W^J parabolic quotient of Coxeter group

$$\mathcal{F} \subseteq W^J \longrightarrow \Delta_{\mathcal{F}}$$

combinatorial properties of $\mathcal{F}$ ($\mathcal{F}$ a Coxeter matroid)

geometric properties of $\Delta_{\mathcal{F}}$

Matroids

Flag matroids

Coxeter groups

Bruhat orderings

Schubert and Richardson variety

Shellability

Lifting properties

Put everything together

Open problems

Matroids

1930's Whitney

A combinatorial way to abstract "linear dependence" among vectors

$S = \{v_1, \dots, v_n\}$

$V = \text{Span } S \qquad \dim V = k$

Exchange property

$B, B' \subset S$ bases of V

$\forall\, b \in B \setminus B'$ $\exists\, b' \in B' \setminus B$ s.t.

$B \setminus \{b\} \cup \{b'\}$ is a basis

$[n] = \{1, \dots, n\}$

$\mathcal{J}_n^k$ set of subsets of $[n]$ of cardinality k

<u>Def.:</u> $\mathcal{M} \subseteq \mathcal{J}_n^k$ is a matroid

$\forall\, I, J \in \mathcal{M}$ $\forall\, a \in I \setminus J$

$\exists\, b \in J \setminus I$: $(I \setminus \{a\}) \cup \{b\} \in \mathcal{M}$.

If $\mathcal{M}$ "comes" from a set of vectors we say $\mathcal{M}$ is representable.

$\mathcal{M}$ represented by $v_1, \dots, v_n \in \mathbb{C}^k$

$$A = \begin{pmatrix} v_1 \,|\, v_2 \,|\, \cdots \,|\, v_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_k \end{pmatrix}$$

$U = \mathrm{Span}\{u_1, \dots, u_k\} \in Gr_k(\mathbb{C}^n)$

Start from $U \in Gr_k(\mathbb{C}^n)$

choose a basis $u_1, \dots, u_k$ of U

$$A = \begin{pmatrix} u_1 \\ \hline u_2 \\ \vdots \\ \hline u_k \end{pmatrix}$$

$\mathcal{M}$ matroid associated to the columns.

Does not depend on the chosen basis.

$$U \longmapsto \mathcal{M}$$

$I \in \mathcal{M}$ means that columns in position I in A are independent.

$C_J = \mathrm{Span}\{e_j : j \in J\}$

$U \cap C_{[n] \setminus I} = \{0\}$

Matroid stratification of the Grassmannian:

U, U' belong to the same stratum if for all $I \subset [n]$

$\dim(U \cap C_I) = \dim(U' \cap C_I)$

Schubert cellular decomposition of $Gr_k(\mathbb{C}^n)$. For every permutation $w \in S_n$ we have such decomposition

U, U' belong to the same w-cell if

$\dim(U \cap C_{\{w_1, w_2, \dots, w_k\}}) = \dim(U' \cap C_{\{w_1, \dots, w_k\}})$

for all k

$w_1 = w(1)$ ....

A first polytope

Torus action on $Gr_k(\mathbb{C}^n)$

$T = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by multiplying coordinates, hence on $Gr_k(\mathbb{C}^n)$

Plucker coordinates

$U \rightsquigarrow A \qquad J \subseteq [n] \qquad |J| = k$

$Pl_J(A) =$ determinant of the $k \times k$ submatrix of A with columns in position J

Moment map

$$\mu : Gr_k(\mathbb{C}^n) \longrightarrow \mathbb{R}^n$$

$$U \longmapsto \frac{\sum_J |Pl_J(A)|^2 e_J}{\sum_J |Pl_J(A)|^2}$$

where $e_J = \sum_{j \in J} e_j$

$\mathcal{M} \subseteq \mathcal{F}_n^k$

$$\Delta_{\mathcal{M}} = \text{Convex Hull}\{e_J : J \in \mathcal{M}\}$$

Theorem (GGMS, 1987). $U \in Gr_k(\mathbb{C}^n)$

$$\mu\left(\overline{TU}\right) = \Delta_{\mathcal{M}}$$

where $\mathcal{M}$ is the matroid associated to U.

<u>Example</u>: $U \subseteq \mathbb{C}^3$: $x_1 - x_3 = 0$

$e_1 + e_3$, e_2 basis of U

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a generic element in the orbit is

$$\begin{pmatrix} a & 0 & c \\ 0 & b & 0 \end{pmatrix} \qquad a,b,c \neq 0$$

is the same as a subspace to

$$\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \end{pmatrix} \qquad \cancel{c \neq 0} \text{ every } c \in \mathbb{C}$$

$J = \{1,2\}$

$$\mu \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \end{pmatrix} = \frac{1 \cdot (e_1 + e_2) + |c|^2 (e_2 + e_3)}{1 + |c|^2}$$

$$= \text{Convex Hull}\,(e_1 + e_2,\ e_2 + e_3)$$

Maximality property

Def.: $I, J \subseteq [n] \qquad |I| = |J| = k$

$I = \{i_1, i_2, \dots, i_k\}$ $\quad i_1 < i_2 < \dots < i_k$

$J = \{j_1, j_2, \dots, j_k\}$ $\quad j_1 < j_2 < \dots < j_k$

$I \leq J$ if

$i_1 \leq j_1 \quad i_2 \leq j_2 \quad \dots \quad i_k \leq j_k$

Example

$I = \{1, 3, 5\} \qquad J = \{2, 3, 6\}$

$I \leq J$

$\{1, 4\}$ & $\{2, 3\}$ are not comparable.

$w \in S_n$

$w_1 <^w w_2 <^w w_3 \quad \dots$

$I \leq^w J$ if $w^{-1} I \leq w^{-1} J$

$w = [3,1,2]$ $\qquad 3 <^w 1 <^w 2$

$\{1,3\} \leq^w \{1,2\}$

Theorem (Gale, 1968): $\mathcal{M} \subseteq \mathcal{F}_n^k$

$\mathcal{M}$ is a matroid if and only if for every $w \in S_n$ there exists $I_w \in \mathcal{M}$ s.t. $I \leq^w I_w$ for all $I \in \mathcal{M}$.

