^2 f(x)[\xi, \xi]$ is coercive on $E^{\mathrm{u}}(\nabla X(x))$.

Let $\mathcal{V}$ be a smooth subbundle of TM, and let $\mathcal{P}$ be a projector onto $\mathcal{V}$: $\mathcal{P}$ is a smooth section of the bundle of endomorphisms of TM such that for every $p \in M$, $\mathcal{P}(p) \in \mathcal{L}(T_p M)$ is a projector onto $\mathcal{V}(p)$. We shall assume the following compatibility conditions between X and $\mathcal{V}$:

(C1) for every $x \in \mathrm{rest}(X)$, the positive eigenspace $E^{\mathrm{u}}(\nabla X(x))$ of the Jacobian of X at x is a compact perturbation of $\mathcal{V}(x)$;

(C2) for every $p \in M$, the operator $(L_X\mathcal{P})(p)\mathcal{P}(p)$ is compact.

Here $L_X\mathcal{P}$ denotes the Lie derivative of $\mathcal{P}$ along X. By (C1), we can define the *relative Morse index of the rest point x with respect to $\mathcal{V}$* to be the integer

$$m(x, \mathcal{V}) := \dim(E^{\mathrm{u}}(\nabla X(x)), \mathcal{V}(x)).$$

Condition (C2) depends only on the subbundle $\mathcal{V}$, and not on the choice of the projector $\mathcal{P}$ onto it. Notice that the subbundle $\mathcal{V}$ is ϕ-invariant (in the sense that $D_2\phi(t, p)\mathcal{V}(p) = \mathcal{V}(\phi(t, p))$ for every $(t, p) \in \Omega(X)$) if and only if $(L_X\mathcal{P})\mathcal{P} = 0$. Condition (C2) is equivalent to the fact that $\mathcal{V}$ is *ϕ-essentially invariant*: $D_2\phi(t, p)\mathcal{V}(p)$ is a compact perturbation of $\mathcal{V}(\phi(t, p))$, for every $(t, p) \in \Omega(X)$. When M is an open subset of the Hilbert space H, and $\mathcal{V}$ is a constant subbundle $\mathcal{V} \equiv V \in \mathrm{Gr}(H)$, so that we can choose $\mathcal{P} \equiv P_V$, there holds

$$(L_X\mathcal{P})\mathcal{P} = [DX, P_V]P_V = (I - P_V)DXP_V. \tag{50}$$

These assumptions have the following consequence.

PROPOSITION 3.2. *Assume that the Morse vector field X satisfies* (C1) *and* (C2) *with respect to the subbundle $\mathcal{V}$. Then for every $x \in \mathrm{rest}(X)$:*

(i) *for every $p \in W^{\mathrm{u}}(x)$, $T_pW^{\mathrm{u}}(x)$ is a compact perturbation of $\mathcal{V}(p)$, with*

$$\dim(T_pW^{\mathrm{u}}(x), \mathcal{V}(p)) = m(x, \mathcal{V});$$

(ii) *for every $p \in W^{\mathrm{s}}(x)$, the pair $(T_pW^{\mathrm{s}}(x), \mathcal{V}(p))$ is Fredholm, with*

$$\mathrm{ind}(T_pW^{\mathrm{s}}(x), \mathcal{V}(p)) = -m(x, \mathcal{V}).$$

So loosely speaking, $W^{\mathrm{u}}(x)$ is essentially parallel to $\mathcal{V}$, while $W^{\mathrm{s}}(x)$ is essentially normal to $\mathcal{V}$.

Let us sketch the proof of the first claim in a simpler case: we assume that M is an open set of the Hilbert space H, and that $\mathcal{V} \equiv V \in \mathrm{Gr}(H)$ is a constant subbundle. Let $p \in W^{\mathrm{u}}(x)$, and let $u(t) := \phi(t, p)$ be the orbit of p. By linearization along u, using the notation of Section 1.2, we have that

$$T_pW^{\mathrm{u}}(x) = W^{\mathrm{u}}_A, \tag{51}$$

where $A(t) := DX(u(t))$. By (C1), $W := T_xW^{\mathrm{u}}(x) = E^{\mathrm{u}}(A(-\infty))$ is a compact perturbation of V. By (C2), the operator $[A(t), P_V]P_V$ is compact for every t, and so is the operator $[A(t), P_W]P_W$. Set

$$B(t) := A(t) - [A(t), P_W]P_W,$$

so that $B(-\infty) = A(-\infty) = DX(x)$, $E^{\mathrm{u}}(B(-\infty)) = W$, and $B(t)W \subset W$ for every t. These facts easily imply that $W^{\mathrm{u}}_B = W$. On the other hand, since $A(t) - B(t)$ is compact for every t, W^{u}_A is a compact perturbation of $W^{\mathrm{u}}_B = W$, hence of V. By (51), $T_p W^{\mathrm{u}}(x)$ is a compact perturbation of V. The formula for its relative dimension with respect to V follows by continuity.

The proof of claim (ii) is simpler. Since the set of Fredholm pairs is open and the index is locally constant, by (C1) the pair $(T_p W^{\mathrm{s}}(x), \mathcal{V}(p))$ is Fredholm of index $-m(x,\mathcal{V})$ for every $p \in W^{\mathrm{s}}(x)$ in a neighborhood of x. The tangent bundle $TW^{\mathrm{s}}(x)$ is ϕ-invariant, and by (C2) the subbundle $\mathcal{V}$ is ϕ-essentially invariant, so these facts remain true for every $p \in W^{\mathrm{s}}(x)$.

By (49), Proposition 3.2 has the following easy corollary.

COROLLARY 3.3. *Assume that the Morse vector field X satisfies* (C1) *and* (C2) *with respect to the subbundle $\mathcal{V}$. Let $x, y \in$* rest(X), *and assume that $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(y)$ meet transversally. Then $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is a submanifold of dimension $m(x,\mathcal{V}) - m(y,\mathcal{V})$.*

We conclude this section by sketching the proof of Theorem 3.1. By the already mentioned embedding theorem of Eells and Elworthy (1970), we can embed M as an open subset of the separable Hilbert space H. By modifying this embedding near the critical points of f, and by using the Morse Lemma (see for instance Palais, 1963), we may assume that f is quadratic near every critical point x:

$$f(x+\xi) = f(x) + \tfrac{1}{2}\langle A(x)\xi, \xi\rangle, \quad \text{for } |\xi| \text{ small},$$

for some self-adjoint invertible operator $A(x)$. Fix a closed linear subspace V of H, with infinite dimension and codimension. By a further modification of the embedding, we may also rotate small neighborhoods of the critical points in such a way that the negative eigenspace $E^{\mathrm{s}}(A(x))$ of the operator $A(x)$ is a compact perturbation of V, of relative dimension $a(x)$. Here we actually need to use Kuiper's theorem (Kuiper, 1965), stating the orthogonal group of H is contractible.

It is now easy to build a vector field X having f as a nondegenerate Lyapunov function, and which satisfies (C1) and (C2) with respect to the constant subbundle V. Indeed, near a critical point x one may choose X to be the linear vector field

$$X(x+\xi) = -\nabla f(x+\xi) = -A(x)\xi, \quad \text{for } |\xi| \text{ small}. \tag{52}$$

Since the negative eigenspace of $A(x)$ is a compact perturbation of V of relative dimension $a(x)$, X satisfies (C1) and x has relative Morse index $m(x,V) = a(x)$. The linear vector field X satisfies also (C2). Indeed by (50),

$$(L_X P_V)P_V = -(I - P_V)A(x)P_V = -P_{V^\perp}P_{E^{\mathrm{s}}(A(x))}A(x)P_V - P_{V^\perp}A(x)P_{E^{\mathrm{u}}(A(x))}P_V,$$

and the operators $P_{V^\perp}P_{E^{\mathrm{s}}(A(x))}$ and $P_{E^{\mathrm{u}}(A(x))}P_V$ are compact because $E^{\mathrm{s}}(A(x))$ is a compact perturbation of V. If $p \in M$ is not a critical point, we may choose

X to be the constant vector field $X(p+\xi) = -\nabla f(p)$, for every ξ so small that $Df(p+\xi)[-\nabla f(p)] < 0$. Every constant vector field trivially satisfies (C2) with respect to the constant subbundle V.

These local definitions of X can be patched together by a smooth partition of unity. In this way one can build a vector field X satisfying (52) near critical points, so that (C1) holds. The set of vector fields satisfying condition (C2) is a module over the ring of real functions, so X satisfies (C2). Having f as a Lyapunov function is a convex condition, so f is a Lyapunov function for X. Up to a small perturbation, we may assume that X also satisfies the Morse – Smale condition. Then Corollary 3.3 implies that $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is a submanifold of dimension $m(x,V) - m(y,V) = a(x) - a(y)$. The fact that X is actually the negative gradient of f near the critical points makes it possible to find a metric g on M such that $X = -\nabla_g f$.

