Lecture 3: $\mathcal{W}$-algebras

1. Overwiew on $\mathcal{W}$-algebras

Recall: The four fundamental physical theories
and the corresponding algebraic structures.

$$
\begin{tikzcd}
PVA - CFT \arrow[d, "\text{Zhu}"] \arrow[r, dotted, bend left, "\text{quantization}"] & QFT - VA \arrow[l, "\text{cl.limit}"] \arrow[d, "\text{Zhu}"'] \\
PA - CM \arrow[u, dotted, bend left, "\text{affiniz.}"] \arrow[r, dotted, bend right, "\text{quantization}"'] & QM - AA \arrow[l, "\text{cl.limit}"'] \arrow[u, dotted, bend right, "\text{affiniz.}"']
\end{tikzcd}
$$

Note: The classical limit corresponds to taking the *associated graded* of a *filtered* AA or VA.
The Zhu algebra [Zhu96] is an AA (resp. PA) associated to a *positive energy* VA (resp. PVA).

$\mathcal{W}$**-algebras** provide a rich family of examples, parametrized by $\mathfrak{g}$: simple Lie algebra, and $f \in \mathfrak{g}$: nilpotent, appearing in all 4 fundamental aspects:

$$
\begin{tikzcd}
\mathcal{W}^{cl}_z(\mathfrak{g}, f) \arrow[rr, dotted, bend left, "\text{quantization}"] \arrow[d, "\text{Zhu}"] & & \mathcal{W}_k(\mathfrak{g}, f) \arrow[ll, "\text{cl.limit}"] \arrow[d, "\text{Zhu}"'] \\
\mathcal{W}^{cl,fin}(\mathfrak{g}, f) \arrow[u, dotted, bend left, "\text{affiniz.}"] \arrow[rr, dotted, bend right, "\text{quantization}"'] & & \mathcal{W}^{fin}(\mathfrak{g}, f) \arrow[ll, "\text{cl.limit}"'] \arrow[u, dotted, bend right, "\text{affiniz.}"']
\end{tikzcd}
$$

They were introduced separately and played important roles in different areas of math. Only later it became fully clear the relations between them.

Classical finite $\mathcal{W}$-algebra: $\mathcal{W}^{cl,fin}(\mathfrak{g}, f)$

- Poisson algebra of functions on the *Slodowy slice* [Slo80].

Finite $\mathcal{W}$-algebra: $\mathcal{W}^{fin}(\mathfrak{g}, f)$

- First appearance in [Kos78,Lyn79]: $\mathcal{W}^{fin}(\mathfrak{g}, f^{pr}) \simeq Z(U(\mathfrak{g}))$
- [Pre02]: general definition; connection to repr. theory of simple finite-dim Lie alg's, and to theory of primitive ideals.

Classical $\mathcal{W}$-algebra: $\mathcal{W}^{cl}_z(\mathfrak{g}, f)$

- Introduced, for principal f, in [DrSok85] (as PA of functions on M^∞), to study KdV-type integrable equations.
- In the 90's: gener.'s [deGroot,Delduc,Feher, Miramontes...]
- formalization within theory of PVA's: [DS,Kac,Valeri,2013]

$\mathcal{W}$-algebra: $\mathcal{W}_k(\mathfrak{g}, f)$

- First example: Zamolodchikov $\mathcal{W}_3$ (1985) ($= \mathcal{W}(\mathfrak{sl}_3, f^{\mathrm{pr}})$). ("non-linear" ∞-dim Lie algebra, extending Virasoro).
- [Fei.Fre.90], [Kac,Roan,Wak.03] general construction via a quantization of the Drinfeld-Sokolov reduction. Application to representation theory of superconformal algebras.

The links among the four appearances of $\mathcal{W}$-algebras are more recent:

- [Gan,Gin,2002]: finite $\mathcal{W}$-algebra as a quantization of the Slodowy slice: $\mathcal{W}^{fin}(\mathfrak{g}, f) \overset{\text{cl.limit}}{\longrightarrow} \mathcal{W}^{cl,fin}(\mathfrak{g}, f)$.
- [DS,Kac,2006], (indep. [Ara07]): the (H-twisted) Zhu algebra of the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to the finite $\mathcal{W}$-algebra $\mathcal{W}^{fin}(\mathfrak{g}, f)$. (Hence, their categories of irreducible representations are equivalent.)

2. The Poisson structure on the Slodowy slice.

Set up:

- $\mathfrak{g}$: simple Lie algebra.
- $(\cdot\,|\,\cdot)$: non-degenerate symmetric invariant form.
- Identify $\Phi : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, $\qquad a \mapsto (a, \cdot)$
- $f \in \mathfrak{g}$: a nilpotent element. By the Jacobson-Morozov Theorem, we can include it in an $\mathfrak{sl}_2$-triple $(f, h = 2x, e)$.