(maximality property).

<u>Proof</u>: assume maximality property.

$I, J \in \mathcal{M}$

$I = \{i_1, i_2, \dots, i_k\}$ $\qquad i_1 \notin J$

$J \setminus I = \{j_1, \dots, j_\ell\}$

$w = [\ \dots\ i_1, j_1, \dots, j_\ell, i_2, \dots, i_k]$

I_w = maximum with respect to w-order.

since $I \leq^w I_w \Rightarrow i_2, \dots, i_k \in I_w$

J? $J \leq^w I_w$ all elements in I_w are w-greater than j_1 so there is $h : j_h \in I_w$.

Flag matroids

$\underline{k} = (k_1, \dots, k_m)$

a flag of rank $\underline{k}$ in $[n]$ is

$$F^\circ = (F^1 \subset F^2 \subset \dots \subset F^m)$$

$|F^i| = k_i \qquad F^m \subset [n]$

$\mathcal{F}_n^{\underline{k}}$ = set of flags of rank $\underline{k}$ in $[n]$

Partial order

$w \in S_n$

$F^\circ \leq^w G^\circ$ if $F^i \leq^w G^i$ for all i

Def.: $\mathcal{F} \subseteq \mathcal{F}_n^k$ is a flag matroid if $\forall w \in S_n$ $\exists$ $F_w^\circ \in \mathcal{F}$ s.t.

$$F_w^\circ \geq^w F^\circ \quad \forall F^\circ \in \mathcal{F}.$$

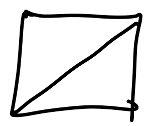
Observation: $\mathcal{F}$ a flag matroid.

$\mathcal{M}_i$ = set of i-th constituents of flags in $\mathcal{F}$

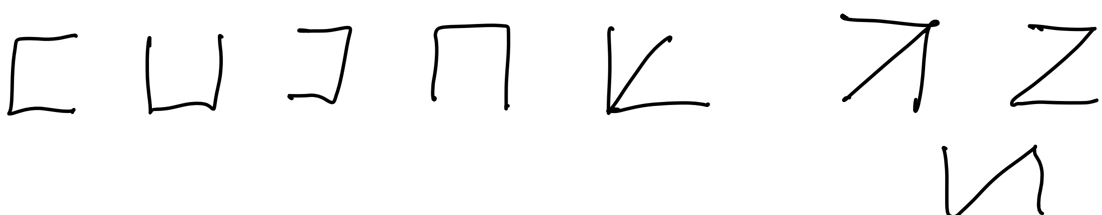
Then $\mathcal{M}_i$ are all matroids.

Def.: $\mathcal{M}$ matroid. A subset $C \subseteq [n]$ is a circuit if it is not contained in any basis of $\mathcal{M}$ but every proper subset of C is.

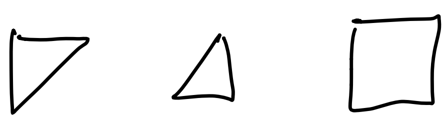
Terminology from graph theory.

G 

Matroid: bases are set of edges in a spanning tree



circuits are the circuits



<u>Def.</u>: $\mathcal{M}'$ is a quotient of $\mathcal{M}$ if every circuit of $\mathcal{M}$ is a union of circuit of $\mathcal{M}'$.

<u>Theorem:</u> $\mathcal{F} \subseteq \mathcal{F}_n^{k}$ with constituents $\mathcal{M}_1, \dots, \mathcal{M}_m$. $\mathcal{F}$ is a flag matroid iff

1. $\mathcal{M}_i$'s are matroids
2. $\mathcal{M}_i$ is a quotient of $\mathcal{M}_{i+1}$
3. $F^\circ = (F^1 \subset F^2 \subset \dots)$ s.t. $F^i \in \mathcal{M}_i$ for all i

$$\Rightarrow F^\circ \in \mathcal{F}$$

<u>Example</u>: $abc = (\{a\} \subset \{a,b\} \subset \{a,b,c\})$

$$\Uparrow$$

$$\mathcal{F}_4^{1,2,3}$$

$\mathcal{F} = \{123, 124, 142, 143, 321, 341, 412, 413, 421, 431\}$

$\mathcal{M}_1 = \{\{1\}, \{3\}, \{4\}\}$

$\mathcal{M}_2 = \{\{12\}, \{14\}, \{23\}, \{34\}, \{24\}\}$

$\mathcal{M}_3 = \{\{123\}, \{124\}, \{134\}\}$

Circuits in $\mathcal{M}_3$: $\{2,3,4\}$

" " $\mathcal{M}_2$: $\{\{13\}, \{124\}, \{234\}\}$

" " $\mathcal{M}_1$: $\{\{2\}, \{13\}, \{14\}, \{34\}\}$

$\mathcal{M}_2' = \{12, 13, 14, 24, 34\}$ circuits become

$\{23, 124, 134\}$

for $w = [1,4,2,3]$ the w-max in $\mathcal{M}_3$ is $\{1,2,3\}$

the w-max in $\mathcal{M}_2'$ is $\{3,4\}$

so there is no w-max in $\mathcal{J}$.

Representable flag matroids

$U^0 = U_1 \subset U_2 \subset \dots \subset U_m$ subspaces of $\mathbb{C}^n$

with $\dim(U_i) = k_i$

A matrix s.t. the first k_i rows span U_i

For all $I \subset [n]$ $|I| = k_i$

$Pl_I(A) = \det \ldots$

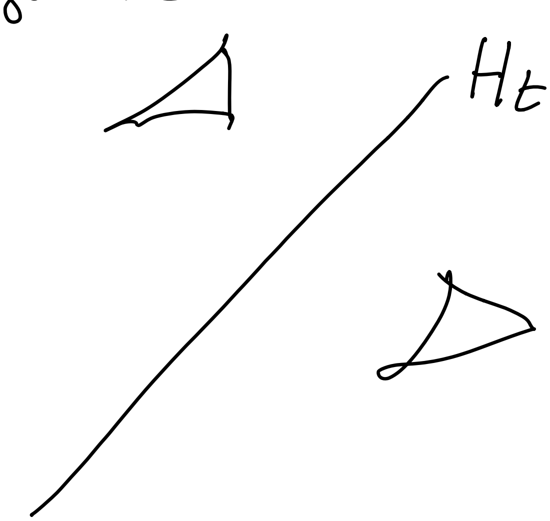
$\mathcal{F} = \{ F^\circ = (F^1 \subset F^2 \subset \ldots) : Pl_{F^i}(A) \neq 0 \}$

is a flag matroid.