We refer to Abbondandolo and Majer (2004) for details on how to keep g uniformly equivalent to a given metric, and on how to obtain that $W^{\mathrm{u}}(x_i) \cap W^{\mathrm{s}}(y_i)$ is nonempty for every $i = 1, \dots, k$.

3.4. ESSENTIAL SUBBUNDLES

It is readily seen that if X satisfies (C1) and (C2) with respect to a subbundle $\mathcal{V}$, then it satisfies (C1) and (C2) also with respect to a subbundle $\mathcal{W}$ which at every point is a compact perturbation of $\mathcal{V}$. This fact suggests the possibility of weakening the structure, fixing only an *essential subbundle* of TM.

In order to make this precise, we need to introduce the essential Grassmannians of a Hilbert space. See again Abbondandolo and Majer (2003b) for a complete discussion. The *essential Grassmannian of H* is the quotient of $\mathrm{Gr}(H)$ by the equivalence relation

$$\{(V,W) \in \mathrm{Gr}(H) \times \mathrm{Gr}(H) \mid V \text{ is a compact perturbation of } W\},$$

and it is denoted by $\mathrm{Gr}_{\mathrm{e}}(H)$. This space can also be seen as the space of symmetric projectors in the Calkin algebra $\mathcal{L}(H)/\mathcal{L}_c(H)$ ($\mathcal{L}_c(H)$ denotes the closed ideal of compact operators). Notice that the finite-dimensional and the finite-codimensional spaces represent two points in $\mathrm{Gr}_{\mathrm{e}}(H)$. We shall actually be interested in the complementary $\mathrm{Gr}^*_{\mathrm{e}}(H)$ of these two points, that is in the quotient of $\mathrm{Gr}_{\infty,\infty}(H)$.

The (0)-*essential Grassmannian* $\mathrm{Gr}_{(0)}(H)$ is the quotient of $\mathrm{Gr}(H)$ by the stronger equivalence relation

$$\{(V,W) \in \mathrm{Gr}(H) \times \mathrm{Gr}(H) \mid \mathrm{V} \text{ is a compact perturbation of } W \text{ and } \dim(V,W) = 0\}.$$

Again, $\mathrm{Gr}^*_{(0)}(H)$ denotes the quotient of $\mathrm{Gr}_{\infty,\infty}(H)$. The Bott periodicity theorem (see Bott, 1959), and the fact that the group of automorphisms of H which are compact perturbations of the identity is homotopy equivalent to the infinite general linear group $\mathrm{GL}(\infty) = \varinjlim \mathrm{GL}(n)$ (see Palais, 1965), allow to determine the homotopy type of the essential Grassmannian, proving the following result.

THEOREM 3.4. *The quotient projection* $\mathrm{Gr}_{\infty,\infty}(H) \to \mathrm{Gr}^*_{(0)}(H)$ *is a fiber bundle with contractible total space. The quotient projection* $\mathrm{Gr}^*_{(0)}(H) \to \mathrm{Gr}^*_{\mathrm{e}}(H)$ *is a universal covering. The space* $\mathrm{Gr}^*_e(H)$ *is path connected, its fundamental group is infinite cyclic, and if* $i \geq 2$,

$$\pi_i(\mathrm{Gr}^*_{\mathrm{e}}(H)) \cong \pi_{i-2}(\mathrm{GL}(\infty)) = \begin{cases} \mathbb{Z} & \text{if } i \equiv 1, 5 \bmod 8, \\ \mathbb{Z}_2 & \text{if } i \equiv 2, 3 \bmod 8, \\ 0 & \text{if } i \equiv 0, 4, 6, 7 \bmod 8. \end{cases}$$

Since the tangent bundle of an infinite-dimensional Hilbert manifold is always trivial (by the already mentioned Kuiper's theorem; Kuiper, 1965), a subbundle $\mathcal{V}$ of TM can be identified with a map $\mathcal{V}\colon M \to \mathrm{Gr}(H)$. Similarly, an *essential subbundle* (respectively a (0)-*essential subbundle*) of TM can be identified with a map $\mathcal{E}\colon M \to \mathrm{Gr}_{\mathrm{e}}(H)$ (resp. $\mathcal{E}\colon M \to \mathrm{Gr}_{(0)}(H)$).

By Theorem 3.4, an essential subbundle $\mathcal{E}$ of TM can be lifted to a (0)-essential subbundle if and only if the homomorphism

$$\mathcal{E}_*\colon \pi_1(M) \to \pi_1(\mathrm{Gr}^*_{\mathrm{e}}(H)) = \mathbb{Z}$$

vanishes. A (0)-essential subbundle $\mathcal{E}$ of TM can be lifted to a true subbundle of TM if and only if all the homomorphisms

$$\mathcal{E}_*\colon \pi_i(M) \to \pi_i(\mathrm{Gr}^*_{(0)}(H))$$

vanish (a condition which has to be checked only for $i \equiv 1, 2, 3, 5 \bmod 8$).

If the vector field X satisfies (C1) and (C2) with respect to a (0)-essential subbundle $\mathcal{E}$ of TM, then the relative Morse index $m(x, \mathcal{E})$ can still be defined, and the conclusions of Proposition 3.2 and of Corollary 3.3 still hold (with the obvious changes).

If the vector field X satisfies (C1) and (C2) with respect to an essential subbundle, there is no relative Morse index. In this case the transverse intersection $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is finite-dimensional, but different components may have different dimension. More precisely, the dimension of the connected component containing p depends on the homotopy class of the orbit of p, seen as a curve from $(\overline{\mathbb{R}}, -\infty, +\infty)$ into (M, x, y). It is actually possible to construct an example of a gradient-like vector field on $S^1 \times H$, which satisfies (C1) and (C2) with respect to a nonliftable essential subbundle, and has two critical points x and y such that the intersection $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is transverse and consists of two connected components of different dimension.

3.5. ORIENTATIONS

We recall that in the case of finite Morse indices, an arbitrary choice of the orientation of all the unstable manifolds — or equivalently of the finite-dimensional spaces $T_x W^{\mathrm{u}}(x)$ — determines an orientation of each transverse intersection of unstable and stable manifolds. Now $T_x W^{\mathrm{u}}(x)$ is infinite-dimensional, so it does not carry orientations. The right object to orient turns out to be the Fredholm pair $(T_x W^{\mathrm{s}}(x), \mathcal{V}(x))$.

In order to deal with this question, we need to introduce the *determinant bundle*

$$\mathrm{Det}(\mathrm{Fp}(H)) \to \mathrm{Fp}(H)$$

on the space of Fredholm pairs (see Abbondandolo and Majer, 2003b, for more details). It is a real line bundle, whose fiber at $(V, W) \in \mathrm{Fp}(H)$ is

$$\mathrm{Det}(V, W) := \mathrm{Det}(V \cap W) \otimes \Big(\mathrm{Det}(H/(V + W))\Big)^*,$$

where $\mathrm{Det}(Z) := \Lambda^{\dim Z}(Z)$ denotes the space of top degree in the exterior algebra of the finite-dimensional vector space Z. Defining a bundle structure for this object is not immediate, because the maps $(V, W) \mapsto V \cap W$ and $(V, W) \mapsto V + W$ are not continuous. We just mention the key ingredients in the constructions. The intersection map $(V, W) \mapsto V \cap W$ is continuous on the space of transverse pairs, while the sum $(V, W) \mapsto V + W$ is continuous on the space of pairs with intersection (0). Then the bundle structure near a Fredholm pair (V_0, W_0) can be constructed by fixing a finite-dimensional space Z such that $Z + V_0 + W_0 = H$ and $Z \cap V_0 = (0)$, and by replacing each pair (V, W) in a neighborhood of (V_0, W_0) by $(Z + V, W)$. Such a replacement turns out to be possible because of the existence of an exact sequence

$$0 \to V \cap W \to (Z + V) \cap W \to Z \to \frac{H}{V + W} \to 0.$$

We recall that an exact sequence of finite-dimensional vector spaces

$$0 \to Z_1 \to \cdots \to Z_k \to 0$$

induces a natural isomorphism

$$\bigotimes_{i \text{ odd}} \mathrm{Det}(Z_i) \cong \bigotimes_{i \text{ even}} \mathrm{Det}(Z_i).$$

The space of Fredholm operators from H_1 to H_2, denoted by $\mathcal{F}(H_1, H_2)$ is "contained" in the space of Fredholm pairs of $H_1 \times H_2$. Indeed, the operator $A \in \mathcal{L}(H_1, H_2)$ is Fredholm if and only if the pair $(\mathrm{graph}\, A, H_1 \times (0)) \in \mathrm{Gr}(H_1 \times H_2) \times \mathrm{Gr}(H_1 \times H_2)$ is Fredholm, and the index is the same. The pullback of the determinant bundle on $\mathrm{Fp}(H)$ by the map

$$\mathcal{F}(H_1, H_2) \to \mathrm{Fp}(H_1 \times H_2), \quad A \mapsto (\mathrm{graph}\, A, H_1 \times (0)),$$

is the *determinant bundle on the space of Fredholm operators*, as defined by Quillen (1985).