The corresponding Slodowy slice is the affine space:

$$\mathcal{S} = \Phi(f + \mathfrak{g}^e) \subset \mathfrak{g}^*$$

Claim: $\mathcal{S} \subset \mathfrak{g}^*$ is a Poisson submanifold.

Exercise 1: If $\xi \in \mathcal{S}$, the sympl. form ω_ξ on the sympl. leave $\mathrm{Ad}^* G(\xi)$ restricts to a sympl. form on $T_\xi \mathcal{S} \cap T_\xi \mathrm{Ad}^* G(\xi)$
Exercise 2: $\mathcal{S}$ intersects transversally the symplectic leafs: $T_\xi \mathcal{S} \cap T_\xi \mathrm{Ad}^* G(\xi) = T_\xi \mathfrak{g}^*$ (i.e. $\mathrm{ad}^*(\mathfrak{g})(\xi) \cap \Phi(\mathfrak{g}^e) = \mathfrak{g}^*$).
The **Claim** follows by these two exercises

In order to *quantize* the theory, we shall describe the Slodowy slice $\mathcal{S}$ as a Hamiltonian reduction of the Poisson manifold $\mathfrak{g}^*$ (with the Kirillov-Kostant Poisson bracket).

Recall: the general procedure of Hamiltonian reduction:

$$\text{Ham.red.}(M, \chi, N) = \mu^{-1}(\chi)/N$$

where N is a Lie group with a Hamiltonian action on M and momentum map $\mu : M \to \mathfrak{n}^*$, and $\chi \in \mathfrak{n}^*$ is $\mathrm{ad}^* N$-invariant.

Set up:

- $\mathrm{ad}\, x$-eignespace decomposition: $\mathfrak{g} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_i$
- On $\mathfrak{g}_{\frac{1}{2}}$ we have the non-degenerate skewsymmetric form

$$\omega(u, v) = (f|[u, v])$$

and we let $\ell \subset \mathfrak{g}_{\frac{1}{2}}$ be a maximal isotropic subspace.

Exercise 3: Check that ω is a non-degenerate skewsymmetric form on $\mathfrak{g}_{\frac{1}{2}}$.

- Consider the nilpotent subalgebra $\mathfrak{n} = \ell \oplus \mathfrak{g}_{\geq 1} \subset \mathfrak{g}$ (and the corresponding unipotent Lie group N).
- Consider the coadjoint action of N on $\mathfrak{g}^*$.

Exercise 4: Prove that the coadjoint action of N on the Poisson manifold $\mathfrak{g}^*$ is a Hamiltonian action, with momentum map $\mu : \mathfrak{g}^* \to \mathfrak{n}^*$ given by restriction.
Exercise 5: The dual of the momentum map $\mu^* : \mathfrak{n} \to \mathfrak{g}$ is the inclusion map.

- Let $\chi = (f|\cdot)|_{\mathfrak{n}} \in \mathfrak{n}^*$.
- Its preimage via the momentum map is $\mu^{-1}(\chi) = \Phi(f + \mathfrak{n}^{\perp})$

Exercise 6: Check that $\chi([\mathfrak{n}, \mathfrak{n}]) = 0$. Use this to prove that χ is $\mathrm{ad}^* N$-invariant.

- Hence, we have the corresponding Hamiltonian reduction:

$$\mathrm{Ham.Red.}(\mathfrak{g}^*, N, \chi) = \mu^{-1}(\chi)/N = \Phi(f + \mathfrak{n}^{\perp})/N$$

Proposition [Gan, Ginzburg, 2001]: The adjoint action

$$N \times (f + \mathfrak{g}^e) \xrightarrow{\sim} f + \mathfrak{n}^\perp$$

is an isomorphism of affine varieties.

Exercise 7: Prove it.

Conclusion: It follows that

$$\text{Ham.Red.}(\mathfrak{g}^*, N, \chi) = \Phi(f + \mathfrak{n}^\perp)/N \simeq \Phi(f + \mathfrak{g}^e) = \mathcal{S}$$

(It is not hard to check that the Poisson structure is the same.)