Finite reflection groups

$\mathbb{R}^n$

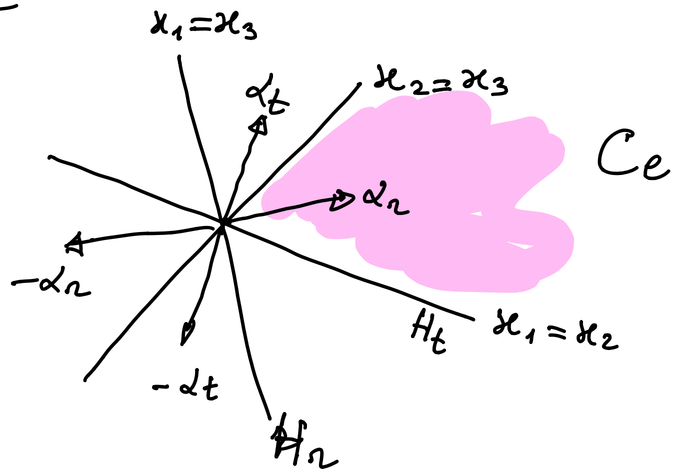
A reflection t is a linear isometry which fixes a hyperplane H_t pointwise and sends an orthogonal vector to H_t to its negative



Def: a finite reflection group is a finite group generated by reflections.

T = set of all reflections

<u>Example</u> $\mathbb{R}^3$



$W = S_3$

Connected components of $\mathbb{R}^n \setminus \bigcup H_t$ are called chambers.

<u>Fact</u>: the action of W on chambers is simply transitive

Choose a chamber : call it fundamentale and denote it by C_e

$$wC_e = C_w$$

For each $t \in T$ take two opposite vectors orthogonal to H_t and call α_t the one pointing towards C_e

$$\Phi_+ = \{\alpha_t : t \in T\}$$

In the case of S_n if we choose C_e as the chamber containing

$$(n, n-1, \dots, 3, 2, 1)$$

The positive roots are $e_i - e_j$, $i<j$.

$$\Phi_+^w = \{\text{roots pointing towards } C_w\}$$

if $W = S_n$

$$\Phi_+^w = \{e_{w(i)} - e_{w(j)} \ , \ i<j\}$$

Fact: can choose $S \subset T$
$\{\alpha_s : s \in S\}$ are linearly independent.

For all $t \in T$

α_t is a non negative linear combination of α_s, $s \in S$.

S generates W

(W, S) is a Coxeter system.

$W = S_n$ $\quad C_e \ni (n, n-1, \dots, 1)$

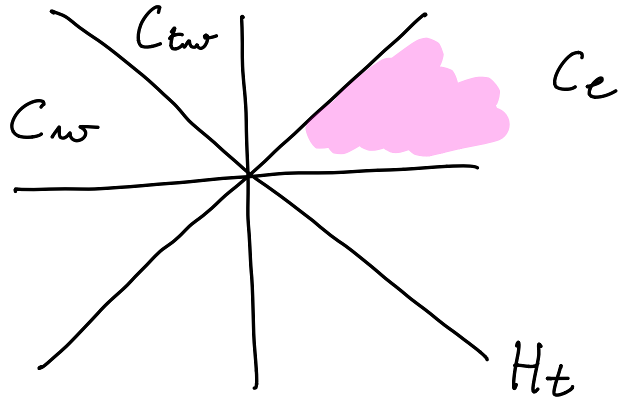
$S = \{s_1, \dots, s_{n-1}\}$ $\quad s_i = (i, i+1)$

Length

$\ell(w) =$ minimal ℓ : w is a product of ℓ elements in S

$=$ # hyperplanes H_t which separate C_w from C_e

$\mathbb{R}^2$



$\ell(w) = 3$

Bruhat order

transitive closure of the relation

$u = tv$ for some $t \in T$

$\ell(u) < \ell(v) \Rightarrow u < v$ in Bruhat order

we have

$tw < w \Leftrightarrow H_t$ separates C_w from C_e

$\ell(w) = \#\{t \in T : tw < w\}$

Corollary: $t \in T$ $w \in W$ $p \in C_w$

$t \in D(w)$ i.e. $tw < w \iff t(p) = p + c\alpha_t$

$c > 0$.

Prop.: $v = s_1 \cdots s_\ell$ $\ell = \ell(v)$

Then $u \leq v$ iff there exist

$1 \leq i_1 < i_2 < \cdots < i_k \leq \ell$:

$u = s_{i_1} s_{i_2} \cdots s_{i_k}$

All Weyl group are finite reflection groups.

Other cases: - dihedral groups

- H_3

- H_4

Parabolic quotients

(W, S) $J \subseteq S$

W_J = subgroup generated by J

W^J = set of W_J left cosets

$W^J \longleftrightarrow \{ w \in W : ws > w \quad \forall s \in J \} = W^J$

$u, v \in W^J$ we say $u \leq v$ if the minimal representative of u is $\leq$ than the " " " v.

$J = S \setminus \{ k_1, \dots, k_m \}$

$w \in W^J$ iff

$w_1 < \dots < w_{k_1} \quad w_{k_1+1} < \dots < w_{k_2} \quad \dots$

Bijection between

$$W^J \longleftrightarrow \mathcal{F}_n^{\underline{k}}$$

Example $\underline{k} = (3, 5, 8) \quad n = 9$

$J = S \setminus \{ s_3, s_5, s_8 \}$

$w = [2, 5, 9, 1, 7, 3, 4, 8, 6] \in W^J$

$F^\circ = (\{2,5,9\}, \{1,2,5,7,9\}, \{1,2,3,4,5,7,8,9\})$

$w(F^\circ(u)) = F^\circ(wu)$

Ehresmann criterion (1930's)

$u, v \in S_n^{S \setminus \{s_{k_1}, s_{k_2}, \dots, s_{k_m}\}}$

$u \leq v \iff F^\circ(u) \leq F^\circ(v)$

$wu \leq wv \iff wF^\circ(u) \leq wF^\circ(v) \iff$

$$F^\circ(u) \leq^{w^{-1}} F^\circ(v)$$

Example

$u = [2,1,4,3] \qquad v = [3,4,2,1]$

$u \leq v$

<u>Definition</u> : $\mathcal{F} \subseteq W^J$. Then

$\mathcal{F}$ is a Coxeter matroid if
for all $w \in W$ $\quad w\mathcal{F} \subseteq W^J$
has a maximum in Bruhat order.

