W-algebras and Hamiltonian equations

Alberto De Sole

Università di Roma La Sapienza

Dobbiaco, february 17-22, 2019

Lecture 1: Review of classical and quantum Hamiltonian mechanics

General Motivation: we will be mainly concerned with the role and applications of the concept of symmetry in mechanics, from a purely algebraic point of view.

Recall that:

- **classical mechanics** deals with the dynamics of particles, rigid bodies, etc,
- **classical field theory** deals with the dynamics of continuous media (fluids, plasma), and fields such as the electromagnetic field, gravity, etc. It can be viewed as an infinite dimensional analogue of the classical mechanics.

We shall deal with the Hamiltonian formulation of both theories.

1. Standard Hamiltonian equations.

In **classical mechanics**, the state of a physical system is a point (q, p) in a $2n$-dimensional space of states M ($q = (q_1, \ldots, q_n)$ = positions, $p = (p_1, \ldots, p_n)$ = momenta).

The dynamics (=time evolution) of the physical system is described in terms of the Hamiltonian function $H(q, p)$ (=energy of the system). The time evolution $(q(t), p(t))$, $t > 0$, is solution of the Hamilton equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \ , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \, .$$

<u>**Example**</u>: Newton's law of dynamics $F = ma$ defines a Hamilton evolution, where $F = -\nabla U$ is the force field associated to an energy potential $U(q)$, and the Hamiltonian $H = E_k + U$ is the sum of the kinetic energy $E_k = \frac{|p|^2}{2m}$ and the potential energy $U(q)$

The space of physical observables is the space $C^\infty(M)$ of all smooth functions of the space of states M.
For example: the coordinates q_i and p_i, the energy $H(q,p)$, are physical observables.
$C^\infty(M)$ is a commutative associative algebra (with respect to the usual product of functions). It has the Poisson bracket

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right).$$

The Hamiltonian equations can then be rewritten in terms of the Poisson bracket:

$$\frac{dq_i}{dt} = \{H, q_i\}, \quad \frac{dp_i}{dt} = \{H, p_i\}$$

By the chain rule of derivation, for $f \in C^\infty(M)$:

$$\frac{df}{dt} = \{H, f\}$$

2. Hamiltonian evolution on a Poisson manifold

The formulation of Hamilton equations in terms of the Poisson bracket allows us generalize the theory.

The space of states of a classical mechanical system is a Poisson manifold: a smooth manifold M (NOT nec. even dim.) with a Poisson bracket $\{\cdot\,,\,\cdot\}$ on $C^\infty(M)$, satisfying skewsymmetry

$$\{f, g\} = -\{g, f\}$$

the Jacobi identity

$$\{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\}$$

and the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

(Equivalently: $C^\infty(M)$ is a Poisson algebra.)

Exercise 0: the standard Poisson bracket satisfies these axioms

The Hamilton equations associated to the Hamiltonian H are:

$$\frac{df}{dt} = \{H, f\}$$

Example: The dynamics of of a rigid body about its center of mass in absence of external forces is:

(1)

$$\dot{\Pi}_1 = \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3\,,\ \dot{\Pi}_2 = \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1\,,\ \dot{\Pi}_3 = \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2$$

$\Pi_i = I_i\Omega_i$, $i = 1, 2, 3$, =angular momenta (Ω_i = angolar velocities, I_i = moments of inerzia).

It has the Hamiltonian form with Poisson bracket

$$\{f, g\}(\Pi) = \Pi \cdot (\nabla f \times \nabla g) \tag{2}$$

and Hamiltonian function

$$H(\Pi) = \frac{1}{2}\Big(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\Big) \tag{3}$$

Exercise 1: Check that (2) is a Poisson bracket, and that (1) are the Hamiltonian equations for H in (3).

Example: **The Kirillov-Kostant Poisson structure on $\mathfrak{g}^*$**

A generalization of the rigid body Poisson bracket is the Kirillov-Kostant bracket associated to a Lie algebra $\mathfrak{g}$.