By passing to the corresponding algebras of (polynomial) functions, we get the *Hamiltonian reduction* definition of the classical finite $\mathcal{W}$-algebra:

$$
\begin{aligned}
W^{cl,fin}(\mathfrak{g}, f) &= \mathbb{C}[\mathcal{S}] \\
&= \left(\mathbb{C}[\mathfrak{g}^*]/\mathbb{C}[\mathfrak{g}^*]\{f \text{ vanish. on } \mu^{-1}(\chi)\}\right)^{\operatorname{ad}\mu^*(\mathfrak{n})} \\
&= \left(S(\mathfrak{g})\Big/S(\mathfrak{g})\{n-(f|n)\}_{n\in\mathfrak{n}}\right)^{\operatorname{ad}\mathfrak{n}} = \mathcal{N}/\mathcal{I}
\end{aligned}
$$

where $\mathcal{N} = \left\{x \in S(\mathfrak{g}) \,\middle|\, \{\mathfrak{n}, x\} \subset \mathcal{I}\right\}$ and $\mathcal{I} = S(\mathfrak{g})\{n-(f|n)\}_{n\in\mathfrak{n}}$

3. The quantum finite $\mathcal{W}$-algebra via quantized Hamiltonian reduction

To define the *finite $\mathcal{W}$-algebra*, we want to quantize the *classical finite $\mathcal{W}$-algebra.*

First, we quantize the symmetric algebra $S(\mathfrak{g})$, by taking the universal enveloping algebra $U(\mathfrak{g})$.

Then, we quantize the Hamiltonian reduction of $S(\mathfrak{g})$, to get:

$$W^{fin}(\mathfrak{g}, f) = \Big(U(\mathfrak{g}) \Big/ U(\mathfrak{g})\{n - (f|n)\}_{n\in\mathfrak{n}}\Big)^{\mathrm{ad}\,\mathfrak{n}} = \mathcal{N}/\mathcal{I}$$

where $\mathcal{N} = \{x \in U(\mathfrak{g}) \,\big|\, [\mathfrak{n}, x] \subset \mathcal{I}\}$, and $\mathcal{I} = U(\mathfrak{g})\{n - (f|n)\}_{n\in\mathfrak{n}}$.

Exercise 8: $\mathcal{N}$ is a subalgebra of $U(\mathfrak{g})$, and $\mathcal{I}$ is its ideal. So, the quotient $\mathcal{N}/\mathcal{I}$ is a well defined algebra.

We want to see that, indeed, $W^{fin}(\mathfrak{g}, f)$ is a *quantization* of $W^{cl,fin}(\mathfrak{g}, f)$.

We define the following Kazhdan filtration of the universal enveloping algebra $U(\mathfrak{g})$: for $a \in \mathfrak{g}_i$, we let $\Delta(a) = 1 - i$ (we call this the "conformal weight" of a). Then, we let

$$F_n U(\mathfrak{g}) = \mathrm{Span}\left\{ a_1 \dots a_s \,\middle|\, \Delta(a_1) + \cdots + \Delta(a_s) \leq n \right\}$$

Exercise 9: We have: $\Delta([a, b]) = \Delta(a) + \Delta(b) - 1$. Hence, we have a filtration of the algebra $U(\mathfrak{g})$, and the associated graded is the Poisson algebra $S(\mathfrak{g})$.

Note: $n - (f|n)$ is "homogeneous" w.r.t. conf. weight. The Kazhdan filtration of $U(\mathfrak{g})$ induces a filtr on $\mathcal{W}^{fin}(\mathfrak{g}, f)$, and:

Proposition [Gan Ginzburg 01]: $\mathrm{gr}\, \mathcal{W}^{fin.}(\mathfrak{g}, f) \simeq \mathcal{W}^{cl.fin.}(\mathfrak{g}, f)$.

Exercise 10: prove it.

4. The quantum affine $\mathcal{W}$-algebra

The quantum affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is a vertex algebra.

It is not known how to define it via quantized Hamiltonian reduction. There is a cohomological definition, via the so called BRST cohomology.

It was first defined by [Feigin and Frenkel.1990] for even nilpotent f, and generalized by [Kac, Roan and Wakimoto, 2003].

To every vertex algebra (conformal, positive energy) V, there is associated an associative algebra called its Zhu algebra $Zhu(V)$, which describes its representations. In the sense that there is an equivalence of categories

$$\{\text{positive energy repr's of } V\} \leftrightarrow \{\text{fin.dim. repr's of Zhu}(V)\}$$

We proved in [D.S., Kac 2006] that $Zhu\mathcal{W}^k(\mathfrak{g}, f) \simeq \mathcal{W}^{fin.}(\mathfrak{g}, f)$.

5. The classical affine $\mathcal{W}$-algebra via "affine" Hamiltonian reduction

Set up. It is the same as before:
$\mathfrak{g}$: a semisimple Lie algebra. $(e, h, f) \in \mathfrak{g}$: an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.
$\mathfrak{g} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_i$: decomposition in eigenspaces of $\frac{1}{2}(\operatorname{ad} h)$.

