Monodromy of projective hypersurfaces

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Galois group

We will work over $\mathbb{C}$.
Let $f : X \to Y$ a finite morphism between irreducible varieties of the same dimension; to f we can associate the Galois group

$$G = Gal(K/K(Y))$$

where K is the Galois closure of the extension field $K(X)/K(Y)$ and $K(Y)$ is the field of rational functions on Y.

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The monodromy group of linear projections

In particular our maps will be the followings:
let X be a irreducible, reduced, projective hypersurface of dimension n and degree d.
Take $p \notin X$ a point and let π_p be the restriction to X of the linear projection from p:

$$\pi_p : X \subset \mathbb{P}^{n+1} \to \mathbb{P}^n$$

that is a finite map of degree d.
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In the preimage of a general point $y \in U$ there are d distinct points $\Gamma := \{q_1, \ldots, q_d\}$ corresponding to the intersection of the line $< p, y >$ with X.
We can associate permutations of the general fiber Γ to loops in U centered in y: e.g. if we fix $\tilde{\gamma}(0) \in \Gamma$ where $\tilde{\gamma}$ is the lift of a loop γ centered in y, we can define a permutation inside Γ sending

$$\tilde{\gamma}(0) \mapsto \tilde{\gamma}(1) \in \Gamma$$

Hence we can define a map

$$\mu : \pi_1(U, y) \to S_d$$

Definition

The monodromy group of π_p is $M(\pi_p) := \mu(\pi_1(U)) \leq S_d$.

Proposition

The monodromy group $M(\pi_p)$ is isomorphic to the Galois group G.

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Properties

- $M(\pi_p)$ is a transitive subgroup of S_d
- A map π_p is said *decomposable* if it admits a non trivial factorization, i.e.

$$X \xrightarrow{\alpha} X' \xrightarrow{\beta} \mathbb{P}^n$$

  with $deg(\alpha), deg(\beta) > 1$.
  If $M(\pi_p) = S_d$ then π_p is indecomposable.
  Being indecomposable is equivalent to say that $M(\pi_p)$ is primitive, i.e. it does not preserve non trivial blocks.
- Conversely, if π_p is indecomposable and $M(\pi_p)$ containes a trasposition, then it is the whole S_d.

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Non uniform projections

Definition

We will say that a point $p \notin X$ is *uniform* if $M(\pi_p) = S_d$; *non uniform* otherwise.

We are interested in computing the dimension of the locus inside $\mathbb{P}^{n+1}$ of the non uniform points. Indeed:

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Main Theorem

Theorem (CCM)

Let X be a irreducible, reduced, smooth, non developable, projective hypersurface of dimension n and degree d; let $p \in \mathbb{P}^{n+1} \setminus X$ be a point and $\pi_p : X \to \mathbb{P}^n$ be the projection from p.
Then X admits at most a finite number of non uniform points.

X is non developable if its Gauss map has maximal rank, i.e. the dual of X is an hypersurface.

Remark

The smoothness hypotesis can be relaxed.

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Known results

- Pirola and Schlesinger showed the result for plane curves. Moreover, they allow $p \in X \subset \mathbb{P}^2$.
  ('Monodromy of projective curves'. J. Algebraic Geom, 2005)

- Cuzzucoli, Moschetti and Serizawa proved the result for smooth surfaces in $\mathbb{P}^3$.
  ('Non-uniform projections of surfaces in $\mathbb{P}^3$', Le matematiche, 2017)

Traspositions in the monodromy group

Recall that if $M(\pi_p)$ is primitive and containes a trasposition then $M(\pi_p) = S_d$.

We first study the existence of traspositions.

Proposition

In the above setting, take a point $y \in \mathbb{P}^n$ suh that $\pi_p^{-1}(y)$ is made by $d-1$ distinct points (i.e. y is a simple branch point). Then there is a trasposition in $M(\pi_p)$.

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Simple branch points correspond to simple tangent lines to the hypersurface X; then we study families of lines tangent to X by means of the classical theory of focal loci due to Segre.

Lemma

There are at most a finite number of points p such that there are no transposition inside $M(\pi_p)$.

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The same holds also for singular hypersurfaces.

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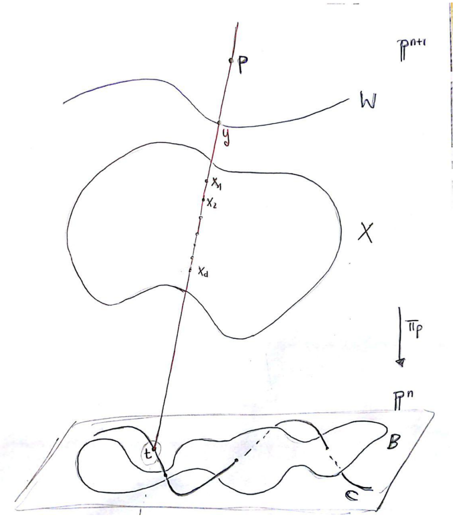
Primitive monodromy group

Let W be the closure of the locus of non uniform points in $\mathbb{P}^{n+1}$. From an induction argument we know that $\dim(W) \leq 1$. Assume $\dim(W) = 1$.

The general point of W contains a trasposition in its monodromy group, hence to be non uniform it must be non primitive.
To conclude we have to show that the general point of W has primitive monodromy group.



The picture is the following:

take a general point $p \in \mathbb{P}^{n+1}$ and project from it.
Let B be the branch locus of the projection π_p and $C := \pi_p(W)$ be the image of the curve W.

We have a map

$$\begin{tikzcd} \pi_1(C \setminus B, t) \arrow[r] & \pi_1(\mathbb{P}^n \setminus B, t) \arrow[d, "\mu"] \\ & M(\pi_p) = S_d \end{tikzcd}$$

If C and B intersect transversally everywhere than we have

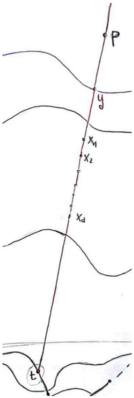
Nori's Lemma

$\pi_1(C \setminus B) \twoheadrightarrow \pi_1(\mathbb{P}^n \setminus B)$

Proof of the main theorem

If we are in the previous situation, we are able to study the monodromy of π_p just looking at

$$\mu : \pi_1(C \setminus B) \longrightarrow S_d$$



Consider a general fiber over C of our projection from p; the line will meet W in a point y.

Note that the d points $x_1 \ldots, x_d$ in which the line meets X are the same if we project from p or from y.

Work in progress

- Cukiermann showed that for $X \subset \mathbb{P}^2$ general curve, $W = \emptyset$.
  Our first aim is to generalize this result for general hypersurfaces in every dimension.
- Estimate the dimension of the locus of non uniform centers of projections for higher codimension varieties.

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