Let $n \in \mathbb{N}$, and let

$$\mathrm{Det}(\mathrm{Gr}_{n,\infty}(H)) \to \mathrm{Gr}_{n,\infty}(H)$$

be the real line bundle whose fiber at $Z \in \mathrm{Gr}_{n,\infty}(H)$ is $\mathrm{Det}(Z)$. Let $\mathcal{S}$ be the set of all $(Z,(V,W))$ in

$$\Big(\bigcup_{n\in\mathbb{N}} \mathrm{Gr}_{n,\infty}(H)\Big) \times \mathrm{Fp}(H)$$

such that $Z \cap V = (0)$, and let $\mathrm{Det}(\mathcal{S}) \to \mathcal{S}$ be the restriction to $\mathcal{S}$ of the tensor product of the bundles $\bigcup_{n\in\mathbb{N}} \mathrm{Det}(\mathrm{Gr}_{n,\infty}(H))$ and $\mathrm{Det}(\mathrm{Fp}(H))$. The map

$$\mathcal{S} \to \mathrm{Fp}(H), \quad (Z,(V,W)) \mapsto (Z+V,W),$$

is continuous, and can be lifted to a continuous morphism between the corresponding determinant bundles:

$$S : \mathrm{Det}(\mathcal{S}) \to \mathrm{Det}(\mathrm{Fp}(H)).$$

The construction of such a morphism is based on the exact sequence

$$0 \to V \cap W \to (Z+V)\cap W \to \frac{Z+V}{V} \cong Z \to \frac{H}{V+W} \to \frac{H}{Z+V+W} \to 0.$$

The morphism S is associative, meaning that if Z and Y are finite-dimensional linear subspaces of H such that $Z \cap Y = (Z+Y)\cap V = (0)$, the diagram

$$\begin{tikzcd} \mathrm{Det}(Y)\otimes\mathrm{Det}(Z)\otimes\mathrm{Det}(V,W) \arrow[r, "\mathrm{id}\otimes S"] \arrow[d, "S\otimes\mathrm{id}"'] & \mathrm{Det}(Y)\otimes\mathrm{Det}(Z+V,W) \arrow[d, "S"] \\ \mathrm{Det}(Y+Z)\otimes\mathrm{Det}(V,W) \arrow[r, "S"] & \mathrm{Det}(Y+Z+V,W) \end{tikzcd}$$

commutes.

An orientation of a finite-dimensional space Z can be defined as an orientation of the line $\mathrm{Det}(Z)$; similarly, an *orientation of the Fredholm pair* (V,W) is an orientation of the line $\mathrm{Det}(V,W)$. The morphism S allows to sum orientations: if $(Z,(V,W)) \in \mathcal{S}$, the orientations of two objects among

$$Z, \quad (V,W), \quad (Z+V,W),$$

determines an orientation of the other object.

Let us go back to the question of orienting the intersections between unstable and stable manifolds. The assumption is that the vector field X satisfies (C1) and (C2) with respect to a subbundle $\mathcal{V}$ of TM. By assumption (C1), the pair

$(T_x W^s(x), \mathcal{V}(x))$ is Fredholm, for every $x \in \mathrm{rest}(X)$. Let us choose an orientation $o(x)$ of such a Fredholm pair, for every rest point x, in an arbitrary way.

Now let x, y be rest points such that $W^u(x)$ and $W^s(y)$ have transversal intersection. Let $p \in W^u(x) \cap W^s(y)$. By Proposition 3.2(ii), the pair $(T_p W^s(y), \mathcal{V}(p))$ is Fredholm. Choose a closed complement V of $T_p(W^u(x) \cap W^s(y))$ in $T_p W^s(y)$. By transversality, V is also a complement of $T_p W^u(x)$ in $T_p M$. It is a general fact in this case that the backward evolution of V with respect to the differential of the flow converges to $T_x W^s(x)$:

$$\lim_{t \to -\infty} D_2\phi(t, p)V = T_x W^s(x).$$

Therefore, the Fredholm pair $(V, \mathcal{V}(p))$ inherits by continuity an orientation from the orientation $o(x)$ of $(T_x W^s(x), \mathcal{V}(x))$. On the other hand, the Fredholm pair $(T_p W^s(y), \mathcal{V}(p))$ inherits an orientation from the orientation $o(y)$ of $(T_y W^s(y), \mathcal{V}(y))$. The last two objects among

$$T_p(W^u(x) \cap W^s(y)), \quad (V, \mathcal{V}(p)),$$
$$(T_p(W^u(x) \cap W^s(y)) + V, \mathcal{V}(p)) = (T_p W^s(y), \mathcal{V}(p))$$

are then oriented, so they induce an orientation of the first space. The construction continuously depends on p, hence it determines an orientation of $W^u(x) \cap W^s(y)$. We shall see in Section 3.7 that the orientations defined here satisfy a suitable coherence property.

3.6. COMPACTNESS

In the case of finite Morse indices, we have seen that the (PS) condition together with the positive completeness of X implies that $W^u(x) \cap \{f \geq a\}$ is precompact. Now the unstable manifold is infinite-dimensional, so this cannot be true, but we can hope $W^u(x) \cap W^s(y)$ to be precompact. However, assumptions (C1) and (C2) are not sufficient to get this result: $W^u(x) \cap W^s(y)$ may consist, for instance, of infinitely many flow lines going from x to y, with no cluster points besides x and y. We need to strengthen condition (C2), a local assumption, into a more global condition.

We recall that that the Hausdorff distance of two subsets A, B of a complete metric space (W, d) is the number

$$\mathrm{dist}_{\mathcal{H}}(A, B) := \max\left\{\sup_{a\in A} \inf_{b\in B} d(a,b), \sup_{b\in B} \inf_{a\in A} d(a,b)\right\} \in [0, +\infty],$$

and that the *Hausdorff measure of noncompactness of A* is the number

$$\beta_W(A) := \inf\{r > 0 \mid A \text{ can be covered by finitely many balls of radius } r\} \in [0, +\infty],$$

so that A is precompact if and only if $\beta_W(A) = 0$. The function β is continuous with respect to the Hausdorff distance. Moreover, $\beta_W(A)$ coincides with the Hausdorff distance of A from the space of compact subsets of W:

$$\beta_W(A) = \inf\{\mathrm{dist}_{\mathcal{H}}(A, K) \mid K \subset W \text{ compact}\}. \tag{53}$$

In the case of a normed vector space W, β has also the following properties:

$$\beta_W(\lambda A) = |\lambda|\beta_W(A), \quad \beta_W(A + B) \leq \beta_W(A) + \beta_W(B),$$
$$\beta_W(\mathrm{conv}(A)) = \beta_W(A). \tag{54}$$

Let $\mathcal{E}$ be an essential subbundle of TM, different from the trivial essential subbundles $[(0)]$ and $[TM]$. We shall assume that $\mathcal{E}$ admits a *global presentation*: there exists a smooth map $Q : M \to N$ into a Hilbert manifold such that for every $p \in M$, $DQ(p)$ has finite-codimensional range, and $\ker DQ(p)$ belongs to the equivalence class $\mathcal{E}(p)$. For instance, $\mathcal{E}$ could be the equivalence class of a subbundle which is the vertical space of a submersion Q.