Construction: (Affine Hamiltonian reduction)

- We start with the Affine PVA: $\mathcal{V}(\mathfrak{g}) = S(\mathbb{F}[\partial]\mathfrak{g})$, with

$$[a_\lambda b] = [a, b] + (a|b)\lambda\,, \;\; a, b \in \mathfrak{g}\,,$$

- Consider the differential algebra ideal $\langle a - (f|a) \rangle_{a \in \mathfrak{g}_{\geq 1}}$.
The quotient $\mathcal{V}(\mathfrak{g})/\langle a - (f|a) \rangle_{a \in \mathfrak{g}_{\geq 1}}$ is NOT a PVA.
- If we take invariants w.r.t. λ-action of the Lie conformal algebra $\mathbb{C}[\partial]\mathfrak{g}_{\geq \frac{1}{2}}$, we get a PVA.

Definition: The classical affine $\mathcal{W}$-algebra is

$$\mathcal{W}(\mathfrak{g}, f) = \big(\mathcal{V}(\mathfrak{g})/\langle a - (f|a)\rangle_{a\in\mathfrak{g}_{\geq 1}}\big)^{\mathrm{ad}_\lambda(\mathbb{F}[\partial]\mathfrak{g}_{\geq \frac{1}{2}})} = \mathcal{N}/\mathcal{I}$$

where

$$\mathcal{N} = \big\{x \in \mathcal{V}(\mathfrak{g}) \,\big|\, \{\mathfrak{g}_{\geq \frac{1}{2}}\,{}_\lambda x\} \subset \mathcal{I}[\lambda]\big\}$$

and

$$\mathcal{I} = \langle a - (f|a)\rangle_{a\in\mathfrak{g}_{\geq 1}} \qquad \text{(diff. alg. ideal)}$$

Exercise 11: Check that $\mathcal{N}$ is a Poisson vertex subalgebra of $\mathcal{V}(\mathfrak{g})$ and $\mathcal{I}$ is its idea. Hence, $\mathcal{W}(\mathfrak{g}, f)$ is a Poisson vertex algebra.

Structure Thm: as a differential algebra, $\mathcal{W}(\mathfrak{g}, f)$ is isomorphic to the algebra of differential polynoamials in finitely many variables w_i, $i = 1, \dots, \dim(\mathfrak{g}^f)$ (Premet's generators):

$$W(\mathfrak{g}, f) \simeq \mathbb{F}[w_i^{(n)} \mid {\scriptstyle i = 1, \dots, \dim(\mathfrak{g}^f) \atop n \in \mathbb{Z}_+}]$$

Note: the same is true for all other types of W-algebras.

Natural questions:

Problem 1: Find explicit formulas for generators $\{w_i\}_{i=1}^{\dim(\mathfrak{g}^f)}$.

Problem 2: Find explicit formulas for the λ-brackets among generators: $\{w_i{}_\lambda w_j\} \in \mathbb{F}[\lambda]W(\mathfrak{g}, f)$.

Problem 3: Construct an integrable hierarchy of Hamilt. eq's for the PVA structure of $\mathcal{W}(\mathfrak{g}, f)$.

Example / Exercise 12: $W(\mathfrak{sl}_2, f) \simeq \mathcal{V}(\mathrm{Vir})$; corresponding integrable hierarchy: KdV.

GOAL:

For a classical Lie algebra $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N, \mathfrak{so}_N, \mathfrak{sp}_N$ and arbitrary nilpotent $f \in \mathfrak{g}$ we have a new method, based on the notions of Adler type operators and generalized quasideterminants, which gives a complete answer to all three problems at the same time, for every nilpotent element f.

6. Lax equations

Definition [P. Lax 1968] Let $L = L(t)$, $P = P(t)$ be linear operators, depending on t. The corresponding Lax equation is

$$\frac{dL}{dt} = [P, L] \tag{1}$$

Usually, $L = \partial^n + \ldots$ (pseudodiff. operator) and $P = (L^{k/n})_+$.

Then: [*Lax "theorem"*] Equation (1) is integrable, and $\int \mathrm{Res}_\partial L^{k/n}$, $k \geq 1$, are integrals of motion in involution.

Example: Lax main example:

$$L = \partial^2 + u, \quad P = \partial^3 + 2u\partial + u'.$$

Then $[P, L] = u''' + uu'$, hence

$$\frac{dL}{dt} = [P, L] \Leftrightarrow \mathrm{KdV}: \ \frac{du}{dt} = u''' + uu'.$$

Main Issue (which hasn't been completely resolved to date): the Lax Equation (1) should be selfconsistent.

Example: consider the operators $L = \partial^3 + u,\ P = (L^{k/3})_+$. Then the Lax equation (1) for $k = 1$ is: $\frac{du}{dt_1} = u'$, but for $k = 2$ it is

$$\frac{du}{dt} = 2u'\partial + u''$$

which is inconsistent.

