III Il baricentro

Se un filo ha la forma di una curva γ e
1) densità lineare di massa data da $\delta = \delta(x,y,z)$,
allora, se $\underline{m}$ è la massa totale del filo, le coordinate del baricentro sono date dalle formule:

$$\bar{x} := \frac{1}{m}\int_\gamma x\delta\, ds \quad \bar{y} := \frac{1}{m}\int_\gamma y\delta\, ds \quad \bar{z} := \frac{1}{m}\int_\gamma z\delta\, ds$$

2) Se il filo è omogeneo, possiamo scrivere
$\delta = 1 \Rightarrow$

$$\bar{x} = \frac{1}{l}\int_\gamma x\, ds \ ; \quad \bar{y} = \frac{1}{l}\int_\gamma y\, ds \ ; \quad \bar{z} = \frac{1}{l}\int_\gamma z\, ds$$

Calcolare le coordinate del baricentro:

① $\gamma: \vec{r}(t) = 4t\,\vec{i} - 3\cos t\,\vec{j} + 3\sin t\,\vec{k} \qquad t \in [0,1]$.

$\delta(x,y,z) = x$

$\vec{r}\,'(t) = (4\ ;\ 3\sin t,\ 3\cos t)$

$$\boxed{ds = \sqrt{16 + 9\sin^2 t + 9\cos^2 t}\, dt = \sqrt{25}\, dt = 5\,dt}$$

$$m = \int_\gamma x \cdot ds = \int_0^1 4t \cdot 5\,dt = \frac{20t^2}{2}\Big|_0^1 = \underline{10}$$

1) $$\int_\gamma x\delta\, ds = \int_0^1 4t \cdot 4t \cdot 5\,dt = \frac{80t^3}{3}\Big|_0^1 = \frac{80}{3}$$

$$\boxed{\overline{x} = \frac{80}{3} \Big| 10 = \frac{8}{3}}$$

$= 43 =$

2) $\int_\gamma y\delta \, ds = \int_\gamma xy \, ds = \int_0^1 (-12t\cos t) \cdot 5\, dt =$

$= -60 \int_0^1 t\cos t \, dt = -60 \int_0^1 t \, d(\sin t) =$

per parti

$= -60 \cdot t \cdot \sin t \Big|_0^1 + 60 \int_0^1 \sin t \, dt =$

$= -60 \cdot 1 \cdot \sin 1 + 0 - 60\cos t \Big|_0^1 = -60\sin 1 - 60\cos 1 + 60\cos 0 =$

$= 60(1 - \sin 1 - \cos 1)$

$$\boxed{\overline{y} = \frac{60}{10}(1 - \sin 1 - \cos 1) = 6 \cdot (1 - \sin 1 - \cos 1)}$$

3) $\int_\gamma z\delta \, ds = \int_\gamma xz \, ds = \int_0^1 (12t \sin t) \, 5 \cdot dt =$

$= 60 \int_0^1 t\sin t \, dt = -60 \int_0^1 t \, d(\cos t) = -60 \cdot t \cdot \cos t \Big|_0^1 +$

$+ 60 \int_0^1 \cos t \, dt = -60 \cdot 1 \cdot \cos 1 + 0 + 60 \sin t \Big|_0^1 =$