We shall assume that N is endowed with a complete Riemannian metric, and we shall consider the induced metric on TN. The new assumption on the vector field X is:

(C3) (i) $\|DQ \circ X\|_\infty < +\infty$;

(ii) for every $q \in N$ there exists $\delta = \delta(q) > 0$ and $c = c(q) \geq 0$ such that

$$\beta_{TN}\Big(DQ(X(A))\Big) \leq c\,\beta_N(Q(A)) \quad \forall A \subset Q^{-1}(B_\delta(q)).$$

Let us restate this condition in a simple situation: assume that M is an open set of the Hilbert space H, and that $\mathcal{E}$ is the equivalence class of a constant subbundle $V \in \mathrm{Gr}(H)$. Then we can choose the global presentation to be the orthogonal projector onto $W := V^\perp$, $Q := P_W$. Denote by (X_V, X_W) be the two components of X with respect to the orthogonal splitting $H = V \oplus W$. Condition (C3)(i) says that X_W is bounded, while (C3)(ii) is equivalent to: for every $\xi \in W$ there exist $\delta > 0$ and $c \geq 0$ such that

$$\beta_W(X_W(A)) \leq c\,\beta_W(P_W A) \quad \forall A \subset M \cap (V \times (B_\delta(\xi) \cap W)). \tag{55}$$

In particular, if $A \subset M$ is such that $P_W A$ is precompact, then also $X_W(A) = P_W X(A)$ is required to be precompact. Thus, for every $\xi \in M$ the map $\eta \mapsto (I - P_V)X(\xi + P_V\eta)$ is a compact map in a neighborhood of 0. Therefore, the differential of this map at 0, namely

$$(I - P_V)DX(\xi)P_V = (L_X P_V)(\xi)P_V$$

is compact. Hence (C3) implies (C2): the simple situation — $M \subset H$, $\mathcal{E}$ constant, Q projector — in which we have checked this fact is indeed the general local situation, and (C2) is a local assumption.

Notice that in general (55) is strictly stronger than the fact that for every $\xi \in W$, X_W should map $(\xi + V) \cap M$ into a precompact set, because (55) involves a Lipschitz control on the measure of noncompactness. However, these conditions are equivalent under a mild Lipschitz assumption on X. See Abbondandolo and Majer (2003b, Proposition 7.9) for a precise statement (in the case of a general map Q). The main result of this section is the following compactness theorem.

THEOREM 3.5. *Let $\mathcal{E}$ be an essential subbundle of TM with a global presentation $Q\colon M \to N$ into a complete Riemannian Hilbert manifold. Assume that the Morse vector field X is complete, has a nondegenerate Lyapunov function f, (X, f) satisfies* (PS). *Assume also that X satisfies* (C1)–(C3). *Then for every pair of critical points x, y, the intersection $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is precompact.*

Let us sketch the proof. It is useful to introduce the following notion: a subset $A \subset M$ is said *essentially vertical* if $Q(A)$ is precompact. The proof is then based on the following steps:

(i) if A is essentially vertical and $t \geq 0$, then $\phi([0,t] \times A)$ is essentially vertical;

(ii) each local unstable manifold $W^{\mathrm{u}}_{\mathrm{loc},r}(x)$ is essentially vertical;

(iii) each local stable manifold $W^{\mathrm{s}}_{\mathrm{loc},r}(x)$ has precompact intersection with every essentially vertical subset.

Let us prove (i) under the simplifying assumption that the target of the map Q is a Hilbert space E, and that the constants appearing in condition (C3)(ii) are uniform: c does not depend on q, and we can take $\delta = +\infty$. So (C3)(ii) becomes

$$\beta_E\Big(DQ(X(B))\Big) \leq c\beta_E(Q(B)) \quad \forall B \subset M. \tag{56}$$

Let $A \subset M$ be an essentially vertical set, that is $\beta_E(Q(A)) = 0$. Since Q takes value in a Hilbert space, there holds

$$Q(\phi(t,p)) = Q(p) + t \cdot \frac{1}{t}\int_0^t DQ(\phi(s,p))[X(\phi(s,p))]\, ds,$$

from which we deduce that

$$Q(\phi([0,t] \times A)) \subset Q(A) + [0,t]\ \overline{\mathrm{conv}}\Big(DQ(X(\phi([0,t] \times A))\Big).$$

Then, by the properties (53) of the measure of noncompactness β and by (56),

$$\begin{aligned}\beta_E\Big(Q(\phi([0,t] \times A))\Big) &\leq \beta_E(Q(A)) + t\beta_E\Big(\overline{\mathrm{conv}}\Big(DQ(X(\phi([0,t] \times A))\Big)\Big)\\ &= t\beta_E\Big(DQ\Big(X(\phi([0,t] \times A))\Big)\Big) \leq tc\beta_E\Big(Q(\phi([0,t] \times A))\Big).\end{aligned}$$

By the above inequality, $\beta_E\Big(Q(\phi([0,t] \times A))\Big)$ vanishes for every $t < 1/c$, and by iteration, for every $t \geq 0$. This proves (i).

Since (ii) and (iii) are local statements, we may assume that the rest point x is the origin of the Hilbert space H, and that Q is the orthogonal projector with kernel V, a constant local representative of the essential subbundle $\mathcal{E}$. By (C1), $E^{\mathrm{u}} := E^{\mathrm{u}}(\nabla X(0))$ is a compact perturbation of V. This fact easily implies that a bounded set $A \subset H$ is essentially vertical if and only if its projection $P^{\mathrm{s}}A$ on $E^{\mathrm{s}} := E^{\mathrm{s}}(\nabla X(0))$ is precompact. In particular, the graph of a map $\sigma\colon E^{\mathrm{u}}(r) \to E^{\mathrm{s}}(r)$ is essentially vertical if and only if the map σ is compact. So (ii) can be restated by saying that the map $\sigma^{\mathrm{u}}\colon E^{\mathrm{u}}(r) \to E^{\mathrm{s}}(r)$ whose graph is the local unstable manifold (see Theorem 1.12) is compact. By the graph transform method (see Shub, 1987, Chapter 5), σ^{u} is the fixed point of the contraction F, mapping every 1-Lipschitz map $\sigma \in \mathrm{Lip}_1(E^{\mathrm{u}}(r), E^{\mathrm{s}}(r))$ into the map $F(\sigma) \in \mathrm{Lip}_1(E^{\mathrm{u}}(r), E^{\mathrm{s}}(r))$, whose graph is the ϕ-evolution at time 1 of the graph of σ, intersected with $E^{\mathrm{u}}(r) \times E^{\mathrm{s}}(r)$. So claim (i) implies that the contraction F maps the closed nonempty subspace of compact maps into itself, hence the fixed point σ^{u} is a compact map, proving (ii). Claim (iii) is an immediate consequence of the fact that $W^{\mathrm{s}}_{\mathrm{loc},r}(x)$ is the graph of a continuous map $\sigma^{\mathrm{s}}\colon E^{\mathrm{s}}(r) \to E^{\mathrm{u}}(r)$.

Let us see how claims (i), (ii), and (iii) allow to conclude, in the case in which there are no rest points in the strip where $f(y) < f(p) < f(x)$. Let $(p_n) \subset W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$. We must prove that (p_n) has a converging subsequence. We can assume that x and y are not limit points of (p_n). Then we can find $s_n < 0 < t_n$ such that

$$\phi(s_n, p_n) \in W^{\mathrm{u}}_{\mathrm{loc},r}(x) \cap \{f = f(x) - \epsilon\}, \quad \phi(t_n, p_n) \in W^{\mathrm{s}}_{\mathrm{loc},r}(y) \cap \{f = f(y) + \epsilon\},$$

for some small $\epsilon > 0$. The fact that the are no rest points in the strip $\{f(y) < f < f(x)\}$ implies that $(t_n - s_n)$ is bounded: otherwise by Remark 2.1, we could find a sequence $r_n \in [s_n, t_n]$ such that $(Df(\phi(r_n, p_n))[X(\phi(r_n, p_n))])$ tends to zero, and by (PS) the sequence $(\phi(r_n, p_n))$ would have a subsequence converging to a rest point in the strip $\{f(y) + \epsilon \leq f \leq f(x) - \epsilon\}$, a contradiction. By claim (ii), the set $\{\phi(s_n, p_n) \mid n \in \mathbb{N}\}$ is essentially vertical. By claim (i) and by the fact that $(t_n - s_n)$ is bounded, also the set $\{\phi(t_n, p_n) \mid n \in \mathbb{N}\}$ is essentially vertical. But the latter set is contained in the local stable manifold of y, so by claim (iii) it is precompact. Since (t_n) is bounded, also the sequence (p_n) is compact.

In the general case, one needs the following stronger versions of (ii) and (iii): there exist arbitrarily small neighborhoods U of the rest point x such that if (p_n) converges to x then:

(ii′) if $t_n \geq 0$ and $\phi(t_n, p_n)\partial U$ then the set $\{\phi(t_n, p_n) \mid n \in \mathbb{N}\}$ is essentially vertical;

(iii′) if $s_n \leq 0$ and $\phi(s_n, p_n)\partial U$ then the set $\{\phi(s_n, p_n) \mid n \in \mathbb{N}\}$ has compact intersection with any essentially vertical subset.

