$\boxed{\text{II parte}}$

I Integrali di linea (curvilinei, di prima specie)

=1=

$\boxed{1}$ Sia $r : [a,b] \to \mathbb{R}^m$ ($m = 2$ o 3) un arco di curva regolare di sostegno γ e sia f una funzione a valori reali, definita in un sottoinsieme A di $\mathbb{R}^m$ contenente γ. Allora

$$\int_\gamma f\,ds = \int_a^b f(r(t)) \cdot |r'(t)|\,dt$$

I Calcolare l'integrale:

① $\int_\gamma \sqrt{1-y^2}\,ds \qquad \gamma : \vec{r}(t) = (\sin t, \cos t) \quad t \in [0,\pi]$

$$\sqrt{1-y^2} = \sqrt{1-\cos^2 t} = \sqrt{\sin^2 t} = |\sin t|$$

$$t \in [0,\pi] \Rightarrow \sin t \geq 0 \Rightarrow \sqrt{1-y^2} = \sin t$$

$$\vec{r}\,'(t) = (\cos t, -\sin t)$$

$$ds = \sqrt{\cos^2 t + \sin^2 t}\,dt = \sqrt{1}\,dt = dt$$

$$\Rightarrow \int_\gamma \sqrt{1-y^2}\,ds = \int_0^\pi \sin t\,dt = -\cos t \Big|_0^\pi =$$

$$= -\cos\pi + \cos 0 = -(-1)+1 = \underline{\underline{2}}$$

(2) $\int_\gamma \frac{x}{1+y^2}\,ds$ $\quad \gamma: \vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$

$t \in [0, \frac{\pi}{2}]$ $\quad$ Risp. $\frac{\pi}{4}$

(3) $\int_\gamma \sqrt{x^2+y^2}\,ds$ $\quad \vec{r}(t) = 2(\cos t + t\sin t)\vec{i} + 2(\sin t - t\cos t)\vec{j}$

$t \in [0, 2\pi]$ $\quad$ Risp. $\frac{4}{3}\left[(1+4\pi^2)^{3/2} - 1\right]$

(4) $\int_\gamma (x^2+y^2)\,ds$ $\quad \gamma: \begin{cases} x = a(t\sin t + \cos t) \\ y = a(\sin t - t\cos t) \end{cases}$

$0 \le t \le 2\pi$

(5) $\int_\gamma y^2\,ds$ $\quad \gamma: \begin{cases} x = t \\ y = e^t \end{cases}$ $\quad t \in [0; \log 2]$ $\quad$ Risp. $\frac{5^{3/2} - 2^{3/2}}{3}$

(6) $\int_\gamma x\,ds$ $\quad \gamma: \begin{cases} x = t \\ y = t^2 \end{cases}$ $\quad t \in [0, a],\ a > 0$

Risp. $\frac{1}{12}\left[(1+4a^2)^{3/2} - 1\right]$

$\dot{\vec{r}}(1, 2t)$ $\quad ds = \sqrt{1+4t^2}\,dt$

$$\int_\gamma x\,ds = \int_0^a t \cdot \sqrt{4t^2+1}\,dt = \frac{1}{2}\int_0^a (4t^2+1)^{\frac{1}{2}}\,d(t^2) =$$

$$= \frac{1}{2\cdot 4}\int_0^a (4t^2+1)^{\frac{1}{2}}\,d(4t^2+1) = \frac{1}{8}\,\frac{(4t^2+1)^{3/2}}{3/2}\Bigg|_0^a =$$

$$= \frac{1}{12}\left[(1+4a^2)^{3/2} - 1\right]$$

(7) $\int_\gamma \sqrt{z}\, ds$ $\quad \gamma(t): \begin{cases} x = \cos t \\ y = \sin t \\ z = t^2 \end{cases}$ $\quad t\in[0,\pi]$

Risp. $\frac{1}{12}\left[(1+4\pi^2)^{3/2}-1\right]$

(8) $\int_\gamma \frac{1}{x}\, ds$ $\quad \gamma: \vec{r}(t) = (t,\ t\log t)$ $\quad t\in[1,2]$

$$\frac{1}{x} = \frac{1}{t}$$

$$\dot{\vec{r}}(t) = \left(1,\ \log t + t\cdot\frac{1}{t}\right) = (1,\ \log t+1)$$

$$ds = \sqrt{1+(\log t+1)^2}\ dt$$

$$I = \int_\gamma \frac{1}{x}\, ds = \int_1^2 \frac{1}{t}\sqrt{1+(\log t+1)^2}\, dt = \int_1^2 \sqrt{1+(\log t+1)^2}\ d(\log t+1) =$$