$= -60\cos 1 + 60\sin 1 - 0 = 60(\sin 1 - \cos 1)$

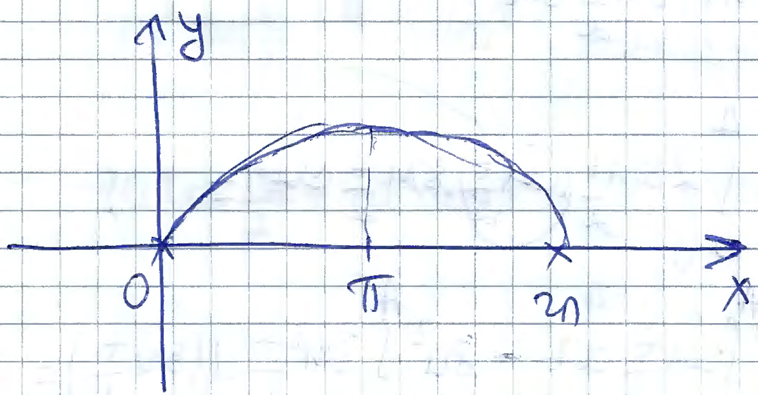
$$\boxed{\overline{z} = \frac{60}{10}(\sin 1 - \cos 1) = 6(\sin 1 - \cos 1)}$$

$\Rightarrow B\left(\frac{8}{3}, 6(1 - \sin 1 - \cos 1), 6(\sin 1 - \cos 1)\right)$

(2) $\gamma: \vec{r}(t) = a(t-\sin t)\vec{i} + a(1-\cos t)\vec{j}$, $a>0$

γ è linea materiale omogenea

a) $t \in [0, 2\pi]$ b) $t \in [0, \pi]$



Per punto a) possiamo dire subito che $\bar{x} = \pi$ (simmetria). Questo non possiamo dire per il punto b)

$\dot{x} = a(1-\cos t)$ $\dot{y} = a\sin t$

$ds = \sqrt{a^2(1-2\cos t+\cos^2 t)+a^2\sin^2 t}\, dt = \sqrt{a^2(2-2\cos t)}\, dt =$

$= \sqrt{a^2 \cdot 2 \cdot 2\sin^2\frac{t}{2}}\, dt = 2a\left|\sin\frac{t}{2}\right| dt$

$0 \le t \le 2\pi \Rightarrow 0 \le \frac{t}{2} \le \pi$ $\sin\frac{t}{2} \ge 0$ per punto a)

$0 \le t \le \pi \Rightarrow 0 \le \frac{t}{2} \le \frac{\pi}{2}$ $\sin\frac{t}{2} \ge 0$ per punto b)

$\Rightarrow \boxed{ds = 2a\sin\frac{t}{2}\, dt}$

a) $l = \int_0^{2\pi} 2a\sin\frac{t}{2}\, dt = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = -4a\cos\pi + 4a\cos 0 =$

$= 4a + 4a = 8a$

b) $l = \int_0^{\pi} 2a\sin\frac{t}{2}\, dt = -4a\cos\frac{\pi}{2} + 4a\cos 0 = \underline{4a}$

$$\int_\gamma x\,ds = \int_0^{2\pi \text{ o } \pi} a(t-\sin t)\cdot 2a\sin\frac{t}{2}\,dt =$$ =45=

2π o π = A

$$= 2a^2\int_0^{2\pi \text{ o } \pi} t\sin\frac{t}{2}\,dt - 2a^2\int_0^{} \sin t\cdot\sin\frac{t}{2}\,dt =$$

per parti

$$= -4a^2\int_0^A t\, d\left(\cos\frac{t}{2}\right) - 2a^2\int_0^A 2\sin\frac{t}{2}\cos\frac{t}{2}\cdot\sin\frac{t}{2}\,dt =$$

$$= -4a^2\cdot t\cdot\cos\frac{t}{2}\Big|_0^A + 4a^2\int_0^A \cos\frac{t}{2}\,dt - 8a^2\int_0^A \sin^2\frac{t}{2}\,d\left(\sin\frac{t}{2}\right) =$$

$$= -4a^2\cdot A\cdot\cos\frac{A}{2} + 4a^2\cdot 0 + 8a^2\sin\frac{t}{2}\Big|_0^A - \frac{8a^2}{3}\sin^3\frac{t}{2}\Big|_0^A =$$

$$= -4a^2\cdot A\cdot\cos\frac{A}{2} + 8a^2\sin\frac{A}{2} - 0 - \frac{8a^2}{3}\sin^3\frac{A}{2} + 0 =$$

