Lecture 2: Vertex algebras and Poisson vertex algebras

Going from a *finite* to an *infinite* number of degrees of freedom, we pass from:

classical & quantum mechanics

to

Classical Field Theory & Quantum Field Theory

Question: what are the corresponding correct algebraic structures?

In conformal field theory, the algebraic structures describing chiral fields are known as vertex algebras (they were introduced by [Borcherds, 1986]).

Their quasi-classical limits are the Poisson vertex algebras (introduced by [DS, Kac, 2006]): they are associated to classical field theory in the same way as Poisson algebras are associated to classical mechanics.

1. Review of Quantum Field Theory

Basic postulates:

- The space of states is a Hilbert space V,
- The vacuum is a state $|0\rangle \in V$.
- The physical observables are operator valued distributions $\Phi(x)$, functions of x in the space-time M, with values in $\operatorname{End} V$, called the quantum fields.

Einstein's relativity principle states that:
No signal can travel at speed higher than the speed of light $c = 1$.
Hence:
If $x, y \in M$ are at *space-like* distance, $|x - y|^2 < 0$, then measures $\Phi(x)$ and $\Psi(y)$ must be independent.

Heisenberg principle of uncertainty states that:
If the observable $\Phi(x)$ and $\Psi(y)$ are independent, then they commute: $[\Phi(x), \Psi(y)] = 0$

(So, they can be simultaneously diagonalized, i.e. they can be measured simultaneously without uncertainty.)

In dimension $\dim M = 1+1$: combining the Relativity and the Uncertainty principles, we get locality of chiral fields.

1. We make the chance of variables

$$z = x_0 - x_1\,, \quad \bar{z} = x_0 + x_1 \quad \Rightarrow \quad |x|^2 = z\bar{z}$$

2. By the Relativity and Uncertainty principles, we get:

$$(1) \qquad (z-w)(\bar{z}-\bar{w}) < 0 \;\Rightarrow\; [\Phi(z,\bar{z}),\Psi(w,\bar{w})] = 0$$

3. A chiral field is $\Phi(z)$ (depending only on z, not on $\bar{z}$).
4. Hence, equation (1) for chiral fields becomes:

$$z - w \neq 0 \quad \Rightarrow \quad [\Phi(z),\Psi(w)] = 0$$

This in turn is equivalent to locality:

$$(z-w)^N[\Phi(z),\Psi(w)] = 0 \;\text{ for }\; N >> 0$$

Creation and annihilation operators

Usually a quantum (chiral) field $\Phi(z)$ is expanded in its Fourier modes, as a (formal) power series:

$$\Phi(z) \;=\; \sum_{n\in\mathbb{Z}} \Phi_n z^n \;, \qquad \Phi_n \in \mathrm{End}\, V$$

1. $\Phi_n,\; n<0$ are annihilation operators ($v \in V$):

$$\Phi_n|0\rangle = 0 \;,\; \forall n<0 \qquad \text{and} \qquad \Phi_n v = 0 \;,\;\; \forall n << 0$$

2. $\Phi_n,\; n \geq 0$ are the creation operators:

$$\Phi_n|0\rangle \;=\; |\Phi_n\rangle \neq 0 \;,\; \forall n \geq 0$$

Equivalently:

- $\Phi(z)|0\rangle \in V[[z]]$ is a *Taylor* series in z;
- $\Phi(z)v \in V((z))$ (*Laurent* series in z).

2. Quantum field theory and vertex algebras

The notion of a *vertex algebra* describes the algebraic structure of chiral fields in a quantum field theory in $1+1$-dimension.

Let: V, the space of states; $|0\rangle \in V$ the vacuum; $T \in \operatorname{End} V$, the translation operator.

Definition: A quantum field is a series $\Phi(z) = \sum_{n\in\mathbb{Z}} \Phi_n z^n$ $\in \operatorname{End} V[[z, z^{-1}]]$ such that, for all $v \in V$,

$$\Phi(z)v \in V((z)) \quad \text{or, equivalently,} \quad \Phi_n(v) = 0 \text{ for } n << 0$$

Definition: A (pre)vertex algebra $(V, |0\rangle, T, \mathcal{F})$ is a (complete) collection of quantum fields $\mathcal{F} = \{\Phi^\alpha(z)\}_{\alpha\in J}$ satisfying:

