Formality of arrangement complements

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Arrangements: topology, combinatorics, and stability
Pisa, Feb. 3, 2016

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Co-arrangements (and more generally bi-arrangements) are fascinating objects, between geometry, topology, combinatorics and number theory.

Classical Hodge theory

Definition

A *pure Hodge structure* of weight w is the datum of a finite-dimensional $\mathbb{Q}$-vector space H and a direct sum decomposition

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=w} H^{p,q} \qquad \text{with} \qquad \overline{H^{p,q}} = H^{q,p} \text{ for every } p,q.$$

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The *Hodge filtration* $F^p H_{\mathbb{C}} = \bigoplus_{r \geqslant p} H^{r,s}$ satisfies $H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{w-p+1} H_{\mathbb{C}}}$.

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Theorem (Hodge '41)

Let X be a smooth and projective complex variety. Then for every w, $H^w(X) := H^w(X, \mathbb{Q})$ carries a functorial pure Hodge structure of weight w.

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Application

The *odd* Betti numbers of a smooth projective variety are *even*.

Example : S a compact Riemann surface of genus g, $\dim H^1(S) = 2g$.

$$H^1(S, \mathbb{C}) = H^{0,1} \oplus H^{1,0} \ , \ \overline{H^{0,1}} = H^{1,0}.$$

Mixed Hodge structures

Definition (Deligne)

A *mixed Hodge structure* is the datum of a f. d. $\mathbb{Q}$-vector space H together with

– an increasing filtration $W_\bullet H$, the *weight filtration*;

– a decreasing filtration $F^\bullet H_\mathbb{C}$, the *Hodge filtration*;

such that for every w, $F^\bullet$ induces a pure Hodge structure of weight w on $\mathrm{gr}_w^W H = W_w H / W_{w-1} H$.

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– Mixed Hodge structures form a $\mathbb{Q}$-linear abelian category. Key lemma:

Let $f : H \to H'$ be a morphism of mixed Hodge structures. If H and H' are *pure of different weights*, then $f = 0$.

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$\rightsquigarrow$ tool to prove that spectral sequences degenerate.

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Example

- $H^1(\mathbb{C}^*)$ is a pure Hodge structure of weight 2.
- For S a compact Riemann surface of genus g and $p_0, \ldots, p_r \in S$, $H^1(S - \{p_0, \ldots, p_r\})$ has weights 1 and 2, of respective dimensions $2g$ and r.

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Leitmotiv

All "natural" constructions are compatible with mixed Hodge structures.

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What is important

- There is a canonical weight filtration on the cohomology of complex varieties.
- Morally, every natural construction is compatible with this filtration.
- No non-zero morphism between pure cohomology groups of different weights.

Purity implies formality

Definition (Quillen, Sullivan)

A differentiable manifold U is *formal* if the dga $(E^\bullet(U), d)$ of differential forms on U is quasi-isomorphic to its cohomology:

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Applications: computation of (rational) homotopical invariants of U. Examples:

(a) Smooth projective complex varieties (Deligne–Griffiths–Morgan–Sullivan ’75).

(b) Complements of hyperplane arrangements (Brieskorn ’72).

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Theorem (D. '16)

Let U be a smooth complex variety such that one of the following conditions is satisfied:

(1) *for every k, $H^k(U)$ is pure of weight k;*

(2) *for every k, $H^k(U)$ is pure of weight $2k$.*

Then U is formal.

$(1) \Longrightarrow (a)$ and $(2) \Longrightarrow (b), (c)$.

Ingredients for the proof

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Definition

A meromorphic differential form on X with poles along D is *logarithmic* if in local coordinates $(z_1, \ldots, z_n)$ in which $D = \{z_1 = 0\} \cup \cdots \cup \{z_r = 0\}$ it can be written as a linear combination of

$$\eta \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}}$$

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Logarithmic forms and mixed Hodge theory (Griffiths, Deligne)

- The hypercohomology of the complex of sheaves of logarithmic forms is $H^\bullet(U, \mathbb{C})$.
- The Hodge and weight filtrations on $H^\bullet(U, \mathbb{C})$ come from filtrations on these complexes of sheaves.

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- We conclude with a general proposition, which does not use purity (but is a consequence of Deligne's mixed Hodge theory).

Proposition

Every global logarithmic form is closed : $d = 0$ in $\Omega^\bullet(X,D)$.

Generalizes the "maximum principle": every global holomorphic function on the compact complex manifold X is constant.