Poniamo $\log t+1 = z \Rightarrow$ $\quad t=1 \quad z=1$

$\quad t=2 \quad z=\log 2+1$

$$\boxed{\begin{aligned} &\log t = z-1 \qquad t = e^{z-1} \\ &dt = e^{z-1}\, dz \end{aligned}}$$ in questo caso non ci serve

$$= \underbrace{\int_1^{\log 2+1} \sqrt{1+z^2}\ dz}_{\text{per parti}} = z\cdot\sqrt{1+z^2}\,\Big|_1^{\log 2+1} - \int_1^{\log 2+1} z\cdot\frac{1\cdot \cancel{2}z}{\cancel{2}\sqrt{1+z^2}}\, dz =$$

$$= (\log 2+1)\cdot\sqrt{(\log 2+1)^2+1} - \sqrt{2} - \int_1^{\log 2+1}\frac{(z^2+1)-1}{\sqrt{z^2+1}}\, dz =$$

$$= (\log 2+1)\sqrt{1+(\log 2+1)^2} - \sqrt{2} - \int_1^{\log 2+1}\sqrt{z^2+1}\, dz + \int_1^{\log 2+1}\frac{1}{\sqrt{z^2+1}}\, dz \Longrightarrow$$

$$I = (\log 2+1)\cdot\sqrt{1+(\log 2+1)^2} - \sqrt{2} - I + \ln\left|z+\sqrt{z^2+1}\right|\Big|_1^{\log 2+1} \Rightarrow$$

$$2I = (\log 2+1)\cdot\sqrt{1+(\log 2+1)^2} - \sqrt{2} + \log\left(\log 2+1+\sqrt{(\log 2+1)^2+1}\right) -$$
$$\log(1+\sqrt{2})$$

$$\Rightarrow I = \frac{1}{2}(\log 2+1)\cdot\sqrt{1+(\log 2+1)^2} - \frac{1}{2}\sqrt{2} + \frac{1}{2}\log\left(1+\log 2+\sqrt{(\log 2+1)^2+1}\right) - \frac{1}{2}\log(1+\sqrt{2})$$

(9) $\int_\gamma (x+z)\,ds \qquad \gamma: \vec{r}(t) = \left(t, \frac{3\sqrt{2}}{2}t^2, t^3\right) \qquad t\in[0,1]$

$$x+z = t+t^3 = t(t^2+1)$$

$$\dot{\vec{r}}(t) = (1, 3\sqrt{2}t, 3t^2)$$

$$ds = \sqrt{1+18t^2+9t^4}\;dt$$

$$\int_\gamma (x+z)\,ds = \int_0^1 t\,(t^2+1)\cdot\sqrt{9t^4+18t^2+1}\;dt =$$

$$= \frac{1}{2}\int_0^1 (t^2+1)\cdot\sqrt{9t^4+18t^2+1}\;d(t^2) =$$

poniamo $t^2 = z \qquad t=0 \quad z=0$
$\qquad\qquad\qquad\qquad t=1 \quad z=1$

$$= \frac{1}{2}\int_0^1 (z+1)\cdot\sqrt{9z^2+18z+1}\;dz =$$

Poniamo $z = p - \frac{18}{18} = p-1$

$dz = dp \qquad p = z+1 \qquad z=0 \quad p=1$
$\qquad\qquad\qquad\qquad\qquad z=1 \quad p=2$

$$= \frac{1}{2}\int_1^2 (p-1+1)\cdot\sqrt{9(p-1)^2+18(p-1)+1}\;dp =$$

$$= \frac{1}{2}\int_1^2 p\cdot\sqrt{9p^2-18p+9+18p-18+1}\,dp =$$

$$= \frac{1}{2}\int_1^2 p\cdot\sqrt{9p^2-8}\,dp = \frac{1}{4}\int_1^2 (9p^2-8)^{\frac{1}{2}}\,d(p^2) =$$

$$= \frac{1}{4\cdot 9}\int_1^2 (9p^2-8)^{\frac{1}{2}}\,d(9p^2-8) = \frac{1}{\underset{18}{36}}\cdot\frac{(9p^2-8)^{3/2}}{3/2_{1}}\Bigg|_1^2 =$$

$$= \frac{1}{54}\left[(36-8)^{3/2}-(1)^{3/2}\right] = \frac{1}{54}\left(\sqrt{28^3}-1\right) =$$

$$= \frac{1}{54}\left(\sqrt{28^2\cdot 4\cdot 7}-1\right) = \frac{1}{54}\left(56\sqrt{7}-1\right)$$

(10) a) $\int_\gamma y^2 ds$ b) $\int_\gamma y\, ds$ $\boxed{\text{Risp. } 0}$

$$\gamma: \begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases} \qquad 0 \le t \le 2\pi \qquad a>0$$

a) $y^2 = a^2(1-\cos t)^2 = \cancel{a^2\left(2\sin^2\frac{t}{2}\right)^2} = \cancel{4a^2\sin^4\frac{t}{2}}$