$$= -4a^2\cdot A\cdot\cos\frac{A}{2} + \frac{8a^2}{3}\left(3\sin\frac{A}{2} - \sin^3\frac{A}{2}\right)$$

punto a) $A = 2\pi$

$$I = -4a\cdot 2\pi\cdot\cos\pi + \frac{8a^3}{3}\underbrace{(3\sin\pi - \sin^3\pi)}_{=0} = 8a\pi$$

punto b) $A = \pi$

$$I = -4a^2\cdot\pi\cdot\cos\frac{\pi}{2} + \frac{8a^2}{3}\left(3\sin\frac{\pi}{2} - \sin^3\frac{\pi}{2}\right) =$$

$$= 0 + \frac{8a^2}{3}(3-1) = \frac{16}{3}a^2$$

punto a) $\bar{x} = \frac{8a\pi}{8a} = \pi$

b) $\bar{x} = \frac{16}{3}a^2 / 4a = \boxed{\frac{4}{3}a}$

$$\int_\gamma y\,ds = \int_0^A a(1-\cos t)\cdot 2a\sin\frac{t}{2}\,dt = \qquad = 46 =$$

$$= 2a^2\int_0^A \sin\frac{t}{2}\,dt - 2a^2\int_0^A \cos t\cdot\sin\frac{t}{2}\,dt =$$

$$= -4a^2\cos\frac{t}{2}\Big|_0^A + 4a^2\int_0^A \cos t\, d\left(\cos\frac{t}{2}\right) =$$

$$= -4a^2\cos\frac{A}{2} + 4a^2\cos 0 + 4a^2\int_0^A \left(2\cos^2\frac{t}{2}-1\right) d\left(\cos\frac{t}{2}\right) =$$

$$= -4a^2\cos\frac{A}{2} + 4a^2 + 8a^2\int_0^A \cos^2\frac{t}{2}\, d\left(\cos\frac{t}{2}\right) - 4a^2\int_0^A d\left(\cos\frac{t}{2}\right) =$$

$$= -4a^2\cos\frac{A}{2} + 4a^2 + \frac{8a^2}{3}\cos^3\frac{t}{2}\Big|_0^A - 4a^2\cos\frac{A}{2}\Big|_0^A =$$

$$= -4a^2\cos\frac{A}{2} + 4a^2 + \frac{8a^2}{3}\cos^3\frac{A}{2} - \frac{8a^2}{3}\cos^3 0 - 4a^2\cos\frac{A}{2} +$$

$$+ 4a^2\cos 0 = -8a^2\cos\frac{A}{2} + 8a^2 + \frac{8a^2}{3}\cos^3\frac{A}{2} - \frac{8a^2}{3} =$$

$$= \frac{16}{3}a^2 + \frac{8}{3}a^2\left(\cos^3\frac{A}{2} - 3\cos\frac{A}{2}\right)$$

punto a) $A = 2\pi$

$$I = \frac{16}{3}a^2 + \frac{8}{3}a^2(\cos^3\pi - 3\cos\pi) = \frac{16}{3}a^2 + \frac{8}{3}a^2(-1+3) = \frac{32}{3}a^2$$

$$\bar{y} = \frac{32}{3}a^2/8a = \boxed{\frac{4}{3}a} \Rightarrow B\left(\pi; \frac{4}{3}a\right)$$

punto b) $A = \pi$

$$I = \frac{16}{3}a^2 + \frac{8}{3}a^2\Big(\underbrace{\cos^3\frac{\pi}{2} - 3\cos\frac{\pi}{2}}_{=0}\Big) = \frac{16}{3}a^2$$

$$\bar{y} = \frac{16}{3}a^2/4a = \frac{4}{3}a \Rightarrow B\left(\frac{4}{3}a; \frac{4}{3}a\right)$$

③ $\gamma: \begin{cases} x = \alpha + a\cos t \\ y = \beta + a\sin t \end{cases}$ $a>0$ =47=

$0 \le t \le \frac{\pi}{2}$

γ è un filo omogeneo

④ $\gamma: x^{2/3} + y^{2/3} = a^{2/3}$; $a>0$ (nel piano)

dal punto $M(0,a)$ al punto $N(a,0)$ in senso orario.