Assume $J = \emptyset$

<u>Example:</u> $v = tu \quad t \in T \Rightarrow \{u, v\}$
is a Coxeter matroid

$w \in W \quad \{wu, wv\}$ has a maximum

$$wv = wtu = \underbrace{(wtw^{-1})}_{\in T} wu$$

$\{e, [3,1,2]\} \subset S_3$ $\quad$ not a Coxeter matroid

$w = [1,3,2]$

we obtain $\{[1,3,2], [2,1,3]\}$

Exercise: $\{u, v\}$ is a Coxeter matroid iff $uv^{-1} \in T$.

Main theorem: let $u \leq v$ then the Bruhat interval

$$[u, v] = \{z : u \leq z \leq v\}$$

is a Coxeter matroid.

$W = S_n$ proved by Kodama–Williams 2014

W Weyl group by Williams–Tsukerman 2016

Theorem: given $u \leq v$ then there exists a flag of subspaces of $\mathbb{C}^n$

$U^\circ = (U_1 \subset U_2 \subset \dots \subset U_n)$ s.t.

the flag matroid associated to U° is $[u, v]$.

$Fl(n)$ full flag variety

Schubert cells

$u = [3, 1, 4, 2]$

$$C_u = \left\{ \begin{bmatrix} * & * & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & * & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad * \in \mathbb{C} \right\}$$

$$C'_u = \left\{ \begin{bmatrix} 0 & 0 & 1 & * \\ 1 & * & 0 & * \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad * \in \mathbb{C} \right\}$$

<u>Obs.:</u> $u \qquad u[k] = \{u_1, \dots, u_k\}$ $\qquad I \subset [n]$

$|I| = k$

$$\boxed{\begin{array}{l} Pl_I \not\equiv 0 \text{ in } C'_u \\ \iff u[k] \leq I \end{array}}$$

$$Pl_I \not\equiv 0 \text{ in } C_v \iff v[k] \geq I$$

Richardson cell

$$R_{u,v} = C'_u \cap C_v$$

$$R_{u,v} \neq \emptyset \iff u \leq v$$

$$\overline{R}_{u,v} = \bigcup_{u \leq x \leq y \leq v} R_{x,y}$$

<u>Corollary:</u> $z \in S_n$. Then

$$Pl_{z[k]} \not\equiv 0 \text{ in } R_{u,v} \quad \forall k \in [n]$$

$$\iff \quad u \leq z \leq v$$

<u>Proof:</u> $u \leq z \leq v \qquad R_{z,v} \subset \overline{R_{u,v}}$

$Pl_{z[k]} \not\equiv 0$ in $R_{z,v}$ the same is true in $R_{u,v}$

viceversa if

$Pl_{z[k]} \not\equiv 0$ in $R_{u,v} \quad u[k] \leq z[k] \leq v[k]$

for all $k \iff u \leq z \leq v$ $\square$

If $\mathcal{J} \subseteq W$ we construct a polytope

$\Delta_{\mathcal{J}}$

choose any point p in the chamber opposite to the fundamental chamber.

$$\Delta_{\mathcal{J}} = \text{Convex Hull}\{w(p) : w \in \mathcal{J}\}$$

- if $\mathcal{J} = S_n$ $\quad p = (1, 2, \dots, n)$

  $\Rightarrow \Delta_{\mathcal{J}}$ is the permutohedron.

- if $p = (\underbrace{-1, -1, \dots, -1}_{k}, 0, \dots, 0)$ we obtain the moment map polytope.

<u>Theorem</u>: (Gelfand-Serganova late 1980's)

$\mathcal{J} \subseteq W$. The following are equivalent:

① $\mathcal{J}$ is a Coxeter matroid

② $\Delta_{\mathcal{J}}$ has all edges parallel to roots.

Idea of proof for $W = S_n$

<u>Lemma:</u> $p = (p_1, \dots, p_n)$ $p_1 < p_2 < \dots < p_n$

$u, z, w \in S_n$: $wu < wz$. Then

$z(p) - u(p)$ is a positive sum of w-positive roots.

<u>Proof:</u> $w = e$ $n = 3$

$u_1 < z_1$ $h_1 = p_{z_1} - p_{u_1} > 0$

$\{u_1, u_2\} \leq \{z_1, z_2\}$ $h_2 = p_{z_1} + p_{z_2} - p_{u_1} - p_{u_2} > 0$

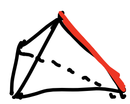
$z(p) - u(p) = h_1(e_1 - e_2) + h_2(e_2 - e_3)$ $\square$

Proof that ① $\Rightarrow$ ② in GS-theorem.

Let ν be an edge with vertices $u(p)$ and $v(p)$.

$f = a$ s.t. the edge v is contained in the hyperplane $f = a$

and the rest of the polytope is contained in the halfspace $f < a$.



can assume that $f(e_i)$ are all distinct.

w s.t.