$\dot{x} = a(1-\cos t) \qquad \dot{y} = a\sin t$

$$ds = \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\,dt = a\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt =$$

$$= a\sqrt{2-2\cos t}\,dt = a\cdot\sqrt{2(1-\cos t)}\,dt =$$

$$= a\sqrt{2\cdot 2\sin^2\frac{t}{2}}\,dt = 2a\left|\sin\frac{t}{2}\right|dt$$

$0 \le t \le 2\pi \Rightarrow 0 \le \frac{t}{2} \le \pi \Rightarrow \sin\frac{t}{2} \ge 0 \Rightarrow$

$$\int_\gamma y^2 ds = \int_0^{2\pi} a^2(1-\cos t)^2 \cdot 2a \sin\frac{t}{2}\, dt =$$

$$= 2\cdot 2a^3 \int_0^{2\pi} \left[1-\left(2\cos^2\frac{t}{2}-1\right)\right]^2 \cdot \sin\frac{t}{2}\, d\left(\frac{t}{2}\right) =$$

$$= -4a^3 \int_0^{2\pi} \left(1-2\cos^2\frac{t}{2}+1\right)^2 d\left(\cos\frac{t}{2}\right) =$$

$$= -4a^3 \int_0^{2\pi} \left(4-8\cos^2\frac{t}{2}+4\cos^4\frac{t}{2}\right) d\left(\cos\frac{t}{2}\right) =$$

$$= -16a^3 \left(\cos\frac{t}{2} - \frac{2\cos^3\frac{t}{2}}{3} + \frac{\cos^5\frac{t}{2}}{5}\right)\Bigg|_0^{2\pi} =$$

$$= -16a^3\left(\cos\pi - \frac{2}{3}\cos^3\pi + \frac{1}{5}\cos^5\pi - \cos 0 + \frac{2}{3}\cos 0 - \frac{1}{5}\cos 0\right) =$$

$$= -16a^3\left(-1+\frac{2}{3}-\frac{1}{5}-1+\frac{2}{3}-\frac{1}{5}\right) = -16a^3\left(-2\overset{15}{}-\frac{2}{5}\overset{3}{}+\frac{4}{3}\overset{5}{}\right) =$$

$$= \frac{-16a^3(-30-6+20)}{15} = \frac{256a^3}{15}$$

(11) $\int_\gamma xy\, ds$ $\quad \gamma: \begin{cases} x = a\,\mathrm{ch}\,t \\ y = a\,\mathrm{sh}\,t \end{cases}$ $\quad a > 0$

$0 \le t \le t_0$

$x \cdot y = a^2\,\mathrm{ch}\,t \cdot \mathrm{sh}\,t$

$\dot{x} = a\,\mathrm{sh}\,t \quad \dot{y} = a\,\mathrm{ch}\,t$

$$ds = \sqrt{a^2 sh^2 t + a^2 ch^2 t}\, dt = a\sqrt{sh^2 t + ch^2 t}\, dt =$$

$$ch^2 t - sh^2 t = 1 \Rightarrow sh^2 t = \cosh^2 t - 1$$

$$ds = a\sqrt{2ch^2 t - 1}$$

$$\int_\gamma xy\, ds = \int_0^{t_0} a^2 sh\, t \cdot ch\, t \cdot a \cdot \sqrt{2ch^2 t - 1}\, dt =$$

$$= a^3 \int_0^{t_0} ch\, t \cdot \sqrt{2ch^2 t - 1}\, d(ch\, t) =$$

$$= \frac{a^3}{2} \int_0^{t_0} \sqrt{2ch^2 t - 1}\, d(ch^2 t) =$$

$$= \frac{a^3}{2.2} \int_0^{t_0} (2ch^2 t - 1)^{1/2}\, d(2ch^2 t - 1) =$$

$$= \frac{a^3}{4} \cdot \left.\frac{(2ch^2 t - 1)^{3/2}}{3/2}\right|_0^{t_0} = \frac{a^3}{6}\left[(2ch^2 t_0 - 1)^{3/2} - (2ch^2 0 - 1)\right] =$$

$$= \frac{a^3}{6}\left[ch^{3/2} 2t_0 - 1\right]$$

(12) $\int_\gamma x^2 y^2\, ds$ $\quad \gamma: \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$ $\quad 0 \le t \le 2\pi$

$R > 0$

(13) a) $\int_\gamma (x^2+y^2+z^2)\,ds$ Risp. $2\pi\sqrt{a^2+b^2}\left(a^2+\frac{4\pi^2 b^2}{3}\right)$

b) $\int_\gamma \frac{ds}{x^2+y^2+z^2}$

$\gamma: \vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\,\vec{k} \quad 0 \le t \le 2\pi \quad a, b > 0$

b) $x^2+y^2+z^2 = a^2\cos^2 t + a^2\sin^2 t + b^2t^2 = a^2+b^2t^2$

$\dot{x} = -a\sin t \quad \dot{y} = a\cos t \quad \dot{z} = b$

$ds = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}\,dt = \sqrt{a^2+b^2}\,dt$

$$\int_\gamma \frac{ds}{x^2+y^2+z^2} = \int_0^{2\pi} \frac{1}{a^2+b^2t^2}\sqrt{a^2+b^2}\,dt =$$

$$= \sqrt{a^2+b^2}\int_0^{2\pi} \frac{1}{a^2\left(1+\frac{b^2t^2}{a^2}\right)}\,dt = \frac{\sqrt{a^2+b^2}}{a^2}\cdot\frac{a}{b}\int_0^{2\pi}\frac{1}{1+\left(\frac{bt}{a}\right)^2}\,d\left(\frac{bt}{a}\right)$$