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The Orlik–Solomon algebra

$$A_\bullet(\mathcal{A}) := E_\bullet(\mathcal{A}) \,/\, I_\bullet(\mathcal{A})$$

- $E_\bullet(\mathcal{A})$ is the exterior algebra on one generator e_i for each hyperplane;
- $I_\bullet(\mathcal{A})$ is the homogeneous ideal generated by
  $\sum_{s=1}^{k}(-1)^{s-1}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_s}}\wedge\cdots\wedge e_{i_k}$ $\quad$ ($\{i_1,\ldots,i_k\}$ a set of dependent hyperplanes).

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Theorem (Brieskorn '72, Orlik–Solomon '80)

There is an isomorphism of algebras : $H^\bullet(\mathbb{C}^n-\mathcal{A})\cong A_\bullet(\mathcal{A})$.

Stratum of $\mathcal{A}$ = intersection of some of the hyperplanes in $\mathcal{A}$. Notation $|S| := \mathrm{codim}(S)$.

There is a decomposition $A_r(\mathcal{A}) = \displaystyle\bigoplus_{|S|=r} A_S(\mathcal{A})$ into S-local components.

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- Generalizes the notion of a normal crossing divisor (*coordinate* hyperplanes).
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The Orlik–Solomon datum

For S a stratum of $\mathcal{A}$ (a connected component of intersections of irreducible components), one can define the S-local Orlik–Solomon vector space $A_S(\mathcal{A})$ by working in local coordinates around any point of S.

- Differentials $A_S(\mathcal{A}) \to A_{S'}(\mathcal{A})$.
- Product morphisms $A_S(\mathcal{A}) \otimes A_{S'}(\mathcal{A}) \to A_T(\mathcal{A})$.
- $\dim(A_S(\mathcal{A})) = (-1)^{|S|}\mu(X, S)$ (μ: the Möbius function of the poset of strata).

Warning: in general, $\bigoplus_S A_S(\mathcal{A})$ is not a dga.

The Orlik–Solomon spectral sequence

Theorem (Looijenga '92, Bibby '13, D. '15)

Let X be a smooth complex variety, not necessarily projective, and $\mathcal{A}$ a hypersurface arrangement in X. Then there is a spectral sequence

$$E_1^{-p,q} = \bigoplus_{|S|=p} H^{q-2p}(S)(-p) \otimes A_S(\mathcal{A}) \implies H^{-p+q}(X - \mathcal{A})$$

in the category of mixed Hodge structures.

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- This is (up to a *décalage*) the Leray spectral sequence of the inclusion $X - \mathcal{A} \hookrightarrow X$.
- The *Tate twist* $(-p)$ shifts the weight filtration by $2p$.
- X projective $\implies E_1^{-p,q}$ is pure of weight $q \implies$ spectral sequence degenerates at E_2.

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Corollary (D. '16)

If for every stratum S and every integer k, $H^k(S)$ is pure of weight $2k$, then:

(1) *for every integer k, $H^k(X - \mathcal{A})$ is pure of weight $2k$;*

(2) *$X - \mathcal{A}$ is formal.*

(in this case, the spectral sequence degenerates at E_1)

Complements of hyperplane arrangements are formal

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– The proof of the "purity implies formality" theorem extends to the case X smooth projective, $\mathcal{A}$ a hypersurface arrangement in X (D. '15)

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- One can take $X = \mathbb{P}^n(\mathbb{C})$ and view $\mathcal{A}$ as a projective arrangement.
- The *global* logarithmic differential forms are those written in affine coordinates as linear combinations of

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  where the f_i's are the linear equations of the hyperplanes.
- One then recovers Brieskorn's original proof:

$$(\Omega^\bullet(\mathbb{P}^n(\mathbb{C}), \mathcal{A}), d = 0) \overset{\sim}{\hookrightarrow} (E^\bullet_{\mathbb{C}}(\mathbb{P}^n(\mathbb{C}) - \mathcal{A}), d) .$$

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Definition

A *toric arrangement* $\mathcal{T}$ in $(\mathbb{C}^*)^n$ is a union of finitely many *hypertori* $\{z_1^{k_1} \cdots z_n^{k_n} = a\}$, with $k_i \in \mathbb{Z}$ and $a \in \mathbb{C}^*$.

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– De Concini–Procesi '05: proof for *unimodular* toric arrangements.

– Deshpande–Sutar '14: proof for *deletion-restriction type* toric arrangements.

– Callegaro–Delucchi '15: use of the Leray spectral sequence to determine the cohomology algebra of $(\mathbb{C}^*)^n - \mathcal{T}$.