$f(e_{w_1}) > f(e_{w_2}) > \dots$

f is positive on w-positive roots

$z \in \mathcal{F}$ $z \neq u$

$f(z(p)) \leq f(u(p)) \Rightarrow$

$z(p) - u(p)$ is not a positive linear combination of w-positive roots

by lemma $wu \not\leq wz \Rightarrow$

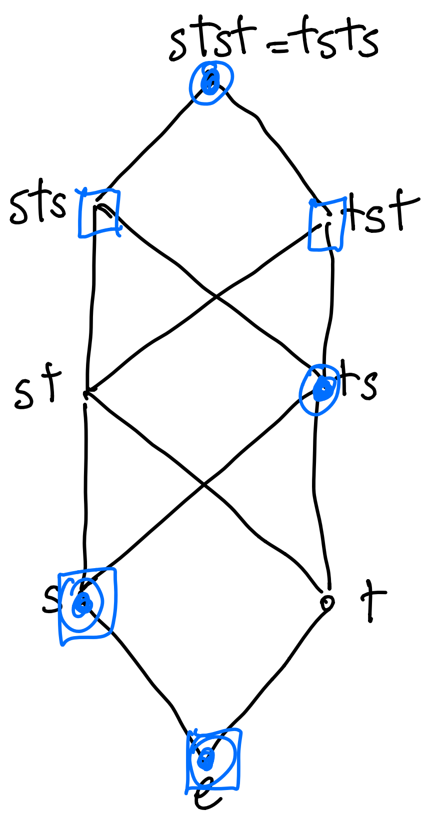
wz is not the maximum of $w\mathcal{F}$

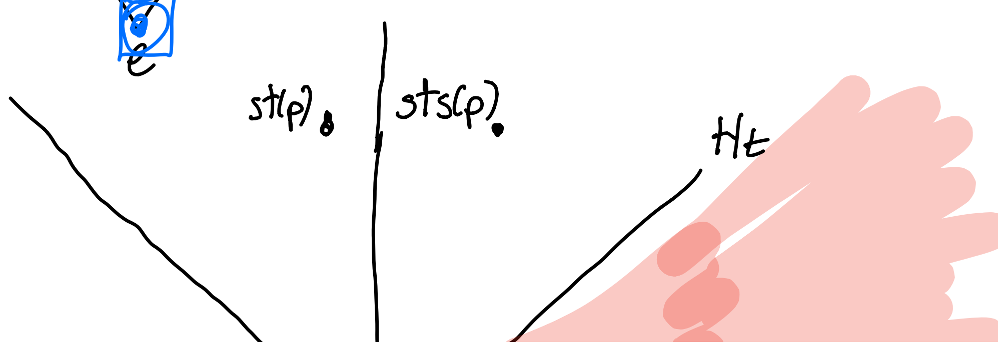
$\Rightarrow$ wu is the maximum of wJ

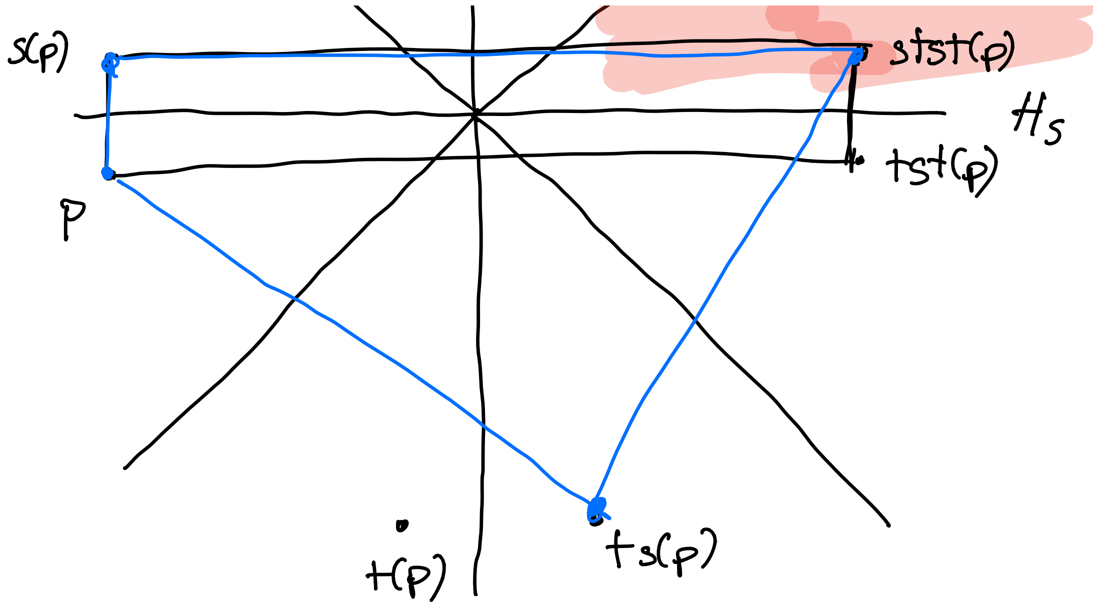
~~...~~ wv would also be a maximum $\square$.

Example

$W = B_2$







Target: prove that Bruhat intervals are Coxeter matroid.

Shellability

Γ simplicial complex

Γ set of subsets of $[n]$ closed with respect to taking subsets

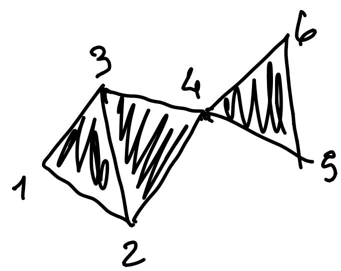
pure of dimension d

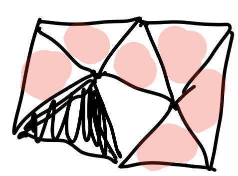
Def.: Γ pure of dimension d is shellable if we can order the facets $F_1, \dots, F_r$ in such way that for all $i<j$
$\exists\, k<j$:

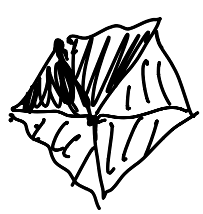
$F_i \cap F_j \subseteq F_k \cap F_j \quad |F_k \cap F_j| = d$