1. vacuum axiom: $\Phi^\alpha(z)|0\rangle \in V[[z]]$ for all α;
2. translation covariance: $[T, \Phi^\alpha(z)] = \partial_z \Phi^\alpha(z)$;
3. locality: $(z-w)^N[\Phi^\alpha(z), \Phi^\beta(w)] = 0$ for $N >> 0$

Example1: affine vertex algebra

Let: $\mathfrak{g}$ a simple Lie algebra; $(\cdot\,,\,\cdot)$ the Killing form.
The affine Kac-Moody algebra is $\hat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K$ with bracket

$$[at^m, bt^n] \;=\; [a,b]t^{m+n} + m\delta_{m,-n}(a|b)K$$

The vacuum module of level k is $V^k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}\,\mathbb{C}_k$ (where $\mathfrak{g}[t] = 0$ and $K = k$ on $\mathbb{C}_k = \mathbb{C}$).

The affine vertex algebra structure on $V^k(\mathfrak{g})$ is given by $|0\rangle = 1$, $T = -\partial_t$ and the collection of fields $\mathcal{F} = \{a(z)\}_{a\in\mathfrak{g}}$, where

$$a(z) \;=\; \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$$

Locality is guaranteed by the Operator Product Expansion:

$$(2) \qquad [a(z), b(w)] = [a,b](w)\delta(z-w) + (a|b)K\partial_w\delta(z-w)$$

where $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}$.
Exercise 1: Check the OPE (2) for the affine vertex algebra $V^k(\mathfrak{g})$. Deduce that the fields $\{a(z)\}_{a\in\mathfrak{g}}$ are pairwise local.

Example 2: Virasoro

The Virasoro Lie algebra is $Vir = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n \oplus \mathbb{C}C$, with

$$[L_m, L_n] \;=\; L_{m+n} + c\frac{m^3-m}{12}\delta_{m,-n}C$$

The vacuum module of central charge c is $Vir^c = \mathrm{Ind}_{Vir_+}\mathbb{C}_c$ (where $L_n = 0$ for $n \geq -1$ and $C = c$ on $\mathbb{C}_c = \mathbb{C}$).

The Virasoro vertex algebra on Vir^c is given by $|0\rangle = 1$, $T = L_0$ and $\mathcal{F} = \{L(z)\}$, where

$$L(z) = \sum_n L_n z^{-n-2}$$

Locality is guaranteed by the Operator Product Expansion:

$$(3) \qquad [L(z), L(w)] = (L'(w) + 2L(w)\partial_w + \frac{1}{2}c\partial_w^3)\delta(z-w)$$

Exercise 2: Check the OPE (3) for the Virasoro vertex algebra Vir^c. Deduce that the field $L(z)$ is local with itself.

3. λ-bracket definition of a vertex algebra

The whole *algebraic structure* of a vertex algebra is encoded in three operations on local quantum fields:

- the derivative $\partial_z \Phi(z)$
- the normally ordered product

$$:\Phi\Psi:(z) \quad = \quad \Phi_+(z)\Psi(z) + \Psi(z)\Phi_-(z)$$

where $\Phi_+(z) = \sum_{n<0} \Phi_n z^{-n-1}$ (creation part) and $\Phi_-(z) = \sum_{n\geq 0} \Phi_n z^{-n-1}$ (annihilation part).

- the λ-bracket, defined as the Fourier transform of the OPE:

$$[\Phi_\lambda \Psi](w) = \mathrm{Res}_z\, e^{\lambda(z-w)}[\Phi(z), \Psi(w)]$$

<u>**Exercise 3**</u>: check that if we have the OPE

$$[\Phi(z), \Psi(w)] = \sum_{n=0}^{N} c_n(w) \partial_w^n \delta(z-w)$$

then the corresponding λ-bracket is

$$[\Phi\ {}_\lambda\ \Psi] = \sum_{n=0}^{N} \frac{\lambda^n}{n!} c_n$$

Dong's Lemma: Let $\mathcal{F}$ be a collection of pairwise local quantum fields.

Let $\widetilde{\mathcal{F}}$ be obtained by adding all derivatives, all normally ordered products, and all coefficients of all λ-brackets of elements in $\mathcal{F}$:

$$\widetilde{\mathcal{F}} \ \ni \ \Phi, \partial\Phi, c_n, :\Phi\Psi:$$

Then, $\bar{\mathcal{F}}$ is again a collection of pairwise local quantum fields.