Examples of L for which the Lax equation (1) is self consistent:

1) $L = \partial^2 + u \quad \Rightarrow$ KdV hierarchy
2) $L = \partial^3 + u\partial + v \quad \Rightarrow$ Boussinesq hierarchy
3) $L = \partial^n + u_1\partial^{n-2} + \cdots + u_{n-1} \quad \Rightarrow$ n-th KdV hierarchy
4) $L = \partial + u\partial^{-1}v \quad \Rightarrow$ NLS hierarchy
5) $L = \partial^2 + u + v\partial^{-1}w \quad \Rightarrow$ Yajima-Oikawa hierarchy

For all these examples the Lax equation

$$\frac{dL}{dt_k} = [(L^{k/n})_+, L_n], \; k = 1, 2, ...$$

is an integrable hierarchy of Hamiltonian PDE, and

$$\int \mathrm{Res}_\partial L^{k/n}$$

are integrals of motion in involution.

Exercise 13: Check that for $L = \partial^2 + u$ and $P = (L^{\frac{3}{2}})_+$, the corresponding Lax equation is the KdV equation.

Main Goal: for each nilpotent $f \in \mathfrak{g}$, we construct a Lax operator $L(\partial)$, such that:

$$\frac{dL}{dt_k} = \left[\left(L^{k/p_1} \right)_+ , \ L \right] \quad (k \in \mathbb{Z}) \tag{2}$$

is an integrable hierarchy of compatible evolution equations, with the infinitely many integrals of motion in involution:

$$\int \mathrm{Res}_\partial \ \mathrm{Tr} \ L^{k/p_1} \quad (k \in \mathbb{Z})$$

***Moreover*:**

1) $L(\partial)$ contains all generators of the W-algebra $W(\mathfrak{gl}_N, f)$;

2) we have an Adler identity for the λ-brackets;

3) all Lax eq.s (2) are Hamiltonian w.r.t. the PVA $W(\mathfrak{gl}_N, f)$

(This solves all our 3 problems at the same time!)

7. First ingredient: Adler type operators

Definition. Let $\mathcal{V}$ be a PVA with λ-bracket $\{\cdot\,{}_\lambda\,\cdot\}$.
$A(\partial) \in \mathrm{Mat}_{N\times N}\,\mathcal{V}((\partial^{-1}))$ is of Adler type (w.r.t. $\{\cdot\,{}_\lambda\,\cdot\}$) if:

$$\{A_{ij}(z)_\lambda A_{hj}(w)\} = A_{hj}(w+\lambda+\partial)(z-w-\lambda-\partial)^{-1}(A_{ik})^*(\lambda-z)$$
$$- A_{hj}(z)(z-w-\lambda-\partial)^{-1}A_{ik}(w)\,.$$

Example. $\mathcal{V} = \mathcal{V}(\mathfrak{gl}_N)$ (with $\{a_\lambda b\} = [a,b] + (a|b)\lambda$). Then:

$$E + \partial 1 = \begin{pmatrix} e_{11}+\partial & e_{21} & \dots & e_{N1} \\ e_{12} & e_{22}+\partial & \dots & e_{N2} \\ \vdots & & \ddots & \vdots \\ e_{1N} & & \dots & e_{NN}+\partial \end{pmatrix} \in \mathrm{Mat}_{N\times N}\mathcal{V}(\mathfrak{gl}_N)$$

is of Adler type. (Notation: $\{e_{ij}\}$ = standard basis of $\mathfrak{gl}_N$.)

Exercise 14: Check this.

Adler type operators are very useful to construct Integrable Systems!

Theorem.[D.S., Kac, Valeri,'15] Let: $\mathcal{V}$ a PVA; $A(\partial)$ an operator of Adler type; $K \geq 1$ s.t. $A(\partial)^{\frac{1}{K}}$ exists. Let

$$\textstyle\int h_n = \int \mathrm{Res}_\partial \mathrm{Tr}(A(\partial)^{\frac{n}{K}}) \in \mathcal{V}/\partial\mathcal{V}\,, \;\; n \in \mathbb{Z}_+$$

They are pairwise in involution:

$$\{\textstyle\int h_m, \int h_n\} = 0 \;\forall m, n$$

Hence, integrable hierarchy of Hamiltonian eq's:

$$\frac{du}{dt_n} = \{\textstyle\int h_n, u\}$$

This hierarchy is equivalently written in Lax form:

$$\frac{dA(\partial)}{dt_n} = \left[\left(A(\partial)^{\frac{n}{K}}\right)_+, A(\partial)\right]\,, \quad n \in \mathbb{Z}_+$$

Idea: To construct integrable systems of Hamiltonian equations, we want Adler operators.