$\|\Gamma\|$ geometric realization

Facets : $\{1,2,3\}, \{2,3,4\}, \{4,5,6\}$



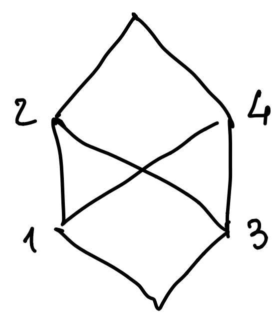




P graded poset

$\Gamma(P)$ order complex

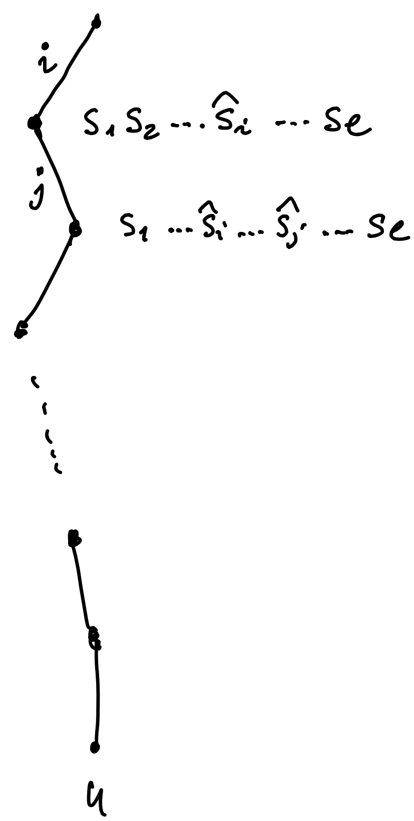
= set of totally ordered subsets of P (chains in P)



Facets of $\Gamma(P)$

$\{1,2\}, \{2,3\}$ $\{1,4\}, \{3,4\}$.

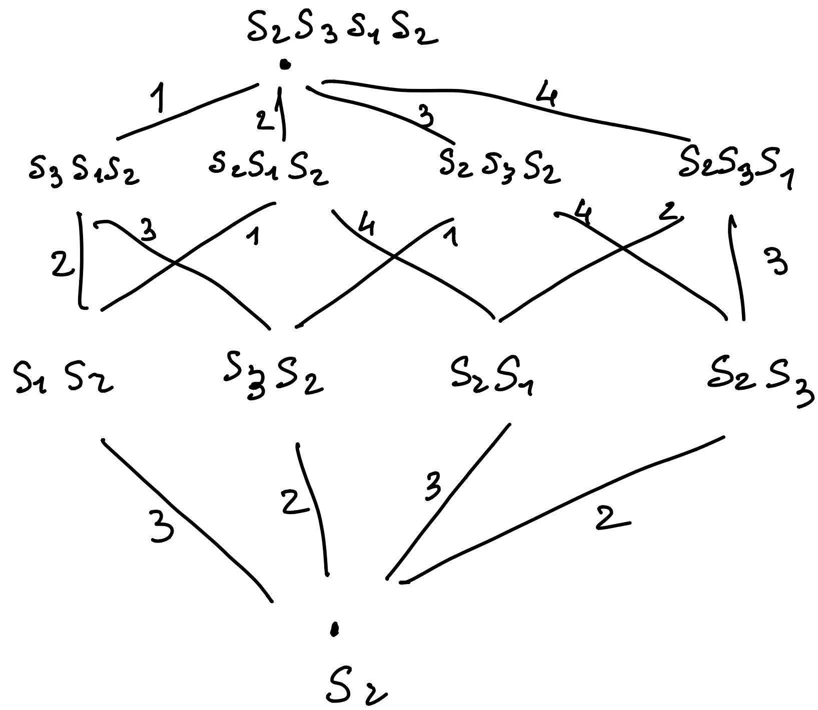
$w = s_1 s_2 \cdots s_\ell$



m a chain

$\lambda(m) = (\lambda_1, \lambda_2, \dots, \lambda_d)$ labels of m from top to bottom

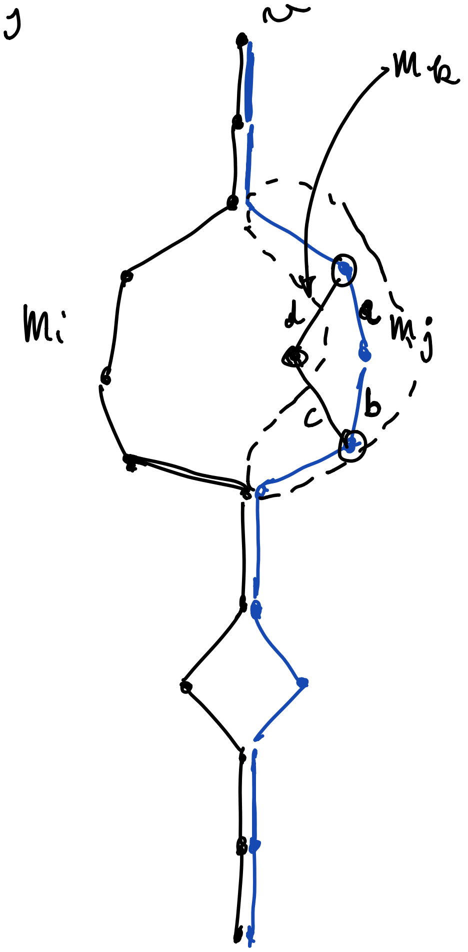
<u>Prop.</u>: there is exactly one maximal chain from u to v with increasing labels. Such chain has labels $\lambda(m)$ lexicographically minimal wrt all chains.



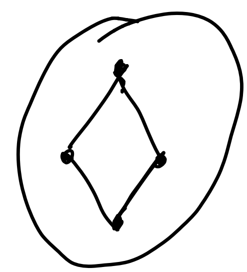
Theorem (Bjorner-Wachs, 1982): Bruhat intervals are shellable.

Proof.: $m_1, m_2, \dots, m_r$ ordered lexicographically wrt labels

$i < j$



$k < j$



$m_i \cap m_j \subset m_k \cap m_j$

$|m_k \cap m_j|$

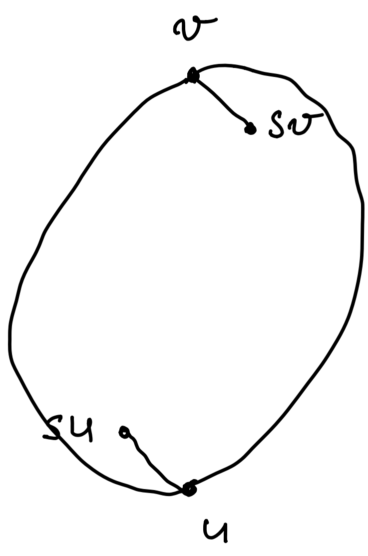
Corollary: Geometric realizations

of Bruhat intervals are homeomorphic to spheres.