$$= \frac{\sqrt{a^2+b^2}}{ab}\operatorname{arctg}\frac{bt}{a}\Big|_0^{2\pi} = \frac{\sqrt{a^2+b^2}}{ab}\Big(\operatorname{arctg}\frac{2\pi b}{a} - \underbrace{\operatorname{arctg}0}_{=0}\Big) =$$

$$= \frac{\sqrt{a^2+b^2}}{ab}\operatorname{arctg}\frac{2\pi b}{a}$$

14) a) $\int_{\gamma_i} (2z - \sqrt{x^2+y^2})\,ds$ b) $\int_{\gamma_i} z\,ds$

c) $\int_{\gamma_i} \sqrt{x^2+y^2}\,ds$ $\gamma_1: \left| \begin{array}{l} x = t\cos t \\ y = t\sin t \\ z = t \end{array}\right.$ $0 \le t \le 2\pi$

$\gamma_2: \left| \begin{array}{l} x = t\cos t \\ y = t\sin t \\ z = t \end{array}\right.$ $0 \le t \le a$

$\sqrt{x^2+y^2} = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t|$ $t>0$

$\Rightarrow \sqrt{x^2+y^2} = t$

$2z - \sqrt{x^2+y^2} = 2t - t = t \Rightarrow$ a) = b) = c)

$\dot{x} = \cos t - t\sin t$ $\dot{y} = \sin t + t\cos t$ $\dot{z} = 1$

$ds = \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1}\,dt =$

$= \sqrt{t^2+2}\,dt$

a) b) c) $\int_{\gamma_2} z\,ds = \int_0^a t\cdot\sqrt{t^2+2}\,dt = \frac{1}{2}\int_0^a (t^2+2)^{1/2}\,d(t^2+2) =$

$= \frac{1}{2}\,\frac{(t^2+2)^{3/2}}{3/2}\Big|_0^a = \underline{\frac{1}{3}\left[(2+a^2)^{3/2} - 2^{3/2}\right]}$

$\Rightarrow$ La risposta $\int_{\gamma_1} z\,ds = \frac{1}{3}\left[(2+4\pi^2)^{3/2} - 2^{3/2}\right]$

(15) $\int_\gamma (x^2+y^2)^n ds$ $\quad \gamma: \begin{cases} x = a\cos t \\ y = a\sin t \end{cases} \quad 0 \le t \le 2\pi$

$a > 0 \quad n \in N$

$x^2+y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$

$(x^2+y^2)^n = a^{2n}$

$\dot{x} = -a\sin t \quad \dot{y} = a\cos t$

$ds = \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = \sqrt{a^2}\, dt = a\,dt$

$\int_\gamma (x^2+y^2)^n ds = \int_0^{2\pi} a^{2n} \cdot a\, dt = a^{2n+1} \int_0^{2\pi} dt =$

$= a^{2n+1} \cdot t \Big|_0^{2\pi} = 2\pi \cdot a^{2n+1}$

(16) a) $\int_\gamma \frac{z^2 ds}{x^2+y^2}$ $\quad$ б) $\int_\gamma \frac{x^2}{x^2+y^2} ds$

$\gamma: \begin{cases} x = a\cos t \\ y = a\sin t \\ z = at \end{cases} \quad 0 \le t \le 2\pi \quad a > 0$

б) $\frac{x^2}{x^2+y^2} = \frac{a^2\cos^2 t}{a^2\cos^2 t + a^2\sin^2 t} = \frac{a^2\cos^2 t}{a^2} = \cos^2 t$

$\dot{x} = -a\sin t \quad \dot{y} = a\cos t \quad \dot{z} = a$

$ds = \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2}\, dt = \sqrt{2a^2}\, dt = a\sqrt{2}\, dt$

$$\int_\gamma \frac{x^2}{x^2+y^2}\,ds = \int_0^{2\pi} a\sqrt{2}\cdot\cos^2 t\,dt = a\sqrt{2}\int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt =$$ $= |1| =$

$$= \frac{a\sqrt{2}}{2}\int_0^{2\pi} dt + \frac{a\sqrt{2}}{2.2}\int_0^{2\pi}\cos 2t\,d(2t) =$$

$$= \frac{a\sqrt{2}}{2}t\Big|_0^{2\pi} + \frac{a\sqrt{2}}{4}\sin 2t\Big|_0^{2\pi} = \frac{\cancel{2}\pi a\sqrt{2}}{\cancel{2}} + \frac{a\sqrt{2}}{4}\underbrace{(\sin 4\pi - \sin 0)}_{=0} =$$

$$= \pi a\sqrt{2}$$

(17) $\int_\gamma xz\,ds$ $\quad \gamma: \begin{cases} x = \ln(1+t^2) \\ y = 2\operatorname{arctg} t - t \\ z = t \end{cases} \quad 0 \le t \le 1$

$$xz = t\cdot\ln(1+t^2)$$

$$\dot{x} = \frac{1}{1+t^2}\cdot 2t \qquad \dot{y} = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2} \qquad \dot{z} = 1$$

$$ds = \sqrt{\frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} + 1}\,dt = \sqrt{\frac{4t^2+1-2t^2+t^4+1+2t^2+t^4}{(1+t^2)^2}}\,dt$$

$$= \sqrt{\frac{2t^4+4t^2+2}{(t^2+1)^2}}\,dt = \sqrt{\frac{2(t^4+2t^2+1)}{(t^2+1)^2}}\,dt = \sqrt{\frac{2(t^2+1)^2}{(t^2+1)^2}}\,dt =$$

$$= \sqrt{2}\,dt$$

$$\int_\gamma x\,z\,ds = \sqrt{2}\int_0^1 t\cdot \ln(1+t^2)\,dt = \frac{\sqrt{2}}{2}\int_0^1 \ln(1+t^2)\,d(t^2+1) =$$