Question: How do we construct new Adler operators?
(So far, only one example: $E + \partial 1 \in \mathrm{Mat}_{N\times N}(\mathfrak{gl}_N)$.)

Answer: we use (generalized) *quasideterminant*

8. Second ingredient: (generalized) quasideterminants

Definition. [Gelfand,Gelfand, Retakh, '05] V: assoc. alg.; $A = (a_{ij}) \in \mathrm{Mat}_{N\times N} V$. The (i,j)-quasideterminant of A is (if $\exists$):

$$|A|_{ij} = a_{ij} - R_i^j (A^{ij})^{-1} C_j^i$$

where: R_i^j = i-th row of A without j-entry; C_j^i = j-th column of A without i-entry; A^{ij} = matrix A without row i and column j.

Exercise 15: $|A|_{ij} = \left(\text{entry } (ji) \text{ of } A^{-1}\right)^{-1}$, (if both inverses exist).

Definition. [DS,Kac,Valeri, '15] Let $I \in \mathrm{Mat}_{N\times M} \mathbb{F}$ and $J \in \mathrm{Mat}_{M\times N} \mathbb{F}$ with $\mathrm{rk}(JI) = M$. The (I,J)-generalized quasideterminant of A is (if it exists):

$$|A|_{IJ} = (JA^{-1}I)^{-1}$$

Theorem/Observation. If $A(\partial)$ is of Adler type for $\mathcal{V}$, then any its generalized quasideterminant $|A(\partial)|_{I,J}$ is again of Adler type.

Exercise 16: prove it.

9) Construction of the Lax operator for $\mathcal{W}(\mathfrak{g}, f)$

Step 1:

Let $\psi : \mathfrak{g} \to \mathrm{End}\, V$ be a finite-dimensional representation of $\mathfrak{g}$ s.t. $(a|b) = \mathrm{tr}_V\ \psi(a)\psi(b)$ is non-degenerate.
Choose a basis $\{u_i\}_{i \in B}$ of $\mathfrak{g}$ and let $\{u^i\}_{i \in B}$ be the dual basis.

The associated ancestor Lax operator is

$$L_V(\partial) = \partial 1_V + \sum_{i \in B} u_i \psi(u^i) \quad \in \mathcal{V}(\mathfrak{g})[\partial] \otimes \mathrm{End}\, V$$

(It is independent of the choice of basis).

Step 2:

The descendant Lax operator $L_{V,f}(\partial)$ for the PVA $W(\mathfrak{g}, f)$ is constructed as follows:

Let $J : V \to V[\Delta]$ be the projection and $I : V[\Delta] \hookrightarrow V$ the inclusion ($\Delta =$ max eigenvalue for $\varphi(x)$).
Let $\rho : \mathcal{V}(\mathfrak{g}) \to V(\mathfrak{g})$ be the differential algebra homomorphism defined by:

$$\rho(a) = \pi_{\leq \frac{1}{2}}(a) + (f|a), \quad a \in \mathfrak{g}.$$

Then $L_{V,f}(\partial)$ is the generalized quasi-determinant:

$$L_{V,f}(\partial) = (J(\rho(L_V(\partial))^{-1} I))^{-1}$$

First Main Theorem

$\forall$ $\mathfrak{g}, V, f$, the descendant Lax operator $L_{V,f}(\partial)$ is an $r_1 \times r_1$ matrix pseudo-differential operator with leading term ∂^{p_1} and coefficients in $W(\mathfrak{g}, f)$:

$$L_{V,f}(\partial) = \partial^{p_1} 1_{r_1 \times r_1} + \ldots \in W(\mathfrak{g}, f)((\partial^{-1})) \otimes \operatorname{End} V[\Delta]$$

(**Note:** $L_{V,f}(\partial)$ encodes all generators of $W(\mathfrak{g}, f)$.)

Second Main Theorem

Let $\mathfrak{g} = \mathfrak{gl}_N$, ψ be its standard representation in $V = \mathbb{F}^N$, $f \in \mathfrak{g}$ nilpotent, associated to the partition $N = p_1 + \cdots + p_s$, ($p_1 \geq \cdots \geq p_s$) and let r_1 be the multiplicity of p_1.
Then $L_{V,f}(\partial)$ satisfies the following Adler identity (based on the famous Adler's map, 1979)

$$\{L(z)_\lambda L(w)\} =$$
$$\big(1 \otimes L(w+\lambda+\partial)\big) i_z (z-w-\lambda-\partial)^{-1} \big(L^*(\lambda-z) \otimes 1\big)\Omega$$
$$- \Omega\big(L(z) \otimes i_z(z-w-\lambda-\partial)^{-1} L(w)\big)$$

where i_z stands for the geometric series expansion for large z, and Ω is the permutation of factors.