Hence: we can assume that $\mathcal{F}$ is closed with respect to the derivative ∂_z, $:\cdot\ \ \cdot:$ and $[\cdot\ {}_\lambda\ \cdot]$; i.e., for $\Phi, \Psi \in \mathcal{F}$:

$$\partial\Phi \ \in \ \mathcal{F}\,, \quad :\Phi\Psi: \in \ \mathcal{F}\,, \quad [\Phi\ {}_\lambda\ \Psi] \ \in \ \mathcal{F}[\lambda]$$

Question: what are the properties of these maps ∂, $:\cdot\ \ \cdot:$ and $[\cdot\ {}_\lambda\ \cdot]$?

We then get an *equivalent definition of a vertex algebra*:

Theorem / Definition [Bakalov, Kac, 2001]:
A vertex algebra is a space of states V, with a vacuum vector $|0\rangle \in V$, a translation operator $\partial \in \operatorname{End} V$, a normally ordered product $:ab: \in V$, and a λ-bracket $[a_\lambda b] \in V[\lambda]$, satisfying the following axioms:

1. vacuum $:a|0\rangle: = :|0\rangle a: = a$
2. translation covariance $\partial(:ab:) = :(\partial a)b: + :a(\partial b):$

(**Rem**: $(V, |0\rangle, \partial, :\ :)$ is a unital differential algebra)

3. sesquilinearity $[\partial a_\lambda b] = -\lambda[a_\lambda b]\,,\ [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$
4. skewsymmetry $[a_\lambda b] = -[b_{-\lambda-\partial} a]$
5. Jacobi identity $[a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]$

(**Def**: $(V, \partial, [\cdot\,_\lambda\,\cdot])$ is a Lie conformal algebra)

6. quasi-associativity $:(:ab:): - :a(:bc:): \; =$
   $= \; :\left(\int_0^{\partial} d\lambda\, a\right)[b_\lambda c]: + :\left(\int_0^{\partial} d\lambda\, b\right)[a_\lambda c]:$
7. quasi-commutativity $:ab: - :ba: \; = \; \int_{-\partial}^{0} d\lambda\, [a_\lambda b]$
8. Wick formula $[a_\lambda :bc:] = \; :[a_\lambda b]c: + :b[a_\lambda c]: + \int_0^{\lambda} d\mu\, [[a_\lambda b]_\mu c]$

To construct examples, we start with a Lie conformal algebra R and we take its *universal enveloping vertex algebra* $V(R)$.

Theorem: given a Lie conformal algebra R, there exists a unique universal enveloping vertex algebra $V(R)$, and we have the PBW Theorem: a basis for $V(R)$ is given by ordered monomials

$$\left\{ :a_{i_1} a_{i_2} \dots a_{i_s}: \;\middle|\; 0 \le i_1 \le \dots \le i_s \right\}$$

Let us review the examples of *affine VA* and *Virasoro VA* in the λ-bracket notation.

Example 1 The affine LCA is $\mathrm{Cur}_k\mathfrak{g} = (\mathbb{C}[\partial] \otimes \mathfrak{g}) \oplus \mathbb{C}|0\rangle$, with λ-bracket ($a, b \in \mathfrak{g}$)

$$[a_\lambda b] = [a, b] + \lambda k(a, b)|0\rangle \tag{4}$$

The affine vertex algebra is $V^k(\mathfrak{g}) = V(\mathrm{Cur}_k\mathfrak{g})$

Exercise 4: check that the λ-bracket (4) defines a structure of a Lie conformal algebra on $\mathrm{Cur}_k\mathfrak{g}$.

Example 2 The Virasoro LCA is $R_c = (\mathbb{C}[\partial]L) \oplus \mathbb{C}|0\rangle$, with λ-bracket

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3|0\rangle \tag{5}$$

$c \in \mathbb{C}$ is the central charge.

The Virasoro vertex algebra is $Vir^c = V(R_c)$

Exercise 5: check that the λ-bracket (5) defines a structure of a Lie conformal algebra on R_c.

4. Classical limit and Poisson vertex algebras

We can apply the *classical limit* procedure, to get algebraic structure of *classical field theory*: this leads to the notion of a *Poisson vertex algebra*.