$\delta(x,y) = x$

⑤ $\gamma: \vec{r}(t) = a\cos t \cdot \vec{i} + a\sin t \cdot \vec{j} + bt \cdot \vec{k}$ $a, b > 0$

$\delta(x,y,z) = x^2+y^2+z^2$

a) $0 \le t \le 2\pi$ b) $0 \le t \le \frac{\pi}{2}$

⑥ $\gamma: \begin{cases} x = e^t \cdot \cos t \\ y = e^t \cdot \sin t \\ z = e^t \end{cases}$

a) $\delta(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$ $0 \le t \le 1$

b) γ è un filo omogeneo $-\infty < t \le 0$

b) $\dot{x} = e^t\cos t - e^t\sin t = e^t(\cos t - \sin t)$

$\dot{y} = e^t\sin t + e^t\cos t = e^t(\sin t + \cos t)$

$\dot{z} = e^t$

$$ds = \sqrt{e^{2t}(\cos^2 t + \sin^2 t - \cancel{2\sin t\cos t} + \sin^2 t + \cos^2 t + \cancel{2\sin t\cos t} + 1)}\,dt =$$

$$= e^t\sqrt{1+1+1}\,dt = \sqrt{3}\cdot e^t\,dt$$

$$\ell = \int_{-\infty}^{0}\sqrt{3}\,e^t\,dt = \sqrt{3}\cdot e^t\Big|_{-\infty}^{0} = \sqrt{3}\,e^0 - \sqrt{3}\lim_{t\to-\infty} e^t =$$

$$= \sqrt{3} - \sqrt{3}\cdot 0 = \sqrt{3}$$

$$\int x\,ds = \sqrt{3}\int_{-\infty}^{0} e^{2t}\cdot\cos t\,dt = \sqrt{3}\,\mathrm{I}$$

$$\mathrm{I} = \underset{\text{per parti}}{\int_{-\infty}^{0} e^{2t}\cdot\cos t\,dt} = \frac{1}{2}\int_{-\infty}^{0}\cos t\,d(e^{2t}) =$$

$$= \frac{1}{2}e^{2t}\cdot\cos t\Big|_{-\infty}^{0} - \frac{1}{2}\int_{-\infty}^{0} e^{2t}\cdot(-\sin t)\,dt =$$

$$= \frac{1}{2}e^0\cdot\cos 0 - \frac{1}{2}\cdot\lim_{t\to-\infty} e^{2t}\cdot\cos t + \frac{1}{4}\int_{-\infty}^{0}\sin t\,d(e^{2t}) =$$

$$= \frac{1}{2} - \frac{1}{2}\cdot 0 + \frac{1}{4}\cdot e^{2t}\sin t\Big|_{-\infty}^{0} - \frac{1}{4}\int_{-\infty}^{0} e^{2t}\cdot\cos t\,dt =$$

perchè $\lim_{t\to-\infty} e^{2t} = 0$ e $|\cos t| \leq 1$

$-1 \leq \cos t \leq 1 \quad |\cdot e^{2t} > 0$

$-e^{2t} \leq e^{2t}\cos t \leq e^{2t}$

$\downarrow 0 \qquad\qquad \downarrow 0$

$= 49 =$

$$= \frac{1}{2} + \frac{1}{4} \underbrace{e^0 \sin 0}_{=0} - \frac{1}{4} \underbrace{\lim_{t\to-\infty} e^{2t}\sin t}_{=0} - \frac{1}{4} I$$