Classical Lie algebras: A similar theorem holds for all classical Lie alg.s: $\mathfrak{sl}_N$, $\mathfrak{so}_N$, $\mathfrak{sp}_N$, with $V = \mathbb{F}^N$. [DSKV, 2018]

(**Note:** The Adler identity encodes all λ-brackets in $W(\mathfrak{g}, f)$.)

As we said, *Adler type operators* are automatically *Lax operators*, i.e. they produce an integrable hierarchy of Hamiltonian eq.s in Lax form [DSKV, 2015-18]. As a corollary:

Third Main Theorem

1) $\int h_n = \int Res_\partial \operatorname{Tr} L_{V,f}(\partial)^{\frac{n}{p_1}} \in W(\mathfrak{g},f)/\partial W$ *are Hamiltonian functionals in involution:*

$$\{\int h_m, \int h_n\} = 0 \quad \text{for all} \quad m,n$$

2) *We thus get an integrable hierarchy of Hamiltonian equations for* $W(\mathfrak{g},f)$

$$\frac{du}{dt_n} = \{\textstyle\int h_n, u\}$$

3) *This hierarchy can be written in Lax form:*

$$\frac{dL_{V,f}(\partial)}{dt_n} = [L_{V,f}(\partial)_+^{\frac{n}{p_1}}, L_{V,f}(\partial)]$$

Historical Remark

- Drinfeld and Sokolov [1985] constructed an integrable Hamiltonian hierarchy of PDE for any simple Lie algebra $\mathfrak{g}$ and its principal nilpotent element f, using Kostant's cyclic elements.
- In [DSKV, 2015] we extended their method for any simple Lie algebra $\mathfrak{g}$ and its nilpotent elements f of "semisimple type".
  (There are very few such elements in classical $\mathfrak{g}$, but about $\frac{1}{2}$ of nilpotents in exceptional $\mathfrak{g}$ are such: 13 out of 20 in E_6, 21 out of 44 in E_7, 27 out of 69 in E_8, 11 out of 15 in F_4, 3 out of 4 in G_2 [Elashvili-Kac-Vinberg, 2013]).
- The Lax operator method generalizes, in case of classical $\mathfrak{g}$, the DS hierarchy to arbitrary nilpotent $f \in \mathfrak{g}$.

10. Examples

Recall: In $\mathfrak{gl}_N$ the nilpotent orbits are parametrized by partitions $N = p_1 + p_2 + \cdots + p_s$, with $p_1 \geq p_2 \geq \cdots \geq p_s$.

Example 1: $2 = 2$

it corresponds to the KdV hierarchy, the simplest equation being:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} \qquad (1895)\ (1877)$$

The first important discovery in theory of integrable systems: KdV is integrable! [Gardner-Green-Kruskal-Miura, 1967]

Example 2: $2 = 1 + 1$

it corresponds to the NLS hierarchy (=AKNS) in two variables u and v, the simplest equation being

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku^2 v \\ \frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial x^2} v - kuv^2 \end{cases} \qquad (1964)$$

Example 3: $3 = 3$: corresponds to the Boussinesq hierarchy, the simplest equation being the Boussinesq equations

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} = \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} \end{cases} \qquad (1872)$$

Example 4: $3 = 1 + 1 + 1$: corresponds to the 3 wave equation.

Example 5: $3 = 2 + 1$: corresponds to the Yajima-Oikawa hiearchy in three variables u, v, w, the simplest equation describing sonic-Langmuir solitons:

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + uw \\ \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - vw \\ \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}(uv) \end{cases} \qquad (1976)$$

Example 6: $N = N$:
it corresponds to the N-th Gelfand-Dickey hierarchy (1975)

Example 7: $N = 2 + 1 + \cdots + 1$:
it corresponds to the N–2-component Yajima-Oikawa hierarchy

Example 8: $N = p + p + \cdots + p$ (r times): corresponds to the p-th $r \times r$-matrix Gelfand-Dickey hierarchy

Exercise 16: check (some of) these examples.

11. Multiplicative Poisson vertex algebras and Hamiltonian difference equations

Parallel Theories:

- PVA $\Rightarrow$ Hamiltonian PDE
- Multiplicative PVA $\Rightarrow$ Hamilt. differential-difference eq.s

The theory of mPVA & Hamiltonian differential-difference equations is much *less developed* then the theory of PVA & Hamiltonian PDE's.