The proof of (ii′) and (iii′) makes use of Proposition 1.17. Then a combination of the argument shown above and the argument in the proof of Theorem 2.2(ii) allows to conclude the proof of Theorem 3.5. □

REMARK 3.6. The requirement that the essential subbundle $\mathcal{E}$ should have a global presentation can be weakened, by replacing the map Q by a suitable family of maps $Q_i\colon M_i \to N_i$, $i \in I$, where $\{M_i\}_{i\in I}$ is an open covering of M. Besides allowing more general essential subbundles, this fact has also the advantage of localizing even more the constants appearing in assumption (C2)(ii).

3.7. TWO-DIMENSIONAL INTERSECTIONS

Assume that the Morse vector field X is complete, has a nondegenerate Lyapunov function f, and that (X, f) satisfies (PS). Assume also that X satisfies (C1) – (C3) with respect to a subbundle $\mathcal{V}$ of TM. In analogy with the finite indices case, we shall say that X satisfies the Morse – Smale property up to order $k \in \mathbb{Z}$ if $W^{\mathrm{u}}(x)$ meets $W^{\mathrm{s}}(y)$ transversally whenever $m(x,\mathcal{V}) - m(y,\mathcal{V}) \le k$.

Let us study what happens when the Morse – Smale condition up to order 2 holds, and x, z are rest points with $m(x,\mathcal{V}) - m(z,\mathcal{V}) = 2$. Let W be a connected component of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(z)$. It is a two-dimensional manifold, and $\mathbb{R}$ acts freely on it. Therefore $W/\mathbb{R}$ is a connected one-dimensional manifold, that is it is either a circle or an interval. In the first case, it is easy to see that $\overline{W} = W \cup \{x,z\}$ is a two-dimensional sphere, and the restriction of ϕ to $\overline{W}$ is topologically conjugated to the exponential flow on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$,

$$\mathbb{R} \times S^2 \ni (t,\zeta) \mapsto e^t\zeta \in S^2.$$

We shall be more interested in the second case, in which $\overline{W}$ is the union of W and two “broken orbits,” with exactly one intermediate rest point. More precisely, the situation is described by the following theorem.

THEOREM 3.7. *Assume that the Morse vector field X is complete, has a nondegenerate Lyapunov function f, and that (X, f) satisfies* (PS). *Assume also that X satisfies* (C1) – (C3) *with respect to a subbundle $\mathcal{V}$ of TM, and has the Morse – Smale property up to order* 2. *Let x, y be rest points with $m(x,\mathcal{V}) - m(z,\mathcal{V}) = 2$, and let W be a connected component of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(z)$ such that $W/\mathbb{R}$ is an interval. Then restriction of the flow ϕ to $\overline{W}$ is topologically conjugated to the product of two shift flows on $\overline{\mathbb{R}}$: there exists a continuous surjective map*

$$h\colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{W}$$

with the following properties:

(i) *$\phi(t, h(u,v)) = h(u+t, v+t)$ for every $(u,v) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, $t \in \mathbb{R}$;*

(ii) *$h(\mathbb{R}^2) = W$, and there exist rest points y, y' with $m(y, \mathcal{V}) = m(y', \mathcal{V}) = m(x, \mathcal{V}) - 1$, and W_1, W_2, W_1', W_2' connected components of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$, $W^{\mathrm{u}}(y) \cap W^{\mathrm{s}}(z)$, $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y')$, $W^{\mathrm{u}}(y') \cap W^{\mathrm{s}}(z)$, respectively, such that $W_1 \cup W_2 \neq W_1' \cup W_2'$, and*

$$\begin{aligned} h(\mathbb{R} \times \{-\infty\}) &= W_1, \quad h(\{+\infty\} \times \mathbb{R}) = W_2, \\ h(\{-\infty\} \times \mathbb{R}) &= W_1', \quad h(\mathbb{R} \times \{+\infty\}) = W_2'; \end{aligned}$$

(iii) *the restrictions of h to $\mathbb{R}^2$, to $\{\pm\infty\} \times \mathbb{R}$, and to $\mathbb{R} \times \{\pm\infty\}$, are diffeomorphisms*;

(iv) *denoting by* deg *the $\mathbb{Z}$-topological degree, referred to the orientations defined in Section* 3.5, *there holds*

$$\deg h = -\deg h|_{\{-\infty\}\times\mathbb{R}} \cdot \deg h|_{\mathbb{R}\times\{+\infty\}} = \deg h|_{\mathbb{R}\times\{-\infty\}} \cdot \deg h|_{\{+\infty\}\times\mathbb{R}}.$$

When $y \neq y'$, h is injective, so it is a conjugacy. When $y = y'$, it may happen that $W_1 = W_1'$, or that $W_2 = W_2'$, but these identities cannot hold simultaneously. Statement (iv) expresses a form of coherence of the orientations defined in Section 3.5.

Let us describe the main idea in the construction of h. By compactness and transversality, we can find a "broken orbit" in the closure of W, with exactly one intermediate rest point y of relative Morse index $m(y, \mathcal{V}) = m(x, \mathcal{V}) - 1$. Let W_1 and W_2 be the corresponding components of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ and $W^{\mathrm{u}}(y) \cap W^{\mathrm{s}}(z)$. Let $p \in W_1$, and let $q \in W_2$. Let A be a small hypersurface in $W^{\mathrm{u}}(x)$ meeting $W^{\mathrm{s}}(y)$ transversally at p, and let B be a small hypersurface in $W^{\mathrm{s}}(z)$ meeting $W^{\mathrm{u}}(y)$ transversally at q. Consider a neighborhood U of y of the form $U = E^{\mathrm{u}}_y(r) \times E^{\mathrm{s}}_y(r)$, where r is so small that the local stable manifold of y is the graph of a θ-Lipschitz map $\sigma^{\mathrm{s}} : E^{\mathrm{s}}_y(r) \to E^{\mathrm{u}}_y(r)$, while the local unstable manifold of y is the graph of a θ-Lipschitz map $\sigma^{\mathrm{u}} : E^{\mathrm{u}}_y(r) \to E^{\mathrm{s}}_y(r)$, for some $\theta < 1$. The forward evolution of A eventually intersects U in the graph of a θ-Lipschitz map from $E^{\mathrm{u}}_y(r)$ to $E^{\mathrm{s}}_y(r)$: there is $t_0 \geq 0$ such that for every $t \geq t_0$

$$\phi(\{t\} \times A) \cap U = \operatorname{graph} \alpha_t : E^{\mathrm{u}}_y(r) \to E^{\mathrm{s}}_y(r), \quad \operatorname{lip}(\alpha_t) \leq \theta,$$

and $\|\alpha_t - \sigma^{\mathrm{u}}\|_\infty \to 0$ for $t \to +\infty$. Similarly, for every $t \leq -t_0$,

$$\phi(\{t\} \times B) \cap U = \operatorname{graph} \beta_t : E^{\mathrm{s}}_y(r) \to E^{\mathrm{u}}_y(r), \quad \operatorname{lip}(\beta_t) \leq \theta,$$

and $\|\beta_t - \sigma^{\mathrm{s}}\|_\infty \to 0$ for $t \to -\infty$. Let $u \geq t_0$ and $v \leq -t_0$. Since $\operatorname{lip}(\alpha_u) \leq \theta < 1$ and $\operatorname{lip}(\beta_v) \leq \theta < 1$, the graphs of α_u and of β_v intersect in exactly one point, and we can define $h(u,v)$ as

$$h(u,v) := (\operatorname{graph} \alpha_u) \cap (\operatorname{graph} \beta_v).$$

This defines h in a neighborhood of $(+\infty, -\infty)$. See Abbondandolo and Majer (2003b, Section 11) for a complete proof.

An analogous argument allows to prove the following result.

PROPOSITION 3.8. *Let x, y, z be rest points such that $m(x,\mathcal{V}) = m(y,\mathcal{V}) + 1 = m(z,\mathcal{V}) + 2$, and let W_1, W_2 be connected components of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$, $W^{\mathrm{u}}(y) \cap W^{\mathrm{s}}(z)$, respectively. Then there exists a unique connected component W of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(z)$ such that $\overline{W_1 \cup W_2}$ belongs to the closure of $\{\overline{\phi(\mathbb{R} \times \{p\})} \mid p \in W\}$ with respect to the Hausdorff distance.*

3.8. THE MORSE COMPLEX

We now dispose of all the ingredients to build the Morse complex. The assumptions are that the Morse C^1 vector field X on the Hilbert manifold M is complete, satisfies (C1) – (C3) with respect to a subbundle $\mathcal{V}$ of TM, with a global presentation $Q \colon M \to N$, that X satisfies the Morse – Smale condition up to order 2, has a nondegenerate Lyapunov function $f \in C^2(M)$, and that the pair (X, f) satisfies (PS).