Poniamo $t^2+1 = x$ $\quad t=0 \quad x=1$
$\quad t=1 \quad x=2$

$$= \frac{\sqrt{2}}{2}\int_1^2 \ln x\,dx = \frac{\sqrt{2}}{2}\,x\ln x\Big|_1^2 - \frac{\sqrt{2}}{2}\int_1^2 \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx =$$

per parti

$$= \frac{\sqrt{2}}{2}\cdot 2\ln 2 - \frac{\sqrt{2}}{2}\cdot 1\cdot \underset{=0}{\ln 1} - \frac{\sqrt{2}}{2}\,x\Big|_1^2 =$$

$$= \sqrt{2}\ln 2 - \frac{\sqrt{2}}{2}\cdot 2 + \frac{\sqrt{2}}{2}\cdot 1 = \sqrt{2}\ln 2 - \frac{\sqrt{2}}{2} =$$

$$= \frac{\sqrt{2}}{2}(2\ln 2 - 1) = \frac{\sqrt{2}}{2}(\ln 4 - 1)$$

(II) Calcolare l'integrale:

(1) $\int_\gamma (x-y)\,ds$ dove γ è il segmento OA $O(0,0)$; $A(4,3)$

γ: OA $\begin{cases} \ni O(0,0) \\ \parallel \overrightarrow{OA}(4,3) \end{cases}$ $\gamma: \begin{cases} x=4t \\ y=3t \end{cases}$ $0\le t\le 1$

$x-y = 4t-3t = t$

$\dot{x}=4 \quad \dot{y}=3 \qquad ds=\sqrt{16+9}\;dt = 5dt$

$$\int_\gamma (x-y)\,ds = \int_0^1 t\cdot 5dt = 5\frac{t^2}{2}\Big|_0^1 = \frac{5}{2}$$

(2) $\int_\gamma \frac{ds}{x-y}$ γ è il segmento AB della retta $g: y=\frac{x}{2}-2$ $A(0,-2)$; $B(4,0)$

$g: x=2y+4$ $y=t \Rightarrow x=2t+4$

AB: $\begin{cases} x=2t+4 \\ y=t \end{cases}$ $-2\le t\le 0$

$y_A=-2 \Rightarrow t=-2$

$y_B=0 \Rightarrow t=0$

$x-y = 2t+4-t = t+4$

$\dot{x}=2 \quad \dot{y}=1 \qquad ds=\sqrt{4+1}\;dt=\sqrt{5}\,dt$

$$\int_\gamma \frac{ds}{x-y} = \int_{-2}^{0}\frac{\sqrt{5}\,dt}{t+4} = \sqrt{5}\int_{-2}^{0}\frac{1}{t+4}\,d(t+4) = \sqrt{5}\ln|t+4|\Big|_{-2}^{0} =$$

$= \sqrt{5}\ln 4 - \sqrt{5}\ln 2 = \sqrt{5}\ln 2$ //

③ $\int_\gamma (x+y)\,ds$

a) γ è il segmento AB $\quad A(-1,-3)$; $B(2,3)$

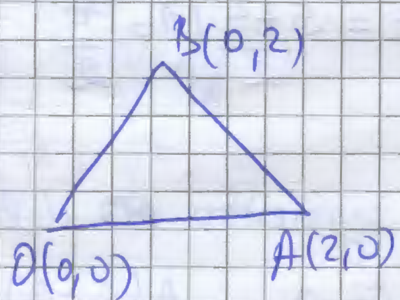
b) γ è la frontiera del dominio D: ΔAOB $\quad A(1,0)$; $O(0,0)$; $B(0,1)$

Risp. $\sqrt{2}+1$

c) γ è la frontiera del dominio D: ΔAOB $\quad A(2,0)$; $O(0,0)$; $B(0,2)$

d) γ è la frontiera del dominio D: $\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2 \end{cases}$

c)