$$\Rightarrow I + \frac{1}{4} I = \frac{1}{2} \qquad \frac{5}{4} I = \frac{1}{2} \qquad \boxed{I = \frac{4}{5}\cdot\frac{1}{2} = \frac{2}{5}}$$

$$\Rightarrow \int x\,ds = \sqrt{3}\cdot\frac{2}{5} = \frac{2\sqrt{3}}{5} \Rightarrow \boxed{\overline{x} = \frac{2\sqrt{3}}{5} / \sqrt{3} = \frac{2}{5}}$$

$$\int_\gamma y\,ds = \int_{-\infty}^{0} e^t(\sin t)\cdot\sqrt{3}e^t\,dt = \sqrt{3}\int_{-\infty}^{0} e^{2t}\cdot\sin t\,dt = \sqrt{3}\,\boxed{I}$$

$$\boxed{I} = \int_{-\infty}^{0} e^{2t}\sin t\,dt = \frac{1}{2}\int_{-\infty}^{0}\sin t\,d(e^{2t}) = \frac{1}{2}e^{2t}\sin t\Big|_{-\infty}^{0} -$$

$$-\frac{1}{2}\int_{-\infty}^{0} e^{2t}\cdot\cos t\,dt = \frac{1}{2}\cdot\underbrace{e^0\sin 0}_{=0} - \frac{1}{2}\underbrace{\lim_{t\to-\infty} e^{2t}\sin t}_{=0} -$$

$$-\frac{1}{4}\int_{-\infty}^{0}\cos t\,d(e^{2t}) = -\frac{1}{4}e^{2t}\cos t\Big|_{-\infty}^{0} + \frac{1}{4}\int_{-\infty}^{0} e^{2t}(-\sin t)\,dt =$$

$$= -\frac{1}{4}e^0\cos 0 + \frac{1}{4}\underbrace{\lim_{t\to-\infty} e^{2t}\cos t}_{=0} - \frac{1}{4}\boxed{I}$$

$$\frac{5}{4}\boxed{I} = \frac{1}{4} \qquad \boxed{I} = \frac{1}{5} \qquad \int_\gamma y\,ds = \frac{\sqrt{3}}{5}$$

$$\boxed{\overline{y} = \frac{\sqrt{3}}{5} / \sqrt{3} = \boxed{\frac{1}{5}}}$$

$\Rightarrow$

$$\int_\gamma z\,ds = \sqrt{3}\int_{-\infty}^{0} e^t \cdot e^t\,dt = \frac{\sqrt{3}}{2} e^{2t}\Big|_{-\infty}^{0} = \frac{\sqrt{3}}{2} e^0 - \ldots = \ldots =$$

$$- \frac{\sqrt{3}}{2} \lim_{t\to-\infty} e^{2t} = \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2}$$

$$\boxed{\bar{z} = \frac{\sqrt{3}}{2} \Big| \sqrt{3} = \frac{1}{2}}$$

$$\Rightarrow \boxed{B\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{2}\right)}$$

IV Momento di inerzia

Se un filo ha la forma di una curva γ e densità lineare di massa data da $\delta = \delta(x,y,z)$, allora il suo momento di inerzia rispetto ad un asse (o a un punto) è dato da

$$I := \int_\gamma d^2 \cdot \delta\,ds \quad \text{dove}$$

$d = d(x,y,z)$ è la distanza di $M(x,y,z)$ $\forall$ punto dalla γ dall'asse (o dal punto)