For any $k \in \mathbb{Z}$, denote by $\mathrm{rest}_k(X)$ the set of rest points x of X of relative Morse index $m(x,\mathcal{V}) = k$, and let $C_k(X)$ be the free Abelian group generated by $\mathrm{rest}_k(X)$. Assume the following finiteness condition:

(C4) for every $k \in \mathbb{Z}$, f is bounded below on $\mathrm{rest}_k(X)$.

For every rest point x, we fix an orientation of the Fredholm pair $(T_x W^{\mathrm{s}}(x), \mathcal{V}(x))$ in an arbitrary way. This choice induces an orientation of all the intersections $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$, for $m(x,\mathcal{V}) - m(y,\mathcal{V}) \le 2$.

Let x, y be rest points of X with $m(x,\mathcal{V}) - m(y,\mathcal{V}) = 1$. Then $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is a 1-dimensional manifold with a free action of $\mathbb{R}$, that is it is the union of the orbits of a discrete set of points. By Theorem 3.5 and by transversality, $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ is compact: otherwise we could find a sequences of orbits in $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ converging to a “broken orbit” from x to y, with at least one intermediate rest point, violating the Morse – Smale condition (up to order 0). Therefore, $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ consists of finitely many orbits W_i, $i = 1, \dots, h$, each of which can be given a sign $\epsilon(W_i) \in \{+1, -1\}$ depending on whether the direction of X agrees or does not agree with the orientation of W_i. In other words, if $W_i = \phi(\mathbb{R} \times \{p\})$, $\epsilon(W_i)$ is the degree of the map $\phi(\cdot, p) \colon \mathbb{R} \to W_i$. We define the integer $n(x, y)$ as

$$n(x,y) = \sum_{i=1}^{h} \epsilon(W_i).$$

By assumption (C4), we can define a homomorphism $\partial_k : C_k(X) \to C_{k-1}(X)$ generatorwise, as

$$\partial_k x = \sum_{y \in \mathrm{rest}_{k-1}(X)} n(x,y) y, \quad \forall x \in \mathrm{rest}_k(X).$$

The results of Section 3.7 imply that these homomorphisms are boundary operators.

PROPOSITION 3.9. *For every $k \in \mathbb{Z}$, $\partial_{k-1}\partial_k = 0$.*

Proof. Let x and z be rest points with $m(x, \mathcal{V}) - m(z, \mathcal{V}) = 2$, and let $\mathcal{S}(x,z)$ be the set of "broken orbits" from x to z with exactly one intermediate rest point, necessarily of relative index $m(z, \mathcal{V}) + 1$. By compactness and transversality, $\mathcal{S}(x,z)$ is a finite set. By Proposition 3.8, for every element $W_1 \cup W_2$ of $\mathcal{S}(x,z)$ there is a unique connected component W of $W^{\mathrm{u}}(x) \cap W^{\mathrm{s}}(y)$ such that $W_1 \cup W_2$ belongs to the closure of $\{\overline{\phi(\mathbb{R} \times \{p\})} \mid p \in W\}$ with respect to the Hausdorff distance. By Theorem 3.7, the closure of W contains exactly one other element $W_1' \cup W_2'$, different from $W_1 \cup W_2$. So there is an involution $W_1 \cup W_2 \mapsto W_1' \cup W_2'$ on $\mathcal{S}(x,z)$, without fixed points, and by Theorem 3.7 (iv),

$$\epsilon(W_1')\epsilon(W_2') = -\epsilon(W_1)\epsilon(W_2). \tag{57}$$

If $m(x, \mathcal{V}) = k$, the coefficient of z in $\partial_{k-1}\partial_k x$ is the number

$$\sum_{y \in \mathrm{rest}_{k-1}(X)} n(x,y) n(y,z) = \sum_{W_1 \cup W_2 \in \mathcal{S}(x,z)} \epsilon(W_1)\epsilon(W_2),$$

which is zero by (57). □

Therefore, the Abelian groups $C_k(X)$ and the homomorphisms ∂_k, for $k \in \mathbb{Z}$, are the data of a chain complex, called the *Morse complex of X*. The construction depends on the choice of the subbundle $\mathcal{V}$, and on the choice of the orientations of $(T_x W^{\mathrm{s}}(x), \mathcal{V}(x))$. Replacing the subbundle $\mathcal{V}$ by a compact perturbations produces a shift in the indices, equal to the relative dimension of the compact perturbation. A change of the orientations produces an isomorphic chain complex, the isomorphism being actually an involution.

When the conditions (C1)–(C3) hold only with respect to a (0)-essential subbundle, there is no orientation theory available, and the above construction produces a chain complex of $\mathbb{Z}_2$-vector spaces.

Replacing the vector field X by another one (still satisfying conditions (C1)–(C3) with respect to the same subbundle $\mathcal{V}$) having the same Lyapunov function f, produces an isomorphic Morse complex: the argument is analogous to the one used in the proof of Theorem 2.26. In particular, the homology of the Morse complex does not depend on the vector field, at it can be denoted by $H_*(f)$, the *Morse homology of f*.

Although in this situation we should not expect the Morse homology $H_*(f)$ to be directly related to the singular homology of M, $H_*(f)$ is still considerably stable with respect to modifications of the function f. For instance, if (X_0, f_0) and (X_1, f_1) satisfy conditions (PS) and (C1) – (C4) (with respect to the same subbundle $\mathcal{V}$), and if $f_1 - f_0$ is bounded, then the corresponding Morse homologies are isomorphic (see Abbondandolo and Majer, 2001, Theorem 1.8, but see also Theorem 1.10).

This fact is a consequence of a more general functorial property of the Morse homology: Morse homology is a functor from the class of Morse functions which are Lyapunov functions of some vector field satisfying (PS) and (C1) – (C4), seen as a small category with the usual order relation, to the category of graded Abelian groups. In other words, to each inequality $f_0 \geq f_1$ is associated a sequence of homomorphisms of Abelian groups

$$\phi_{f_0 f_1} \colon H_k(f_0) \to H_k(f_1), \quad \forall k \in \mathbb{Z},$$

such that $\phi_{f_1 f_2}\phi_{f_0 f_1} = \phi_{f_0 f_2}$ and $\phi_{f f} = \mathrm{id}$. Actually, $\phi_{\theta \circ f f} = \mathrm{id}$, if $\theta \colon \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\theta' > 0$ and $\theta(s) \geq s$. This fact is clearly useful in order to compute the Morse homology of a given function f: if one can squeeze f between two functions, $f_0 \geq f \geq f_1$, the knowledge of the Morse homology of f_0 and f_1 and of the homomorphism $\phi_{f_0 f_1}$ allows to get information on the Morse homology of f. For instance, if $\phi_{f_0 f_1}$ is an isomorphism, then $\phi_{f_0 f}$ is injective and $\phi_{f f_1}$ is surjective, hence the Morse homology of f is at least as rich as the Morse homology of f_0 and f_1.

The construction of the homomorphism $\phi_{f_0 f_1}$ involves the same idea used in the proof of Theorem 2.26: f_0 and f_1 can be used to build a new function $\tilde{f}$ on $\mathbb{R} \times M$, whose boundary operator ∂ is the cone of some homomorphism $\psi_{f_0 f_1}$ from the Morse complex of f_0 to the one of f_1. The $\partial^2 = 0$ formula then implies that $\psi_{f_0 f_1}$ is a chain map, so it induces a homomorphism $\phi_{f_0 f_1}$ in homology.

Bibliographical note

The Morse complex approach for compact manifolds

When M is a compact manifold and X is the negative gradient flow of a smooth function, the relations (36) were proved by Morse (1925), see also Morse (1934; 1947). A classical reference for Morse theory is Milnor (1963). See also the review papers by Bott (1982; 1988).

The dynamical system point of view arose after the seminal work of Smale, see Smale (1960; 1961) and the beautiful foundational paper Smale (1967), and it immediately had influences in topology, see Milnor's book on the *h*-cobordism theorem (Milnor, 1965). In this framework, one can consider Morse – Smale flows, which are dynamical systems more general than gradient-like flows since they may have periodic orbits. The connection between Morse theory for Morse – Smale flows and the homotopy of the underlying manifold has been further clarified by Franks (1979), see also Franks (1980), and Cornea (2002a; 2002b).

Interpreting the boundary homomorphism of a cellular filtration in terms of an algebraic count of the gradient flow lines connecting critical points of index difference 1, was already implicit in a paper by Thom (1949), who however did not clarify the conditions required on the gradient flow. This interpretation was pointed out by Witten (1982), where it is deduced quite indirectly from a relationship between Morse theory and certain deformations of the Laplace – Beltrami operator. The first explicit construction of the Morse complex is due to Floer (1989), see also Salamon (1990). Floer's proof makes use of Conley index theory, a general and powerful method to decompose a dynamical system into simpler invariant sets, see Conley (1978), Conley and Zehnder (1984), and Salamon (1985).