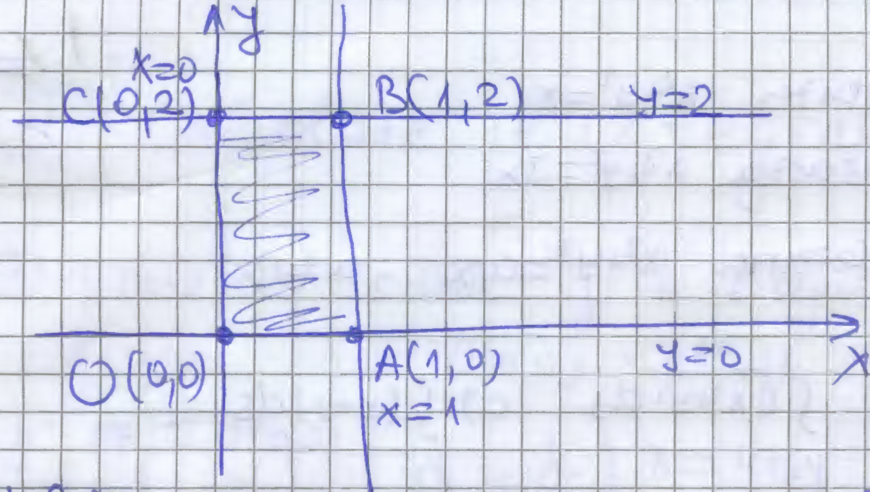
$OA \left| \begin{array}{l} \exists\, O(0,0) \\ \parallel \overrightarrow{OA}(2,0) \end{array} \right. \quad \begin{cases} x = 2t \\ y = 0 \end{cases} \quad 0 \le t \le 1 \quad \boxed{x+y=2t} \quad \dot{x}=2 \ \ \dot{y}=0 \quad \boxed{ds = 2\,dt}$

$OB \left| \begin{array}{l} \exists\, O(0,0) \\ \parallel \overrightarrow{OB}(0,2) \end{array} \right. \quad \begin{cases} x = 0 \\ y = 2t \end{cases} \quad 0 \le t \le 1 \quad \boxed{x+y=2t} \quad \dot{x}=0 \ \ \dot{y}=2 \quad \boxed{ds = 2\,dt}$

$AB \left| \begin{array}{l} \exists\, A(2,0) \\ \parallel \overrightarrow{AB}(-2,2) \end{array} \right. \quad \begin{cases} x = 2-2t \\ y = 2t \end{cases} \quad 0 \le t \le 1 \quad \boxed{x+y=2} \quad \dot{x}=-2 \ \ \dot{y}=2 \quad \boxed{ds = 2\sqrt{2}\,dt}$

$$\int_\gamma (x+y)\,ds = \int_0^1 2t \cdot 2\,dt + \int_0^1 2t \cdot 2\,dt + \int_0^1 2 \cdot 2\sqrt{2}\,dt =$$

$$= \frac{4t^2}{2}\Big|_0^1 + \frac{4t^2}{2}\Big|_0^1 + 4\sqrt{2}\,t\Big|_0^1 = 2 + 2 + 4\sqrt{2} = 4(\sqrt{2}+1)$$



OA: $y=0 \Rightarrow$ OA $\begin{cases} x=t \\ y=0 \end{cases}$ $\quad \boxed{x+y=t}$

$\dot{x}=1 \quad \dot{y}=0$

$0 \le t \le 1 \quad \boxed{ds=dt}$

AB: $\begin{cases} x=1 \\ y=t \end{cases}$ $\quad \boxed{x+y=t+1}$

$\dot{x}=0 \quad \dot{y}=1 \quad \boxed{ds=dt}$

$0 \le t \le 2$

BC: $\begin{cases} x=t \\ y=2 \end{cases}$ $\quad \boxed{x+y=t+2}$

$\dot{x}=1 \quad \dot{y}=0 \quad \boxed{ds=dt}$

$0 \le t \le 1$

OC: $\begin{cases} x=0 \\ y=t \end{cases}$ $\quad \boxed{x+y=t}$

$\dot{x}=0 \quad \dot{y}=1 \quad \boxed{ds=dt}$

$0 \le t \le 2$

$$\int_\gamma (x+y)\,ds = \int_0^1 t\,dt + \int_0^2 (t+1)\,dt + \int_0^1 (t+2)\,dt + \int_0^2 t\,dt.$$

(4) $\int_{\gamma_i} (x^2+y^2)\,ds \qquad \gamma_1$: il segmento AB $\;A(1,-1,2)\;\; B(0,0,0)$

γ_2: la circonferenza $x^2+y^2=R^2$ (nel piano)

(5) γ_1: la circonferenza $x^2+y^2=x$

γ_2: la circonferenza $x^2+y^2=3x$

γ_3: la circonferenza $x^2+y^2=ax$, $a\neq 0$

a) $\int_{\gamma_1} \sqrt{x^2+y^2}\, ds$ b) $\int_{\gamma_3} \sqrt{x^2+y^2}\, ds$ c) $\int_{\gamma_3} (x-y)\, ds$

d) $\int_{\gamma_2} (x-y)\, ds$

b) γ_3: $x^2-ax+y^2=0$

$$\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

I caso $a>0$

il centro $C\left(\frac{a}{2},0\right)$

$R=\frac{a}{2}$

γ_3: $\begin{cases} x=\frac{a}{2}+\frac{a}{2}\cos t \\ y=\frac{a}{2}\sin t \end{cases}$