The underlying Poisson manifold is $\mathfrak{g}^*$. The Poisson bracket on $C^\infty(\mathfrak{g}^*)$ is

$$\{f,g\}(\zeta) = \langle \zeta | [d_\zeta f, d_\zeta g] \rangle \tag{4}$$

where $\langle \cdot \,|\, \cdot \rangle$ is the pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$, and the differential $d_\zeta f \in \mathfrak{g}$ is defined by ($\zeta, \xi \in \mathfrak{g}^*$):

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(f(\zeta + \varepsilon\xi) - f(\zeta)) = \langle \xi | d_\zeta f \rangle$$

Exercise 2: Check that the Kirilov-Kostant Poisson bracket (4) satisfies skewsymmetry, Jacobi identity and the Leibniz rule.
Exercise 3: Check that the Poisson bracket of the rigid body is a special case of the Kirillov-Kostant Poisson bracket when $\mathfrak{g} = \mathfrak{so}_3(\mathbb{R})$

Remark: From a purely algebraic point of view, we can replace smooth functions $C^\infty(\mathfrak{g}^*)$ by polynomial functions $\mathbb{C}[\mathfrak{g}^*] \simeq S(\mathfrak{g})$. This identification maps

$$P = \sum \text{coeff.} a_1 \dots a_s \in S(\mathfrak{g})$$

to the polynomial function

$$P(\zeta) = \sum \text{coeff.} \langle \zeta | a_1 \rangle \dots \langle \zeta | a_s \rangle \, , \quad \zeta \in \mathfrak{g}^*$$

Exercise 4: Under this identification, the Kirillov-Kostant Poisson bracket (4) corresponds to the canonical Poisson bracket of the symmetric algebra $S(\mathfrak{g})$, obtained by extending the Lie algebra bracket of $\mathfrak{g}$ to $S(\mathfrak{g})$ by the Leibniz rules.

3. Hamiltonian equations in classical field theory

Classical field theory describes the evolution of continuous media, such as fluids, strings, electromagnetic field, etc.

The state of the system is described by a field $\varphi(x)$, i.e. a function on $x \in S$ (=the space or space-time).

Hence, the space of states $\mathcal{M} = \mathcal{F}un(S)$ is ∞-dimensional.

Assumptions:

- For simplicity, we assume that S is one-dimensional.
- $\varphi(x)$ is smooth and rapidly decreasing; hence all integrals are defined, and we can perform integration by parts.

The physical observables are local functionals, $F : \mathcal{M} \to \mathbb{R}$:

$$F(\varphi) = \int_S f(\varphi(x), \varphi'(x), \dots, \varphi^{(n)}(x))dx$$

where the density function f depends on $\varphi(x), \dots, \varphi^{(n)}(x)$ only.

To give a Hamiltonian formulation of classical field theory, we assume that the space of states $\mathcal{M}$ is a Poisson manifold.

This means that we have a local Poisson bracket $\{\cdot\,,\,\cdot\}$ on the space $\mathcal{F}(\mathcal{M})$ of local functionals (=physical observables).

It is a Lie algebra bracket on $\mathcal{F}(\mathcal{M})$ of the form

$$\{F, G\}(\varphi) = \int_S \frac{\delta G}{\delta\varphi(x)} K(\varphi(x), \varphi'(x), \ldots, \varphi^{(n)}(x); \partial_x) \frac{\delta F}{\delta\varphi(x)}$$

The Poisson structure $K(\varphi(x), \varphi'(x), \ldots, \varphi^{(n)}(x); \frac{d}{dx})$ is a finite order differential operator. (∂_x = total derivative w.r.t. x). The variational derivative $\frac{\delta F}{\delta\varphi(x)}$ is defined by

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}(F(\varphi + \epsilon\psi) - F(\varphi)) = \int_S \frac{\delta F}{\delta\varphi(x)}\psi(x)dx$$

for every test function $\psi(x) \in \mathcal{M}$.