Weber (1993) contains a concise construction of the Morse complex, by dynamical systems techniques (see also Weber, 2004). A systematic study of the Morse complex of a function as a tool to build a homology theory which satisfies the Eilenberg – Steenrod axioms can be found in Schwarz (1993). Here the methods are closer to those used in Floer homology. The isomorphism with the singular homology is deduced by the fact that all the homology theories which satisfy the Eilenberg – Steenrod axioms are equivalent on compact CW-complexes. A more direct proof of this isomorphism, still in this spirit, can be obtained by interpreting singular homology theory in terms of pseudocycles, see Schwarz (1999).

Banyaga and Hurtubise (2004) presents a self-contained exposition of Morse homology, adopting the dynamical system point of view and providing all the necessary tools from hyperbolic dynamics, as well as applications to Morse theory on Grassmannians and on Lie groups.

The dynamical system point of view is at the basis of Harvey and Lawson's approach to Morse theory in terms of the de Rham – Federer theory of currents (Harvey and Lawson, 2001). The idea is to construct a chain map from the complex of smooth differential forms to the complex of currents, by taking the limit for $t \to +\infty$ of the pullback of a differential form by the flow $\phi(t, \cdot)$. Such a chain map is chain homotopic to the inclusion, and it is a retraction onto the subcomplex of currents spanned by the stable manifolds of the flow. The cohomology of such a subcomplex is then isomorphic to the de Rham cohomology of the manifold, a result which implies the Morse relations (36).

Infinite-dimensional Morse theory

Morse theory for C^2 functions on Hilbert manifolds was developed by Palais (1963) and Smale (1964a; 1964b). The Palais – Smale condition was introduced in these papers. This version of Morse theory has been extensively used in the study of geodesics, see Klingenberg (1978; 1982). The first of these references contains also a description of the cellular complex approach to infinite-dimensional Morse theory, in the case of self-indexing functions. A complete presentation of infinite-dimensional Morse theory including many applications to differential equations can be found in Mawhin and Willem (1989) and Chang (1993).

Morse theory in the case of infinite Morse indices

In simple situations, functions with critical points of infinite Morse index and co-index can be studied by taking finite-dimensional approximations. See, for instance, Chang, (1981; 1993), Conley and Zehnder, (1983; 1984), and Abbondandolo (2001). Another way of overcoming the lack of rigidity due to the presence of critical points of infinite Morse index and co-index is to restrict the class of admissible deformations to more rigid classes, as in Benci and Rabinowitz (1979) and Rabinowitz (1986). In the same spirit, a Morse theory for special classes of functions on a Hilbert space has been introduced by Szulkin (1992), and further refined by Abbondandolo (1997; 2000), Kryszewski and Szulkin (1997), Gęba et al. (1999), and Izydorek (2001). The idea is to develop a *generalized cohomology theory*, which satisfies all the Eilenberg – Steenrod axioms except the dimension axiom. This axiom is replaced by the requirement that suitable infinite-dimensional spheres should have nontrivial cohomology. These generalized cohomologies will be functorial only with respect to restricted classes of continuous maps (the infinite-dimensional sphere is contractible), and it is possible to develop a Morse theory for functions whose gradient flow belongs to such a class.

The idea of forgetting about the whole ambient space and looking only at the gradient flow lines connecting critical points is due to Floer, who applied it to a Cauchy – Riemann type equation which does not even produce a local flow (so the framework is quite different from the setting of these notes). See Floer (1988a; 1988b; 1988c; 1989), and the expository paper Salamon (1999). Angenent and van der Vorst (1999) have used this approach to study the gradient flow of a function associated to a class of elliptic systems. A complete study of the Morse complex approach in the case of functions on a Hilbert space consisting of a compact perturbation of a nondegenerate quadratic form has been carried on by the authors in Abbondandolo and Majer (2001). The results of Abbondandolo and Majer (2003b) summarized in the third part of these notes, allow a much more general setting.

There is a large literature about the Hilbert Grassmannian, and related constructions. In particular, the space of all compact perturbations of an infinite-dimensional and -codimensional closed linear subspace is called *restricted Grassmannian* by some authors (although sometimes this term is reserved for Hilbert-Schmidt perturbations). See for instance Sato (1981), Segal and Wilson (1985), Pressley and Segal (1986), Guest (1997), and Arbarello (2002). The role of these objects in the homotopy theory underlying Floer homology is discussed in Cohen et al. (1995).

References

Abbondandolo, A. (1997) A new cohomology for the Morse theory of strongly indefinite functionals on Hilbert spaces, *Topol. Methods Nonlinear Anal.* **9**, 325 – 382.

Abbondandolo, A. (2000) Morse theory for asymptotically linear Hamiltonian systems, *Nonlinear Anal.* **39**, 997 – 1049.

Abbondandolo, A. (2001) *Morse Theory for Hamiltonian Systems*, Vol. 425 of *Chapman Hall/CRC Res. Notes Math.*, Boca Raton, FL, Chapman & Hall/CRC.

Abbondandolo, A. and Majer, P. (2001) Morse homology on Hilbert spaces, *Comm. Pure Appl. Math.* **54**, 689 – 760.

Abbondandolo, A. and Majer, P. (2003)a Infinite-dimensional Grassmannians, arXiv:math.AT/0307192.

Abbondandolo, A. and Majer, P. (2003)b A Morse complex for infinite-dimensional manifolds. I, *Adv. Math.*, to appear; arXiv:math.DS/0309020.

Abbondandolo, A. and Majer, P. (2003)c Ordinary differential operators on Hilbert spaces and Fredholm pairs, *Math. Z.* **243**, 525 – 562.

Abbondandolo, A. and Majer, P. (2004) When the Morse index is infinite, *Int. Math. Res. Not.* **71**, 3839 – 3854.

Abraham, R. and Robbin, J. (1967) *Transversal Mappings and Flows*, New York, W. A. Benjamin, Inc.

Angenent, S. and van der Vorst, R. (1999) A superquadratic indefinite elliptic system and its Morse – Conley – Floer homology, *Math. Z.* **231**, 203 – 248.

Arbarello, E. (2002) Sketches of KdV, In A. Bertram, J. A. Carlson, and H. Kley (eds.), *Symposium in Honor of C. H. Clemens*, Vol. 312 of *Contemp. Math.*, Salt Lake City, UT, 2000, p. 9 – 69, Providence, RI, Amer. Math. Soc.

Banyaga, A. and Hurtubise, D. (2004) *Lectures on Morse Homology*, Vol. 29 of *Kluwer Texts Math. Sci.*, Dordrecht, Springer.

Benci, V. and Rabinowitz, P. H. (1979) Critical point theorems for indefinite functionals, *Invent. Math.* **52**, 241 – 273.

Bott, R. (1959) The stable homotopy of the classical groups, *Ann. of Math.* (2) **70**, 313 – 317.

Bott, R. (1982) Lectures on Morse theory, old and new, *Bull. Amer. Math. Soc.* (*N.S.*) **7**, 331 – 358.

Bott, R. (1988) Morse theory indomitable, *Inst. Hautes Études Sci. Publ. Math.* **68**, 99 – 114.

Chang, K. C. (1981) Solutions of asymptotically linear operator equations via Morse theory, *Comm. Pure Appl. Math.* **34**, 693 – 712.

Chang, K. C. (1993) *Infinite-dimensional Morse theory and multiple solution problems*, Vol. 6 of *Progr. Nonlinear Differential Equations Appl.*, Boston, MA, Birkhäuser.

Cohen, R. L., Jones, J. D. S., and Segal, G. B. (1995) Floer's infinite-dimensional Morse theory and homotopy theory, In H. Hofer, C. H. Taubes, A. Weinstein, and E. Zehnder (eds.), *The Floer Memorial Volume*, Vol. 133 of *Progr. Math*, p. 297 – 325, Basel, Birkhäuser.

Conley, C. (1978) *Isolated invariant sets and the Morse index*, Vol. 38 of *CBMS Reg. Conf. Ser. Math.*, Providence, RI, Amer. Math. Soc.

Conley, C. and Zehnder, E. (1983) The Birkhoff – Lewis fixed point theorem and a conjecture of V. I. Arnol′d, *Invent. Math.* **73**, 33 – 49.

Conley, C. and Zehnder, E. (1984) Morse-type index theory for flows and periodic solutions for Hamiltonian equations, *Comm. Pure Appl. Math.* **37**, 207 – 253.