$0\leq t\leq 2\pi$

$x^2+y^2=ax=a\left(\frac{a}{2}+\frac{a}{2}\cos t\right)=$

$=\frac{a^2}{2}(1+\cos t)$

$\dot{x}=-\frac{a}{2}\sin t$ $\dot{y}=\frac{a}{2}\cos t$

$ds=\sqrt{\frac{a^2}{4}(\sin^2 t+\cos^2 t)}\,dt=$

II caso $a<0$

il centro $C\left(\frac{a}{2},0\right)$

il raggio $R=\frac{-a}{2}$

γ_3: $x=\frac{a}{2}-\frac{a}{2}\cos t$

$y=-\frac{a}{2}\sin t$

$0\leq t\leq 2\pi$

$x^2+y^2=ax=a\left(\frac{a}{2}-\frac{a}{2}\cos t\right)=$

$=\frac{a^2}{2}(1-\cos t)$

$\dot{x}=\frac{a}{2}\sin t$ $\dot{y}=-\frac{a}{2}\cos t$

$ds=\sqrt{\frac{a^2}{4}(\sin^2 t+\cos^2 t)}\,dt=\frac{|a|}{2}dt=$

$=-\frac{a}{2}\,dt$

$$\int_{\gamma_3} \sqrt{x^2+y^2}\, ds = \int_0^{2\pi} \sqrt{\frac{a^2}{2}(1+\cos t)} \cdot \frac{a}{2}\, dt$$

$$= \frac{a^2}{2\sqrt{2}} \int_0^{2\pi} \sqrt{2\cos^2 \frac{t}{2}}\, dt =$$

$$= \frac{a^2 \cdot \sqrt{2}}{2\sqrt{2}} \int_0^{2\pi} \left|\cos \frac{t}{2}\right| dt =$$

$0 \le t \le 2\pi \quad 0 \le \frac{t}{2} \le \pi$

$t \in [0, \pi] \quad \left|\cos \frac{t}{2}\right| = \cos \frac{t}{2}$

$t \in [\pi, 2\pi] \quad \left|\cos \frac{t}{2}\right| = -\cos \frac{t}{2}$

$$= \frac{a^2}{2}\left[\int_0^{\pi} \cos \frac{t}{2}\, dt - \int_{\pi}^{2\pi} \cos \frac{t}{2}\, dt\right] =$$

$$= \frac{a^2}{2}\left[2\sin \frac{t}{2}\Big|_0^{\pi} - 2\sin \frac{t}{2}\Big|_{\pi}^{2\pi}\right] =$$

$$= \frac{a^2}{2}\left[2.1 - 0 - 2.0 + 2.1\right] =$$

$$= \frac{4a^2}{2} = \underline{\underline{2a^2}}$$

$$\int_{\gamma_3} \sqrt{x^2+y^2}\, ds = = |J| =$$

$$= \int_0^{2\pi} \sqrt{\frac{a^2}{2}(1-\cos t)} \cdot \left(\frac{-a}{2}\right) dt =$$

$$= |a| \cdot \left(\frac{-a}{2}\right) \int_0^{2\pi} \sqrt{\frac{2\sin^2 \frac{t}{2}}{2}}\, dt =$$

$$= -a \cdot \frac{(-a)}{2} \int_0^{2\pi} \left|\sin \frac{t}{2}\right| dt =$$

$0 \le t \le 2\pi \quad 0 \le \frac{t}{2} \le \pi$

$\Rightarrow \sin \frac{t}{2} \ge 0 \Rightarrow \left|\sin \frac{t}{2}\right| = \sin \frac{t}{2}$

$$= \frac{a^2}{2} \int_0^{2\pi} \sin \frac{t}{2}\, dt =$$

$$= \frac{a^2}{2} \cdot \left(-2\cos \frac{t}{2}\Big|_0^{2\pi}\right) = \frac{a^2}{2}\left(-2.(-1) + 2.1\right) =$$

$$= \frac{4a^2}{2} = \underline{\underline{2a^2}}$$

---

(6) $\int_{\gamma} xy\, ds$ a) γ: $\left|\begin{array}{l} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ \text{I quadrante} \end{array}\right.$ l'ellisse b) γ è la circonferenza $\left|\begin{array}{l} x^2+y^2 = R^2 \\ \text{III quadrante} \end{array}\right.$

a) $\left|\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right.$ $\quad xy = ab\sin t\cos t$

$0 \le t \le \frac{\pi}{2}$ $\quad \dot{x} = -a\sin t \quad \dot{y} = b\cos t$

$$ds = \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$$

$$\int_\gamma xy\,ds = \int_0^{\pi/2} (ab \sin t \cos t)\cdot\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt =$$

$$= \int_0^{\pi/2} ab\cdot \sin t\cdot \sqrt{a^2\sin^2 t + b^2(1-\sin^2 t)}\,d(\sin t) =$$

$$= \frac{ab}{2}\int_0^{\pi/2}\sqrt{(a^2-b^2)\sin^2 t + b^2}\,d(\sin^2 t) =$$

$$= \frac{ab}{2(a^2-b^2)}\int_0^{\pi/2}\left[(a^2-b^2)\sin^2 t+b^2\right]^{1/2} d\left[(a^2-b^2)\sin^2 t + b^2\right]$$

$$= \frac{ab}{\cancel{2}(a^2-b^2)}\cdot \frac{\left[(a^2-b^2)\sin^2 t+b^2\right]^{3/2}}{\frac{3}{\cancel{2}}}\Bigg|_0^{\pi/2} =$$

