Multipoint Seshadri constants and explicit Kähler packings of projective complex manifolds.

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Outline

Notation

- X is a projective, complex manifold of dimension n and $p_1, \ldots, p_k$ are distinct points of X.
- L is an ample line bundle on X.
- ω is a Kähler form on X such that $[\omega] = c_1(L) \in H^2(X, \mathbb{Z})$.
- $\pi\colon \tilde{X} = Bl_{p_1,\ldots,p_k}(X) \to X$ the blow up of X at the points $p_1, \ldots, p_k$.
- $\pi^{-1}(p_i) = E_i$ the exceptional divisor corresponding to p_i.

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Definition (Multi-point Seshadri constant)

Using the above notation the $k-$point Seshadri constant

$$\epsilon(X, L; p_1, \dots, p_k) = \text{ sup } \{\epsilon \in \mathbb{Q} > 0 \ : \ \pi^* L - \epsilon \sum_{i=1}^{k} E_i \ \text{ is } \mathbb{Q} - \text{ample } \}.$$

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Exercise

Calculate $\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); p)$.

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Conjecture (Nagata's Conjcture)

Let $p_1, \ldots, p_k$ be points of $\mathbb{P}^2$ in general position then for $k \geq 9$ the multipoint Seshadri constant is given as

$$\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); p_1 \ldots, p_k) = \frac{1}{\sqrt{k}}.$$

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Conjecture (Nagata's conjecture classic version)

Let $p_1, \ldots, p_k$ be points of $\mathbb{P}^2$ in general position and $m_1, \ldots, m_k$ be positive integers. Then for $k \geq 9$ any curve $C \subset \mathbb{P}^2$ of degree d which passes each point p_i with multiplicity m_i satisfies

$$d \geq \frac{1}{\sqrt{k}} \sum_{i=1}^{k} m_i p_i.$$

Remark

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- *Nagata's conjecture is known to be true if* $k = n^2$.
- *It is know to be false if* $k \leq 9$ *with* $k \neq 4$ *or* $k \neq 9$.
- *The conjecture is also false if the points are not chosen in general position. An example here is that all the points are chosen to lie on a line.*

The case when $k = n^2$ was proved and used by Nagata in 1959 in his construction of a counter example to Hilbert's 14th problem.

Kähler packings

Let $B_r^{2n}(0)$ denote a ball in $\mathbb{R}^{2n}$ and ω_{std} the standard Euclidean form. Then $(B_r(0), \omega_{std}) \hookrightarrow (\mathbb{C}^n, \omega_{std})$ such that if $(z_1, \ldots, z_n)$ are coordinates of $\mathbb{C}^n$:

- $B_r^{2n}(0) = \{(z_1, \ldots, z_n) \colon |z_1|^2 + \ldots + |z_n|^2 \leq r^2\}$
- $\omega_{std} = \frac{i}{2\pi} dz_1 \wedge d\bar{z}_1 + \ldots dz_n \wedge d\bar{z}_n$.

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Definition

(X, ω) a Kähler manifold, then a Kähler packing of k disjoint, flat balls of radius r is a holomorphic embedding

$$\phi = (\phi_1, \dots, \phi_k) : \amalg_{i=1}^k (B_r^{2n}(0), \omega_{std}) \hookrightarrow (X, \omega)$$

such that there exists a Kähler form $\omega' \in [\omega]$ with $\phi^* \omega' = \omega_{std}$ and for each point p_i we have $\phi_i(0) = p_i$.

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Definition (The Kähler packing constant)

The k-ball Kähler packing constant

$$\gamma_k = sup\{r \in \mathbb{R} \colon \exists \text{ a Kähler packing of } k \text{ disjoint, flat balls of radius } r.\}$$

Statment of main theorem

Theorem (Eckl '14, Witt-Nystrom '15, Trussiani '18, — '18)

Let X be a Kahler manifold of dimension n, L a ample line bundle on X and $p_1, \ldots, p_k$ be distinct points of X. Then if ω is a Kähler form on X such that $[\omega] = c_1(L)$, we have that

$$\gamma_k(X, \omega; p_1, \ldots, p_k) = \epsilon(X, L; p_1, \ldots, p_k).$$

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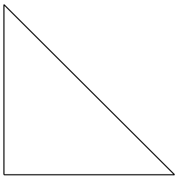
- In 2015 David Witt-Nystrom proved the case when X is of dimension n but only blown up at a single point.
- In 2018 Trussiani and myself proved (independently) that the theorem holds for a projective, complex manifold of any dimension blown up at any number of points.

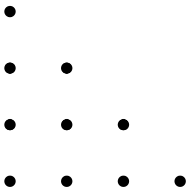
Kähler packings and polytopes

Toric example

- $X = \mathbb{P}^2$
- L a line.

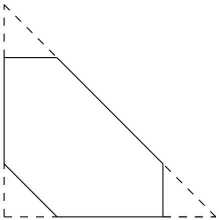
Moment polytope of KL $\supset$ { lattice points } $\leftrightarrow$ Basis of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(kL))$.





$X^a Y^b Z^{k-a-b}$

Blowing up $\mathbb{P}^2$ at $p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1]$ gives the moment polytope of $\pi^* kL - m\sum_{i=1}^{3} E_i$



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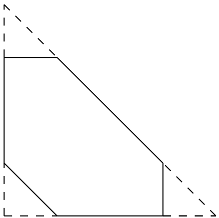

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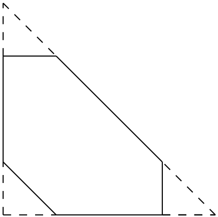

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Idea: Construct a family of Kähler forms ω_δ on X from sections of kL such that contributions of sections not present in $|\pi^* L - m\sum E_i|$ vanish if δ tends to 0.

Difficulty: writing down the ω_δ exactly as we need to determine all the coefficients of monomials $X^a Y^b Z^{k-a-b}$ and how they behave under the limit of $\delta \to 0$.
The embedding $\phi_{i,\delta} \colon B_r^4(0) \hookrightarrow \mathbb{P}^2$ is given by $(z_1, z_2) \mapsto [1 : \delta z_1, \delta z_2]$.
When $\delta = 0$ the limit does not exist so we take δ very small and glue in.
This choice of embedding and Kähler form satisfies the definition of a Kähler packing so we are done.

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Remark

This provides a nice interpretation of the cut off triangles of the moment polytope as we find that they are the shadows under the moment map of the glued in balls.

The End