Complements of toric arrangements are formal

Definition

A *toric arrangement* $\mathcal{T}$ in $(\mathbb{C}^*)^n$ is a union of finitely many *hypertori* $\{z_1^{k_1}\cdots z_n^{k_n}=a\}$, with $k_i\in\mathbb{Z}$ and $a\in\mathbb{C}^*$.

Theorem (D. '16)

Let $\mathcal{T}$ be a toric arrangement in $(\mathbb{C}^*)^n$. Then the complement $(\mathbb{C}^*)^n-\mathcal{T}$ is formal.

- De Concini–Procesi '05: proof for *unimodular* toric arrangements.
- Deshpande–Sutar '14: proof for *deletion-restriction type* toric arrangements.
- Callegaro–Delucchi '15: use of the Leray spectral sequence to determine the cohomology algebra of $(\mathbb{C}^*)^n-\mathcal{T}$.

- For a proof involving logarithmic forms, in the spirit of Brieskorn's proof for hyperplane arrangements: need for a compactification

$$(\mathbb{C}^*)^n-\mathcal{T}\hookrightarrow X$$

  with X smooth and projective, and $X-((\mathbb{C}^*)^n-\mathcal{T})$ a hypersurface arrangement.
- One should recover the De Concini–Procesi and Deshpande–Sutar proofs.

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Arrangement $\mathcal{A} \rightsquigarrow$ cohomology of the complement $H^\bullet(X - \mathcal{A})$.
This is the cohomology of $j_*\mathbb{Q}_{X-\mathcal{A}} \in D(X, \mathbb{Q})$, where $j : X - \mathcal{A} \hookrightarrow X$.

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Poincaré–Verdier duality

$$\mathbb{D}_X(j_*\mathbb{Q}_{X-\mathcal{A}}) \cong j_!\mathbb{Q}_{X-\mathcal{A}}[2n] \qquad (n := \dim_{\mathbb{C}}(X)) .$$

If X is *projective*, then

$$H^k(X, \mathcal{A}) \cong (H^{2n-k}(X - \mathcal{A}))^\vee .$$

In general, there is no such duality, and $H^\bullet(X, \mathcal{A})$ is a *new cohomological invariant*.

Hyperplane co-arrangements (affine setting)

Proposition (D.)

Let $\mathcal{A}$ be an essential affine hyperplane arrangement in $\mathbb{C}^n$.

- *We have natural isomorphisms $H^k(\mathbb{C}^n, \mathcal{A}) \cong (H_k(A_\bullet(\mathcal{A}), d))^\vee$.*
- *This is zero for $k \neq n$.*
- *Thus, the dimension of $H^n(\mathbb{C}^n, \mathcal{A})$ is*

$$(-1)^n \chi(A_\bullet(\mathcal{A})) = (-1)^n \chi(\mathcal{A}, 1) \ .$$

$\chi(\mathcal{A}, q)$ is the *characteristic polynomial* of $\mathcal{A}$:

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Theorem (Zaslavsky '75)

If $\mathcal{A}$ is an essential affine real arrangement, then the number of bounded connected components of the real complement $\mathbb{R}^n - \mathcal{A}$ is $(-1)^n \chi(\mathcal{A}, 1)$.

(This is not surprising!)

Toric co-arrangements

Observations

– The relative cohomology groups $H^\bullet((\mathbb{C}^*)^n, \mathcal{T})$ have natural mixed Hodge structures, which are generally *not pure*: several non-zero graded quotients

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For $a \neq b \in \mathbb{C}^*$, the mixed Hodge structure on $H^1(\mathbb{C}^*, \{a, b\})$ knows about the number
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Proposition (D.)

The dimension of $\mathrm{gr}_0^W H^n((\mathbb{C}^*)^n, \mathcal{T})$ *is* $(-1)^n \chi(\mathcal{T}, 1)$.

Bi-arrangements of cubical-toric type

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- All strata are !-strata: co-arrangement.
- There is a duality inside bi-arrangements, which exchanges $*$-strata and !-strata.
- To each bi-arrangement of hypersurfaces, one associates its *motive*, which is a relative cohomology group.

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Underlying arrangement in $\mathbb{C}^n$ contains the hyperplanes $x_i = 0, 1$ (all corresponding strata are !-strata) and hypertori (the rest of the strata are $*$-strata). The mixed Hodge structure on the corresponding motive knows about numbers such as

$$\int_{[0,1]^n} \frac{dx_1 \cdots dx_n}{1 - x_1 \cdots x_n} = \zeta(n) .$$

Arithmetic properties $\Longleftarrow$ Algebraic/combinatorial properties of the motives.