- Assume that the vertex algebra V has an increasing filtration

$$0 = F^{-1}V \subset F^0V \subset F^1V \subset \cdots \subset V$$

  such that

$$:F^iV \cdot F^jV: \subset F^{i+j}V \quad \text{and} \quad [F^iV_\lambda F^jV] \subset F^{i+j-1}V[\lambda]$$

- Then, we take the associated graded

$$\operatorname{gr} V = \bigoplus_{n \geq 0} \operatorname{gr}^n V \quad \text{where} \quad \operatorname{gr}^n V = F^nV/F^{n-1}V$$

All the *quantum corrections* in the axioms of a vertex algebra disappear, and what we get is a (graded) *Poisson vertex algebra*.

Definition [DS, Kac, 2006]:
A Poisson vertex algebra $\mathcal{V}$ is:

- a unital, commutative, associative, differential algebra,
- a Lie conformal algebra with λ-bracket $\{\cdot\,_\lambda\,\cdot\}$,

and the two structures are related by the following Leibniz rule:

$$\{a_\lambda bc\} = \{a_\lambda b\}c + \{a_\lambda c\}b$$

Exercise 6: Check that $\operatorname{gr} V$ is a graded Poisson vertex alg.

Conversely, given a graded Poisson vertex algebra $\mathcal{V}$, its quantization is a filtered vertex algebra V, such that $\operatorname{gr} V = \mathcal{V}$

Theorem: If R is a Lie conformal algebra, then $\mathcal{V}(R) = S(R)$ has a natural structure of a Poisson vertex algebra. Its quantization is the universal enveloping vertex algebra $V(R)$.

Exercise 7: Prove the above claim.

Example 1: the GFZ PVA is $\mathcal{V} = \mathbb{F}[u, u', u'', \dots]$, with λ-bracket: $[u_\lambda u] = \lambda$ (extended by sesquil. and Leibniz rules).

Example 2: the affine PVA: $\mathcal{V}(\mathfrak{g}) = S(\mathbb{F}[\partial]\mathfrak{g})$, with λ-bracket

$$[a_\lambda b] = [a, b] + (a|b)\lambda\,, \;\; a, b \in \mathfrak{g}\,, \tag{6}$$

(extended by sesquil. and Leibniz rules).
Note: it is the *classical limit* of the affine vertex alg. $V^k(\mathfrak{g})$.

Example 3: the Virasoro-Magri PVA is $\mathcal{V} = \mathbb{F}[L, L', L'', \dots]$, with λ-bracket

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3\,, \tag{7}$$

(extended by sesquil. and Leibniz rules).
Note: it is the *classical limit* of the Virasoro vertex alg. Vir^c.

5. Poisson vertex algebras and Hamiltonian PDE

Notation: a local functional is $\int f \in \mathcal{V}/\partial\mathcal{V}$

Observation: If $\mathcal{V}$ is a PVA, then $\mathcal{V}/\partial\mathcal{V}$ is a Lie algebra, with

$$\{\textstyle\int f, \int g\} = \int \{f_\lambda g\}\big|_{\lambda=0}$$

Definition: the Hamiltonian equation associated to the PVA $\mathcal{V}$, and the Hamiltonian functional $\int h \in \mathcal{V}/\partial\mathcal{V}$, is

(8) $$\frac{du}{dt} = \{h_\lambda u\}\big|_{\lambda=0}$$

An integral of motion for (8) is a local functional $\int g \in \mathcal{V}/\partial\mathcal{V}$ s.t.

$$\{\textstyle\int h, \int g\} = 0 \ \left(\iff \{h_\lambda g\}\big|_{\lambda=0} \in \partial\mathcal{V} \right)$$

GOAL: construct an infinite sequence $\int h_0 = \int h, \int h_1, \int h_2, \dots,$ of lin.indep. integrals of motion in involution:

$$\{\textstyle\int h_m, \int h_n\} = 0 \ \forall m, n \geq 0$$

We then have the integrable hierarchy: $\frac{du}{dt_n} = \{\int h_n, u\}$.

Example: the KdV equation:

$$\frac{\partial u}{\partial t} = 3u\frac{\partial u}{\partial x} + c\frac{\partial^3 u}{\partial x^3} \quad (c \in \mathbb{C})$$

It is Hamiltonian w.r.t. the GFZ PVA and the Hamiltonian functional $\int h = \frac{1}{2}\int(u^3 + cuu'')$:

$$\frac{\partial u}{\partial t} = \{h_\lambda u\}|_{\lambda=0}$$

Exercise 8: Check this (again!) using λ-bracket computations (and the axioms of Poisson vertex algebra).