Exercise 5: Explicit formula for the variational derivative:

$$\frac{\delta F}{\delta\varphi(x)} = \sum_{n\in\mathbb{Z}_+} (-\partial_x)^n \Big(\frac{\partial f}{\partial\varphi^{(n)}(x)}\Big)$$

Example: The Gardner-Faddeev-Zakharov (GFZ) local Poisson-bracket is given by

$$\{F,G\}(\varphi) = \int_S \frac{\delta G}{\delta\varphi(x)}\partial_x\frac{\delta F}{\delta\varphi(x)} \tag{5}$$

i.e., it has Poisson structure $K = \partial_x$.

Exercise 6: Check that the GFZ local Poisson bracket (5) is a Lie algebra bracket on $\mathcal{F}(\mathcal{M})$.

The dynamics (=time evolution) of the physical system is described in terms of the Hamiltonian functional $H(\varphi) \in \mathcal{F}(\mathcal{M})$, describing the energy of the system in the state $\varphi(x)$.

It is given by the Hamiltonian equations

$$\frac{dF}{dt} = \{H, F\}$$

where $F(\varphi)$ is a local functional of $\varphi(x)$.

We can also write an equation for the evolution of the coordinate variable $\varphi(x,t)$. By the definition of variational derivative, we have

$$\frac{dF}{dt} = \int_S \frac{\delta F}{\delta \varphi(x)} \frac{d}{dt} \varphi(x,t)\,,$$

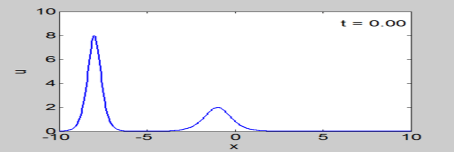
Combining this with the form of the local Poisson bracket, we get

$$\frac{d}{dt}\varphi(x,t) = K(\varphi(x), \varphi'(x), \dots, \varphi^{(n)}(x); \partial_x) \frac{\delta H}{\delta \varphi(x)}$$

Example: The famous Korteveg-de Vries (KdV, 1985) equation

$$\frac{\partial \varphi}{\partial t} = 3\varphi(x,t)\frac{\partial \varphi}{\partial x} + c\frac{\partial^3 \varphi}{\partial x^3} \tag{6}$$

describes the evolution of waves in shallow water.



It is the Hamiltonian equation associated to the GFZ Poisson bracket on $\mathcal{M}$ and the Hamiltonian functional

$$H(\varphi) = \int_S \left(\frac{1}{2}\varphi(x)^3 - \frac{c}{2}\varphi'(x)^2\right)dx$$

Exercise 7: Check the above statement.

4. Quantum mechanics

In quantum mechanics the space of states of a system is a Hilbert space, i.e. a complex vector space V with a positive definite hermitian inner product $\langle \cdot \,|\, \cdot \rangle$.

A state is a normalized vector $|v\rangle \in V$, such that $\langle v|v\rangle = 1$.

The physical observables are operators $A \in \mathrm{End}(V)$, which are selfadjoint: $A = A^\dagger$.

The *physical meaning* is as follows: if the state of the system is an eigenvector $|v\rangle \in V$ of A of eigenvalue $\lambda \in \mathbb{R}$, then the result of the measurement of the observable A is λ.
By the Spectral Theorem, any state $|v\rangle$ is linear combination of orthonormal eigenvectors with real eigenvalues: $|v\rangle = \sum_i c_i |v_i\rangle$, with $A|v_i\rangle = \lambda_i |v_i\rangle$. Then, the result of a measurement of the observable A is λ_i with probability $|c_i|^2$.

The Hamiltonian operator H describes the energy of the system and defines its dynamics.

There are two alternative ways to introduce the dynamics:

- In the Schroedinger picture, the observables do not evolve, while the states evolves by the Schroedinger equation

$$\frac{d}{dt}|\psi_t\rangle = H|\psi_t\rangle \tag{7}$$

- In the Heisenberg picture, the the state does not evolve, while the observables evolve by to the Heisenberg equation:

$$\frac{d}{dt}A(t) = [H, A(t)] \tag{8}$$

  where $[H, A] = HA - AH$ is the commutator.