<u>Corollary</u>: given 2 maximal chains m & m' there exists a sequence $m = m_0, m_1, \dots, m_k = m'$ s.t.
m_i & m_{i+1} differ by exactly one element.

Lifting properties

Standard one:



$u < v$

<u>if</u> $s \in S$:

$sv < v$ &

$su > u$

$\Rightarrow$

$su, sv \in [u, v]$

Generalized lifting property:

<u>Theorem</u> (C-Sentinelli, 2017): Let W be a (possibly infinite) Coxeter group. TFAE

① W is finite and simply laced

② for all $u < v$ $\exists t \in T$:

$u \triangleleft tu \leq v$

$u \leq tv \triangleleft v$

$W = S_n$ ① $\Rightarrow$ ② was proved by Williams-Tzukerman 2015.

<u>Exercise:</u> W finite reflection group. H_t^+: halfspace containing C_e determined by the hyperplane H_t TFAE

a. $t(H_r^+) = H_{r^t}^-$ where $r^t = trt$

b. $r \in D(t)$, i.e. $rt < t$

c. $r \in D(tu) \Leftrightarrow r^t \notin D(u)$ for any $u \in W$

d " " for every $u \in W$

Def.: $\alpha_1, \dots, \alpha_n$ simple roots

dominance order on roots: $\alpha_t \preceq \alpha_r$ if $\alpha_r - \alpha_t$ is a positive sum of simple roots

<u>Main lemma:</u> W finite & simply laced $u < v$ & t a minimal reflection in $D(v) \setminus D(u)$. Let $r \in D(t) \setminus \{t\}$. then

$$r \in D(u) \Leftrightarrow r \in D(v) \Leftrightarrow r^t \notin D(v) \Leftrightarrow r^t \notin D(u)$$

Proof uses Dyer's theory of reflection subgroups

Prove "something" of GLP.

$u < v$ t minimal in $D(v) \setminus D(u)$ then

$\ell(tu) = \ell(u) + 1$.

$\varphi: D(u) \longrightarrow D(tu) \setminus \{t\}$

$$\varphi(r) = \begin{cases} r & r \in D(t) \\ r^t & r \notin D(t) \end{cases}$$

- $r \in D(t)$

$$r \in D(u) \Longleftrightarrow r^t \notin D(u) \Longleftrightarrow r \in D(tu)$$

similarly

- $r \notin D(t)$

$$r \in D(u) \Longleftrightarrow r^t \in D(tu).$$

Patimo (2018): combinatorial invariance:

W, W' finite & simply laced

$u, v \in W \quad u', v' \in W'$:

$[u,v] \cong [u',v']$ as posets

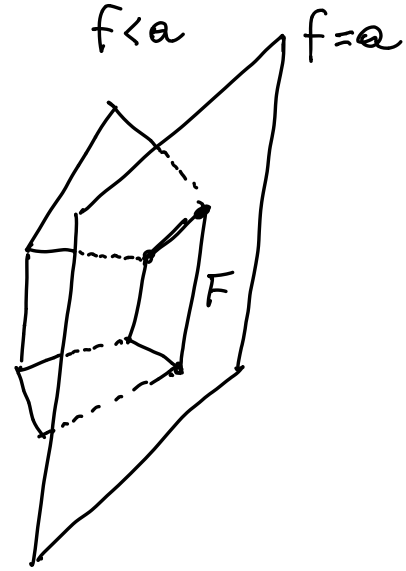
the coefficients of degree 1 in

$P_{u,v}(q)$ & $P_{u'v'}(q)$ agree.

Exercise: B_2 GLP does not hold.

$u < v$ $\qquad \Delta_{[u,v]}$

assume that $x(p)$ $y(p)$, with $x<y$, belong to a face F of $\Delta_{[u,v]}$



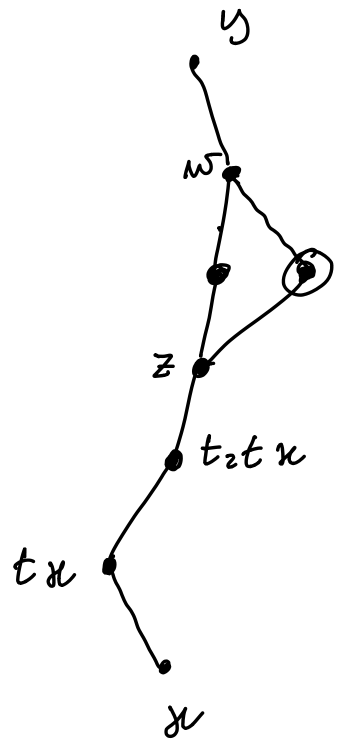
By GLP $\exists t$: $\quad tx \rhd x \quad ty \lhd y$

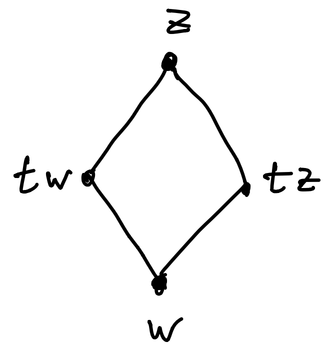
$$t(x(p)) = x(p) + c\,\alpha_t \qquad c>0 \Rightarrow f(\alpha_t) \le 0$$

$$t(y(p)) = y(p) - d\,\alpha_t \qquad d>0 \Rightarrow f(\alpha_t) \ge 0$$

$$\Rightarrow f(\alpha_t) = 0$$

$\Rightarrow t(x(p)) \in F$





<u>Theorem:</u> if ~~GLP holds~~ and $u \le x \le y \le v$ and $x(p), y(p)$ belong to a face F of $\Delta[u,v]$ $\Rightarrow z(p) \in F \quad \forall z \in [x, y]$.

Reflection ordering

It's a total ordering on Φ_+ s.t.

$\forall \alpha, \beta, \gamma \in \Phi_+$ s.t. $\gamma = a\alpha + b\beta \quad a,b>0$

$\Rightarrow \alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$.