Momenti di inerzia rispetto:

all'asse x $\quad d = \sqrt{y^2+z^2}$

all'asse y $\quad d = \sqrt{x^2+z^2}$

all'asse z $\quad d = \sqrt{x^2+y^2}$

al punto $M_0(x_0, y_0, z_0)$

$$d = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

all'origine $O(0,0,0)$

$$d = \sqrt{x^2+y^2+z^2}$$

(4) **Calcolare il momento di inerzia**

(1) $\gamma: \vec{r}(t) = R\cos t\,\vec{i} + R\sin t\,\vec{j} + t\,\vec{k}$ $\quad t \in [0, 4\pi]$. $R > 0$

rispetto all'asse Oz ; γ è omogenea

$$\dot{\vec{r}}(t) = (-R\sin t, R\cos t, 1)$$

$$ds = \sqrt{R^2\sin^2 t + R^2\cos^2 t + 1}\, dt = \sqrt{R^2+1}\, dt$$

$$d = \sqrt{x^2+y^2} = \sqrt{R^2\cos^2 t + R^2\sin^2 t} = \sqrt{R^2} = R$$

$$\delta(x,y,z) = \delta = \text{costante}$$

$$I = \int_\gamma \delta \cdot d^2\, ds = \int_\gamma \delta \cdot (x^2+y^2)\, ds = \int_\gamma \delta \cdot R^2\, ds =$$

$$x^2+y^2 = R^2\cos^2 t + R^2\sin^2 t = R^2$$

$$= R^2\delta \int_0^{4\pi} \sqrt{R^2+1}\, dt = R^2\sqrt{R^2+1}\cdot\delta\cdot t\Big|_0^{4\pi} =$$

$$= 4\pi \cdot \delta \cdot R^2 \cdot \sqrt{R^2+1}$$

=52=

(2) Un filo omogeneo, di densità lineare δ costante, è disposto lungo la curva di equazione $\vec{r}(t) = a(\cos t + t\sin t)\vec{i} + a(\sin t - t\cos t)\vec{j}$, $a > 0$; $t \in [0, 2\pi]$.

Calcolare il momento di inerzia rispetto all'asse z.

(3) $\gamma: \begin{cases} x^2+y^2+z^2=16 \\ z=2\sqrt{3} \\ \text{I ottante} \end{cases}$ $\quad \delta(x,y,z) = xy$

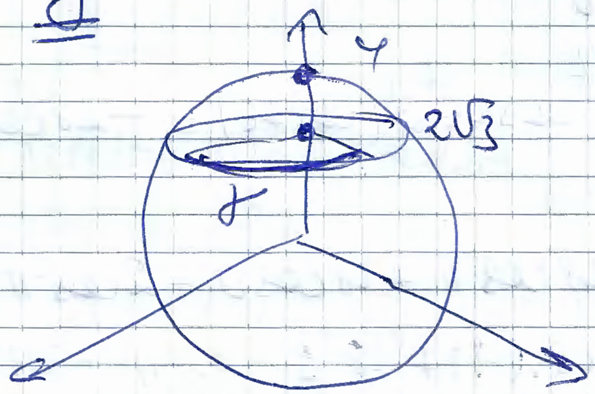
rispetto all'asse $\underline{y}$

$z = 2\sqrt{3} \Rightarrow$

$x^2+y^2+(2\sqrt{3})^2=16$

$x^2+y^2+12=16$

$\boxed{x^2+y^2=4}$



una circonferenza

$\gamma: \begin{cases} x=2\cos t \\ y=2\sin t \\ z=2\sqrt{3} \end{cases}$ $\quad t \in [0, \frac{\pi}{2}]$ (I ottante)

$\dot{x} = -2\sin t \quad \dot{y} = 2\cos t \quad \dot{z} = 0$

$ds = \sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 2dt$

$\delta = xy = 4\sin t\cos t = 2\sin 2t$

$d = \sqrt{x^2+z^2} = \sqrt{4\cos^2 t + 12} = \sqrt{4(3+\cos^2 t)}$

$$I = \int_\gamma d^2 \cdot \delta \, ds =$$

$$= \int_0^{\pi/2} 4(3+\cos^2 t)\cdot(2\sin 2t)\cdot 2dt =$$

$$= \frac{16}{2}\int_0^{\pi/2}\left(3+\frac{1+\cos 2t}{2}\right)\cdot \sin 2t \, d(2t)$$

$$= -\frac{8}{2}\int_0^{\pi/2}(7+\cos 2t)\, d(\cos 2t) = -28\int_0^{\pi/2} d(\cos 2t) -$$