Note: This is the *"quantum version"* of the Hamiltonian equation of classical mechanics.

Exercise 8: Show that the Schroedinger equation (7) and the Heisenberg equation (8) are equivalent in the following sense: the evolution of the matrix elements $\langle\varphi|A|\psi\rangle$ is the same.

5. Algebraic structures of classical and quantum mechanics

From a purely algebraic point of view, when considering a classical mechanic physical system, we ignore completely the underlying configuration space M, and we just retain the algebraic structure of the space of functions $C^\infty(M)$.

What we end up with, is the structure of a *Poisson algebra* for classical mechanics, and the structure of an *associative algebra* for quantum mechanics.

A) In *classical mechanics* $P = C^\infty(M)$ is a Poisson algebra, i.e. a commutative associative algebra, with a Lie algebra bracket $\{\cdot\,,\,\cdot\}$ satisfying the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

The Hamiltonian equations $\frac{df}{dt} = \{H, f\}$, $f, H \in P$, are entirely written in terms of the Poisson algebra structure.

Example: Given a Lie algebra $\mathfrak{g}$, we have the Poisson algebra $S(\mathfrak{g})$ with the canonical Poisson bracket, obtained by extending the Lie algebra bracket of $\mathfrak{g}$ to $S(\mathfrak{g})$ by the Leibniz rule.

B) In *quantum mechanics*, the physical observables are (non commuting) objects in $A = \operatorname{End} V$, which is an associative algebra.

The Hamiltonian equations here become $\frac{da}{dt} = [H, a]$, $a, H \in A$, and they are written in terms of the commutator of A.

The procedure of passing from a quantum mechanics to classical mechanics is known as classical limit.

It can be described in *purely algebraic terms* as follows:

- We assume that the associative algebra A (associated to the quantum system) has an increasing filtration

$$0 = F^{-1}A \subset F^0 A \subset F^1 A \subset \cdots \subset A$$

such that

$$F^i A \cdot F^j A \subset F^{i+j} A \quad \text{and} \quad [F^i A, F^j A] \subset F^{i+j-1}$$

- Then, the associated graded algebra

$$\operatorname{gr} A = \bigoplus_{n \geq 0} \operatorname{gr}^n A \quad \text{where} \quad \operatorname{gr}^n A = F^n A / F^{n-1} A$$

is naturally a graded Poisson algebra:

for $\bar{a} = a + F^{i-1}A \in \mathrm{gr}^i A$ and $\bar{b} = b + F^{j-1}A \in \mathrm{gr}^j A$, their commutative associative product is

$$\bar{a}\bar{b} = ab + F^{i+j-1}A \in \mathrm{gr}^{i+j} A$$

and their Poisson bracket is

$$\{\bar{a}, \bar{b}\} = ab - ba + F^{i+j-2}A \in \mathrm{gr}^{i+j-1} A$$

The Poisson algebra $\mathrm{gr}\, A$ is graded in the sense that

$$\mathrm{gr}^i A \cdot \mathrm{gr}^j A \subset \mathrm{gr}^{i+j} A \quad \text{and} \quad \{\mathrm{gr}^i A, \mathrm{gr}^j A\} \subset \mathrm{gr}^{i+j-1} A$$

Exercise 9: Check that $\mathrm{gr}\, A$ is indeed a graded Poisson algebra.

Conversely, given a graded Poisson algebra P, describing a classical mechanic system, its quantization is, by definition, a filtered associative algebra A, describing a quantum system, such that

$$\operatorname{gr} A = P$$

Note: while the procedure of classical limit is uniquely defined, the quantization is not.

Example: A quantization of the symmetric algebra $S(\mathfrak{g})$ is the universal enveloping algebra $U(\mathfrak{g})$ with the usual polynomial filtration. Indeed, by the PBW Theorem, $\operatorname{gr} U(\mathfrak{g}) \simeq S(\mathfrak{g})$.

Question: What are the algebraic structures of *classical* and *quantum field theory*? We will see it in Lecture 2.