Cornea, O. (2002)a Homotopical dynamics. II. Hopf invariants, smoothing, and the Morse complex, *Ann. Sci. École Norm. Sup.* (4) **35**, 549 – 573.

Cornea, O. (2002)b Homotopical dynamics. IV. Hopf invariants, and Hamiltonian flows, *Comm. Pure Appl. Math.* **55**, 1033 – 1088.

Cornea, O. and Ranicki, A. (2003) Rigidity and glueing for Morse and Novikov complexes, *J. Eur. Math. Soc.* (*JEMS*) **5**, 343 – 394.

Dold, A. (1980) *Lectures on Algebraic Topology*, Vol. 200 of *Grundlehren Math. Wiss.*, Berlin, Springer, 2 edition.

Dugundji, J. (1978) *Topology*, Allyn and Bacon Series in Advanced Mathematics, Boston, MA, Allyn and Bacon Inc., reprinting of the 1966 original.

Eells, J. and Elworthy, K. D. (1970) Open embeddings of certain Banach manifolds, *Ann. of Math.* (2) **91**, 465 – 485.

Floer, A. (1988)a Morse theory for Lagrangian intersections, *J. Differential Geom.* **28**, 513 – 547.

Floer, A. (1988)b A relative Morse index for the symplectic action, *Comm. Pure Appl. Math.* **41**, 393 – 407.

Floer, A. (1988)c The unregularized gradient flow of the symplectic action, *Comm. Pure Appl. Math.* **41**, 775 – 813.

Floer, A. (1989) Witten's complex and infinite-dimensional Morse theory, *J. Differential Geom.* **30**, 207 – 221.

Franks, J. M. (1979) Morse – Smale flows and homotopy theory, *Topology* **18**, 199 – 215.

Franks, J. M. (1980) *Homology and Dynamical Systems*, Vol. 49 of *CBMS Reg. Conf. Ser. Math.*, Providence, RI, Amer. Math. Soc.

Gęba, K., Izydorek, M., and Pruszko, A. (1999) The Conley index in Hilbert spaces and its applications, *Studia Math.* **134**, 217 – 233.

Guest, M. A. (1997) *Harmonic Maps, Loop Groups, and Integrable Systems*, Vol. 38 of *London Math. Soc. Stud. Texts*, Cambridge, Cambridge Univ. Press.

Harvey, F. R. and Lawson, Jr., H. B. (2001) Finite volume flows and Morse theory, *Ann. of Math.* (2) **153**, 1 – 25.

Izydorek, M. (2001) A cohomological Conley index in Hilbert spaces and applications to strongly indefinite problems, *J. Differential Equations* **170**, 22 – 50.

Kato, T. (1980) *Perturbation Theory for Linear Operators*, Berlin, Springer, 2 edition.

Klingenberg, W. (1978) *Lectures on Closed Geodesics*, Vol. 230 of *Grundlehren Math. Wiss.*, Berlin, Springer.

Klingenberg, W. (1982) *Riemannian Geometry*, Vol. 1 of *de Gruyter Stud. Math.*, Berlin, de Gruyter.

Kryszewski, W. and Szulkin, A. (1997) An infinite-dimensional Morse theory with applications, *Trans. Amer. Math. Soc.* **349**, 3181 – 3234.

Kuiper, N. H. (1965) The homotopy type of the unitary group of Hilbert space, *Topology* **3**, 19 – 30.

Lang, S. (1999) *Fundamentals of Differential Geometry*, Vol. 191 of *Grad. Texts in Math.*, New York, Springer.

Lasry, J.-M. and Lions, P.-L. (1986) A remark on regularization in Hilbert spaces, *Israel J. Math.* **55**, 257 – 266.

Mawhin, J. and Willem, M. (1989) *Critical Point Theory and Hamiltonian Systems*, Vol. 74 of *Appl. Math. Sci.*, New York, Springer.

Milnor, J. (1963) *Morse Theory*, Vol. 51 of *Ann. of Math. Stud.*, Princeton, NJ, Princeton Univ. Press.

Milnor, J. W. (1965) *Topology from the Differentiable Viewpoint*, Charlottesville, VA, The University Press of Virginia.

Morse, M. (1925) Relations between the critical points of a real function of *n* independent variables, *Trans. Amer. Math. Soc.* **27**, 345 – 396.

Morse, M. (1934) *The Calculus of Variations in the Large*, Vol. 18 of *Amer. Math. Soc. Colloq. Publ*, Providence, RI, Amer. Math. Soc.

Morse, M. (1947) *Introduction to Analysis in the Large*, Princeton, NJ, Princeton Univ. Press.

Nemirovskiĭ, A. S. and Semenov, S. M. (1973) The polynomial approximation of functions in Hilbert spaces, *Mat. Sb.* (*N.S.*) **21 (92)**, 255 – 277, Russian.

Palais, R. S. (1963) Morse theory on Hilbert manifolds, *Topology* **2**, 299 – 340.

Palais, R. S. (1965) On the homotopy type of certain groups of operators, *Topology* **3**, 271 – 279.

Palis, J. (1968) On Morse – Smale dynamical systems, *Topology* **8**, 385 – 405.

Palis, Jr., J. and de Melo, W. (1982) *Geometric Theory of Dynamical Systems*, New York, Springer.

Poźniak, M. (1991) The Morse complex, Novikov cohomology and Fredholm theory, preprint, University of Warwick.

Pressley, A. and Segal, G. (1986) *Loop Groups*, Oxford Math. Monogr., Oxford, Oxford Univ. Press.

Quillen, D. (1985) Determinants of Cauchy – Riemann operators over a Riemann surface, *Funct. Anal. Appl.* **19**, 31 – 34.

Rabinowitz, P. H. (1986) *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Vol. 65 of *CBMS Reg. Conf. Ser. Math.*, Providence, RI, Amer. Math. Soc.

Salamon, D. (1985) Connected simple systems and the Conley index of isolated invariant sets, *Trans. Amer. Math. Soc.* **291**, 1 – 41.

Salamon, D. (1990) Morse theory, the Conley index and Floer homology, *Bull. Amer. Math. Soc.* **22**, 113 – 140.

Salamon, D. (1999) Lectures on Floer homology, In Y. Eliashberg and L. Traynor (eds.), *Symplectic Geometry and Topology*, Vol. 7 of *IAS/Park City Math. Ser.*, Park City, UT, 1997, p. 143 – 229, Providence, RI, Amer. Math. Soc.

Sato, M. (1981) Soliton equations as dynamical systems on an infinite-dimensional Grassman manifold, *Sūrikaisekikenkyūsho Kōkyūroku* **439**, 30 – 46.

Schwarz, M. (1993) *Morse Homology*, Vol. 111 of *Progr. Math*, Basel, Birkhäuser.

Schwarz, M. (1999) Equivalence for Morse homology, In M. Barge and K. Kuperberg (eds.), *Geometry and Topology in Dynamics*, Vol. 246 of *Contemp. Math.*, Winston-Salem, NC, 1998/San Antonio, TX, 1999, p. 197 – 216, Providence, RI, Amer. Math. Soc.

Segal, G. and Wilson, G. (1985) Loop groups and equations of KdV type, *Inst. Hautes Études Sci. Publ. Math.* **61**, 5 – 65.

Shub, M. (1987) *Global Stability of Dynamical Systems*, New York, Springer.

Smale, S. (1960) Morse inequalities for a dynamical system, *Bull. Amer. Math. Soc.* **66**, 43 – 49.

Smale, S. (1961) On gradient dynamical systems, *Ann. of Math.* (2) **74**, 199 – 206.

Smale, S. (1964)a A generalized Morse theory, *Bull. Amer. Math. Soc.* **70**, 165 – 172.

Smale, S. (1964)b Morse theory and a nonlinear generalization of the Dirichlet problem, *Ann. of Math.* (2) **80**, 382 – 396.

Smale, S. (1965) An infinite-dimensional version of Sard's theorem, *Amer. J. Math.* **87**, 861 – 866.

Smale, S. (1967) Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73**, 747 – 817.

Szulkin, A. (1992) Cohomology and Morse theory for strongly indefinite functionals, *Math. Z.* **209**, 375 – 418.

Thom, R. (1949) Sur une partition en cellules associée à une fonction sur une variété, *C. R. Acad. Sci. Paris* **228**, 973 – 975.

Weber, J. (1993) Der Morse – Witten Komplex, Master's thesis, TU Berlin.

Weber, J. (2004) The Morse – Witten complex via dynamical systems, preprint.

Witten, E. (1982) Supersymmetry and Morse theory, *J. Differential Geom.* **17**, 661 – 692.