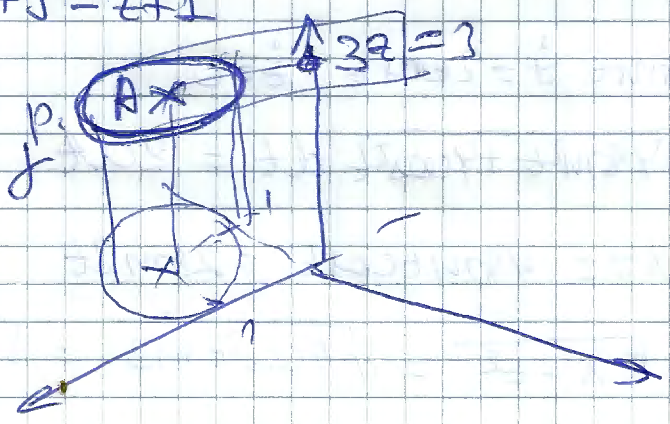
$$- 4\int_0^{\pi/2}\cos 2t \, d(\cos 2t) = -28\cos 2t \Big|_0^{\pi/2} - \frac{4\cos^2 2t}{2}\Big|_0^{\pi/2} =$$

$$= -28\cos\pi + 28\cos 0 - 2\cos^2\pi + 2\cos^2 0 =$$

$$= -28\cdot(-1) + 28\cdot 1 - 2\cdot(-1)^2 + 2\cdot 1^2 = 56 - \cancel{2} + \cancel{2} = 56$$

(4) $\gamma: \begin{cases} (x-1)^2+(y+1)^2 = 1 \\ z = 3 \end{cases}$ rispetto p. $A(1,-1,3)$

$$\delta(x,y,z) = x^2+y^2-z+1$$



= 57 =

$$\gamma: \begin{cases} x = 1+\cos t \\ y = -1+\sin t \\ z = 3 \\ t \in [0, 2\pi] \end{cases}$$

1) $\dot{x} = -\sin t \quad \dot{y} = \cos t \quad \dot{z} = 0$

$ds = \sqrt{\sin^2 t + \cos^2 t}\, dt = dt$

2) $d = \sqrt{(x-1)^2+(y+1)^2+(z-3)^2} =$

$= \sqrt{(\cancel{1}+\cos t-1)^2+(-1+\sin t+1)^2+(3-3)^2} =$

$= \sqrt{\cos^2 t+\sin^2 t} = \sqrt{1} = 1$ (questo risultato possiamo vedere direttamente, perché il punto A è il centro della circonferenza γ.

$\delta(x,y,z) = (1+\cos t)^2 + (-1+\sin t)^2 - 3+1 =$

$= \cancel{1}+2\cos t+\underline{\cos^2 t}\cancel{+1}-2\sin t+\underline{\sin^2 t}\cancel{-2} =$

$= 1+2\cos t-2\sin t$

$$I = \int_\gamma d^2 \cdot \delta\, ds = \int_0^{2\pi} 1^2 \cdot (1+2\cos t-2\sin t) \cdot dt =$$

$$= \int_0^{2\pi} dt + 2\underbrace{\int_0^{2\pi} \cos t\, dt}_{=0} - 2\underbrace{\int_0^{2\pi} \sin t\, dt}_{=0} = t\Big|_0^{2\pi} = 2\pi$$

⑤

$\gamma: \begin{cases} x = 3\cos t \\ y = 3\sin t \\ z = 4t \end{cases} \qquad t \in [0, 2\pi]$

= 55 =

$\delta(x,y,z) = x\,y\,z$ rispetto all'origine.