6. Moment map and Hamiltonian reduction

The concept of **momentum map** is a geometric generalization of the usual *linear* and *angular momentum.*

Example 1: *linear momentum.*
The space of states of a point particle is $M = \mathbb{R}^6 = \{(q,p)\}$. On this, we have the action of the translation group $G = \mathbb{R}^3$,

$$G \times M \to M \quad \text{mapping} \quad a \in G, (q,p) \in M \mapsto (q+a, p)$$

The Lie algebra is $\mathfrak{g} = \mathbb{R}^3$, with infin. action on $C^\infty(M)$:

$$a \in \mathfrak{g}, f(q,p) \in C^\infty(M) \mapsto \frac{d}{d\varepsilon} f(q - \varepsilon a, p)|_{\varepsilon=0} = -\nabla_q f(q,p) \cdot a$$

The RHS of the above action can be written as

$$\nabla_q f(q,p) \cdot a = \{\mu(q,p;a), f(q,p)\} \quad \text{where} \quad \mu(q,p;a) = p \cdot a$$

We thus say that the action of the translation group on M is associated to the momentum map $\mu : M \to \mathfrak{g}^*$, given by

$$\mu(q,p) = \langle p | \cdot \rangle$$

which associates to the state (q,p) its linear momentum p.

Example 2: *angolar momentum.*

On $M = \mathbb{R}^6$ acts also the rotation group $G = SO_3(\mathbb{R})$,

$$G \times M \to M \quad \text{mapping} \quad g \in G, (q,p) \in M \mapsto (gq, gp)$$

The Lie algebra is $\mathfrak{g} = \mathfrak{so}_3(\mathbb{R}) \simeq \mathbb{R}^3$, where we identify

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathfrak{so}_3(\mathbb{R}) \mapsto a = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$$

Its infininitesimal action maps $A \in \mathfrak{g}$, $f(q,p) \in C^\infty(M)$ to

$$\frac{d}{d\varepsilon} f(e^{-\varepsilon A} q, e^{-\varepsilon A} p)|_{\varepsilon=0} = -\nabla_q f(q,p) \cdot Aq - \nabla_p f(q,p) \cdot Ap$$
$$= -\{\mu(q,p;a), f(q,p)\} \quad \text{where} \quad \mu(q,p;a) = a \cdot (q \times p)$$

Exercise 10: Check the above equation.

Hence, the action of the rotation group on M is associated to the momentum map $\mu : M \to \mathfrak{g}^*$, given by

$$\mu(q,p) = \langle q \times p | \cdot \rangle$$

which associates to the state (q,p) its angular momentum $q \times p$.

We generalize the above two examples to get the definition of the *momentum map* μ.

Definition A Hamiltonian action of a Lie group G on a Poisson manifold M,

$$G \times M \to M$$

has a momentum map $\mu : M \to \mathfrak{g}^*$ such that the corresponding infinitesimal action

$$\mathfrak{g} \times C^\infty(M) \to C^\infty(M)$$

is given by

$$(a \cdot f)(x) := \frac{d}{d\varepsilon} f(e^{-\varepsilon a} x)|_{\varepsilon=0} = \{\langle \mu(x), a \rangle, f(x)\}$$

FACT: if G is a group of symmetries for the physical system, i.e. the system in invariant by the action of G, then the momentum $\mu(x)$ is constant, i.e. it does not vary in time.

This is the content of the famous:

Noether's Theorem: Suppose that the Hamiltonian $H(x) \in C^\infty(M)$ is invariant by the Hamiltonian action of the Lie group G:

$$H(gx) = H(x) \quad \text{for all} \quad g \in G$$

Then the value of the momentum map μ is constant of motion:

$$\mu(x_t) = \mu(x_0) \quad \text{for all times} \quad t$$

Proof: For $a \in \mathfrak{g}$, we have

$$\frac{d}{dt}\langle \mu(x_t), a \rangle = \{H, \langle \mu(x_t), a \rangle\}$$

$$= -\{\langle \mu(x_t), a \rangle, H\} = -\frac{d}{d\varepsilon} H(e^{-\varepsilon a} x)|_{\varepsilon=0} = 0$$

.