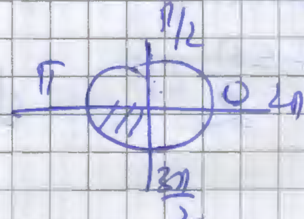
$$= \frac{ab}{3(a^2-b^2)}\left[(a^2-b^2+b^2)^{3/2} - (b^2)^{3/2}\right] = \frac{ab}{3(a^2-b^2)}\left((a^2)^{3/2}-(b^2)^{3/2}\right) =$$

$$= \frac{ab(a^3-b^3)}{3(a^2-b^2)} = \frac{ab\cancel{(a-b)}(a^2+ab+b^2)}{3\cancel{(a-b)}(a+b)} =$$

γ è l'ellisse $\Rightarrow a \neq b \Rightarrow a-b \neq 0$

$$= \frac{ab(a^2+ab+b^2)}{3(a+b)}$$

b) $\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$



$\pi < t < \frac{3\pi}{2}$

(7) $\int_\gamma (x^{4/3} + y^{4/3})\,ds$ $\quad \gamma$: è l'astroide $x^{2/3} + y^{2/3} = a^{2/3}$ $a > 0$

$$\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \\ 0 \le t \le 2\pi \end{cases}$$

$$x^{4/3} + y^{4/3} = (x^{2/3})^2 + (y^{2/3})^2 =$$

$$(a^{2/3} \cdot \cos^2 t)^2 + (a^{2/3} \sin^2 t)^2 = a^{4/3}(\sin^4 t + \cos^4 t)$$

$$\dot{x} = 3a\cos^2 t(-\sin t)$$

$$\dot{y} = 3a\sin^2 t \cos t$$

$$ds = \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\,dt =$$

$$= 3a\sqrt{\cos^2 t \sin^2 t}\,dt = 3a|\sin t||\cos t|\,dt$$

$$\int_\gamma (x^{4/3} + y^{4/3})\,ds = \int_0^{2\pi} 3a|\sin t|\,|\cos t| \cdot a^{4/3}(\sin^4 t + \cos^4 t)\,dt =$$

$$= 3a^{7/3}\int_0^{2\pi}\left[(\sin^2 t + \cos^2 t)^2 - 2\sin^2 t\cos^2 t\right]|\sin t| \cdot |\cos t|\,dt$$

$$= 3a^{7/3}\int_0^{2\pi}(1 - 2\sin^2 t\cos^2 t)|\sin t| \cdot |\cos t|\,dt =$$

$$= 3a^{7/3}\int_0^{\pi/2}(1 - 2\sin^2 t\cos^2 t)\sin t \cdot \cos t\,dt +$$

$$3a^{7/3}\int^{\pi}(1 - 2\sin^2 t\cos^2 t)\sin t\,(-\cos t)\,dt +$$

$$+3a^{7/3}\int_{\pi}^{3\pi/2}(1-2\sin^2t\cos^2t)(-\sin t)(-\cos t)\,dt+$$

$$+3a^{7/3}\int_{3\pi/2}^{2\pi}(1-2\sin^2t\cos^2t)\cdot(-\sin t)\cdot\cos t)\,dt=$$

$$=3a^{7/3}\Big[\int_0^{\pi/2}(1-2\sin^2t\cos^2t)\sin t\cos t\,dt+\int_{\pi}^{3\pi/2}(1-2\sin^2t\cos^2t)\sin t\cos t\,dt$$

$$-\int_{\pi/2}^{\pi}(1-2\sin^2t\cos^2t)\sin t\cos t\,dt-\int_{3\pi/2}^{2\pi}(1-2\sin^2t\cos^2t)\sin t\cos t\,dt\Big]=I$$

$$\int(1-2\sin^2t\cos^2t)\sin t\cos t\,dt=\int(1-2\sin^2t\cos^2t)\sin t\,d(\sin t)=$$

$$=\frac{1}{2}\int\left[1-2\sin^2t(1-\sin^2t)\right]\cdot d(\sin^2t)=$$

$$=\frac{1}{2}\int\left(2(\sin^2t)^2-2\sin^2t+1\right)d(\sin^2t)=$$

$$=\frac{1}{2}\cdot\left[\frac{2(\sin^2t)^3}{3}-\frac{2\cdot(\sin^2t)^2}{2}+\sin^2t\right]=$$

$$=\frac{1}{3}\sin^6t-\frac{1}{2}\sin^4t+\frac{1}{2}\sin^2t$$

$$I=3a^{7/3}\left[\frac{1}{3}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}+\frac{1}{3}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\left(-\frac{1}{3}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}-\frac{1}{3}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}\right)\right]=$$

$$=3a^{7/3}\left[\frac{2}{3}+\frac{2}{3}\right]=3a^{7/3}\cdot\frac{4}{3}=4a^{7/3}$$