There are so far only very partial *classification results* and some well studied *examples of integrable Hamiltonian eq's.*

Definition

A multiplicative Poisson vertex algebra (mPVA) is an algebra $\mathcal{V}$ with an automorphism D, and a λ-bracket $\{f_\lambda g\} \in \mathcal{V}[\lambda, \lambda^{-1}]$ s.t.

$$
\begin{aligned}
\text{(sesquilinearity)}\quad & \{D(f)_\lambda g\} = \lambda^{-1}\{f_\lambda g\}\,, \quad \{f_\lambda D(g)\} = \lambda D\{f_\lambda g\} \\
\text{(skewsymmetry)}\quad & \{f_\lambda g\} = -{}_{\leftarrow}\{g_{\lambda^{-1}D^{-1}} f\}, \\
\text{(Jacobi identity)}\quad & \{f_\lambda \{g_\mu h\}\} - \{g_\mu \{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda\mu} h\}. \\
\text{(Leibniz rule)}\quad & \{f_\lambda gh\} = \{f_\lambda g\}h + g\{f_\lambda h\}.
\end{aligned}
$$

Remark: mPVA $\Leftrightarrow$ "local" Poisson algebra $(\mathcal{V}, S)$ (= a PA $\mathcal{V}$ with an automorphism S, s.t. $\{S^n a, b\} = 0$ for $|n| >> 0$.)

Proof: $\{a_\lambda b\} = \sum_{n\in\mathbb{Z}} \lambda^n \{S^n a, b\}$ is a mPVA structure on $\mathcal{V}$.

Exercise 17: prove it.

Example: The most famous example of a "local" PA is the Faddeev-Takhtajan-Volkov algebra [1986]: $\mathcal{V} = \mathbb{F}[u_n | n \in \mathbb{Z}]$, with $D(u_n) = u_{n+1}$, and Poisson bracket

$$\begin{aligned}\{u_m, u_n\} = u_m u_n \big(& (\delta_{m+1,n} - \delta_{m,n+1})(1 - u_m - u_n) \\ & - u_{m+1}\delta_{m+2,n} + u_{n+1}\delta_{m,n+2}\big).\end{aligned}$$

The corresponding mPVA λ-bracket:

$$\begin{aligned}\{u_\lambda u\} = & \, u(1+\lambda D)u(1+\lambda D)u - u(1+\lambda^{-1}D^{-1})u(1+\lambda^{-1}D^{-1})u \\ & - u(\lambda D - \lambda^{-1}D^{-1})u\end{aligned}$$

Exercise 18: Check this formula.

Basic Lemma. Let $\mathcal{V}$ be a mPVA. Let $\{\cdot\,,\cdot\} = \{\cdot\,_\lambda\,\cdot\}|_{\lambda=1}$.

- $\overline{\mathcal{V}} := \mathcal{V}/(D-1)\mathcal{V}$ (= local functionals) is a Lie algebra with Lie bracket $\{\cdot\,_\lambda\,\cdot\}|_{\lambda=1}$;
- LA representation of $\overline{\mathcal{V}}$ on $\mathcal{V}$ (= functions).

Definition

The Hamiltonian equation associated to the mPVA $\mathcal{V}$ and the Hamiltonian functional $\int h \in \overline{\mathcal{V}}$ is

$$\frac{du}{dt} = \{\textstyle\int h, u\} \ , \quad u \in \mathcal{V}$$

Integrability: $\exists \int h_0 = \int h, \int h_1, \int h_2 \dots$ (lin.ind.) integrals of motion in involution: $\{\int h_m, \int h_n\} = 0 \ \forall m, n$

12. Example: the Volterra lattice eq.

It is the simplest example of a Hamiltonian difference equation.

The Volterra lattice eq. on $\mathcal{V} = \mathbb{F}[u_n | n \in \mathbb{Z}], D(u_n) = u_{n+1}$ is

$$\frac{du_n}{dt} = u_n(u_{n+1} - u_{n-1}), \; n \in \mathbb{Z}$$

It is a Hamiltonian differential-difference equation with Hamiltonian functional $h_1 = \int u$ and multiplicative λ-bracket

$$\{u_\lambda u\}_1 = \lambda u u_1 - \lambda^{-1} u u_{-1}.$$

It is the first equation of the Lax hierarchy $\frac{dL}{dt_n} = [(L^{2n})_+, L]$, for the pseudodifference operator $L = S + uS^{-1}$

Exercise 19: Check these facts.

Hence, it is integrable with integrals of motion $h_m = \int \operatorname{Res} L^{2m}$, (where $\operatorname{Res} \sum_j a_j S^j = a_0$.)

The well-known various versions of: the *Toda lattice* hierarchies, the *Bogoyavlensky lattice* hierarchies, the *discrete KP* hierarchies, and many other integrable Hamiltonian differential-difference equations can be treated along the same lines.

The general theory is work in progress.