This is used to *reduce the # of degrees of freedom* of the system.

This process is known as Hamiltonian reduction:

- take a point $\xi \in \mathfrak{g}^*$ fixed by G: $\mathrm{Ad}^* g(\xi) = \xi \quad \forall g \in G$;
- take the preimage via the moment map $\mu^{-1}(\xi) \subset M$;
- and take the space of G-orbits: $\mu^{-1}(\xi)/G$.

We thus get the Hamiltonian reduction of M:

$$\overline{M} = \mathrm{Ham.red.}(M, \xi, G) = \mu^{-1}(\xi)/G$$

FACT: This is again a Poisson manifold, with Poisson bracket induced by that of M.

We have the induced Hamiltonian function H on $\overline{M}$, defining the *reduced physical system*
(# degrees of freedom: $\dim \overline{M} = \dim M - 2 \dim G$).

After solving the reduced system on $\overline{M}$, it is possible to *reconstruct* the original system on M.

Can describe Hamiltonian reduction in purely algebraic terms:

Geometry	Algebra
▶ Poisson manifold M	▶ Poisson algebra $P = C^\infty(M)$
▶ moment map	▶ Lie algebra homom.

Exercise 11: Check that, if $\mu : M \to \mathfrak{g}^*$ is a momentum map, then $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism.

$\mu : M \to \mathfrak{g}^*$	$\mu^* : \mathfrak{g} \to C^\infty(M)$ or: Poisson algebra homom. $\mu^* : S(\mathfrak{g}) \to P$
▶ fiber of a point $\mu^{-1}(\xi) \subset M$	▶ algebra $C^\infty(\mu^{-1}(\xi)) \simeq$ quotient P/I, where $I = \{f \text{ vanish. on } \mu^{-1}(\xi)\}$

Exercise 12: Check that

$$I = \{\text{funct's } f \in C^\infty(M) \text{ vanishing on } \mu^{-1}(\xi)\} = P\mu^*\{a - \langle \xi, a\rangle \,|\, a \in \mathfrak{g}\} \subset P$$

Check also that, if $\xi \in \mathfrak{g}^*$ is $\mathrm{ad}^*\,\mathfrak{g}$-invariant, then $\{a - \langle \xi, a\rangle \,|\, a \in \mathfrak{g}\} \subset S(\mathfrak{g})$ is invariant by the adjoint action of $\mathfrak{g}$.

▶ space of G-orbits $\mu^{-1}(\xi)/G$	▶ subalgebra of functions f which are const's on G-orbits i.e.: $(P/I)^{\mathrm{ad}\,\mu^*(\mathfrak{g})}$

Exercise 13: f is constant on the G-orbits if and only if

$$(\mathrm{ad}\,\mu^*(a))(f) := \{\mu^*(a), f\} = 0 \quad \text{for all} \quad a \in \mathfrak{g}$$

In conclusion, we arrive at the following *algebraic construction*:

- P: a Poisson algebra;
- $\mathfrak{g}$: a Lie algebra;
- $\mu^* : \mathfrak{g} \to S(\mathfrak{g})$: a Poisson algebra homomorphism;
- $J \subset S(\mathfrak{g})$: a subset invariant by the adjoint action of $\mathfrak{g}$;
- take the corresponding algebra ideal (NOT PA ideal) $I = P\mu^*(J) \subset P$.

The corresponding Hamiltonian reduction is:

$$\text{Ham.red.}(P, J, \mathfrak{g}) = (P/I)^{\text{ad}\,\mu^*(\mathfrak{g})}$$
$$= \{f \in P \,|\, \{\mu^*(a), f\} \in I \text{ for all } a \in \mathfrak{g}\}/I$$

Exercise 14: $\{f \in P \,|\, \{\mu^*(a), f\} \in I \text{ for all } a \in \mathfrak{g}\}$ is a Poisson subalgebra of P and I is its ideal. Hence, $\text{Ham.red.}(P, J, \mathfrak{g})$ is a Poisson algebra.