Reflection orderings exists and for finite Coxeter group they are in bijection with reduced expressions of the longest elements.

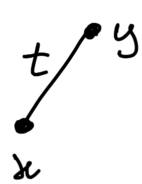
Example: For S_n $\quad \alpha_{ij} = e_i - e_j \quad i<j$

lex order on the indices ij is a reflection ordering

$$\alpha_{ik} = \alpha_{ij} + \alpha_{jk} \qquad i<j<k$$

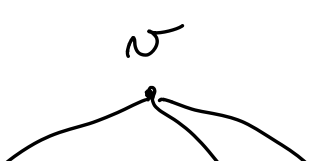
$$\alpha_{ij} < \alpha_{ik} < \alpha_{jk}$$

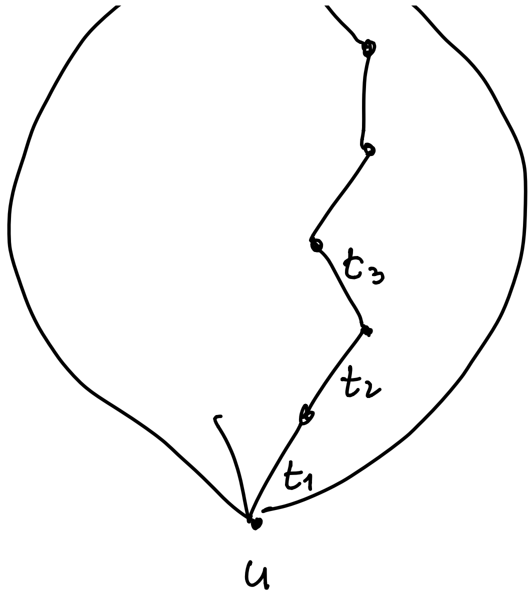
We can transfer this ordering on reflections



$x \lessdot y \Rightarrow$

$y = tx \qquad t \in T$





a maximal chain is increasing is the corresponding labels are

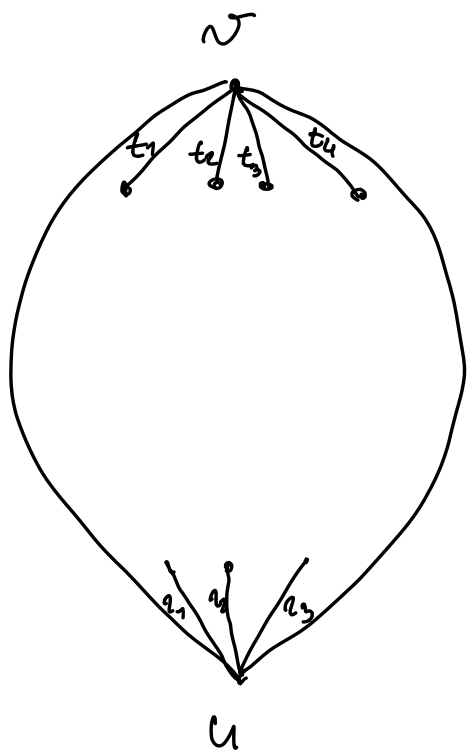
Prop.: $u<v$, for every refl. ordering $\prec$, there exists a unique increasing chain from u to v.

Joint work with D'Adderio-Marietti.

Let $x \in [u,v]$

$U_{u,v}(x) = \{\alpha_t \in \Phi_+ : x \lhd tx \leq v\}$

$D_{u,v}(x) = \{\alpha_t \in \Phi_+ : u \leq tx \lhd x\}$



$D_{u,v}(v) =$

$\{\alpha_{t_1}, \alpha_{t_2}, \alpha_{t_3}, \alpha_{t_4}\}$

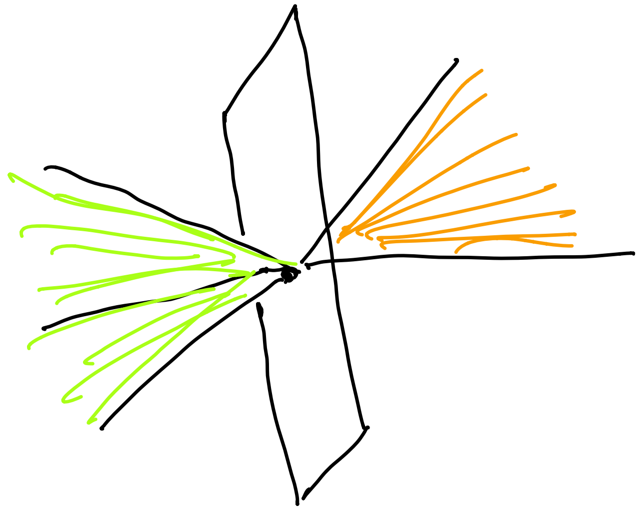
$U_{u,v}(u) = \{\alpha_{r_1}, \alpha_{r_2}, \alpha_{r_3}\}$

GLP: $D_{u,v}(v) \cap U_{u,v}(u) \neq \emptyset$

<u>Theorem:</u> (week GLP): in every Coxeter group $\mathrm{Cone}(D_{u,v}(v)) \cap \mathrm{Cone}(U_{u,v}(u)) \neq \{0\}$

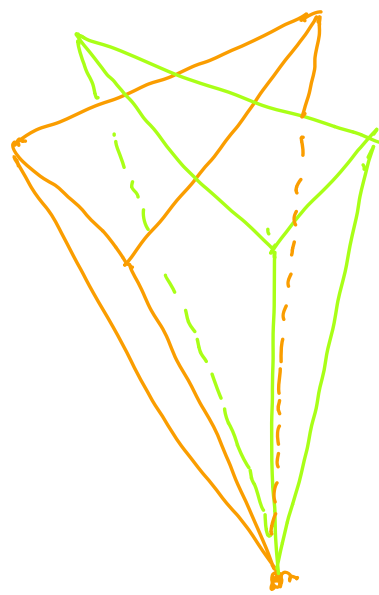
$\mathrm{Cone}(A) = \{$ non negative linear comb. of elements in $A\}$

Idea :

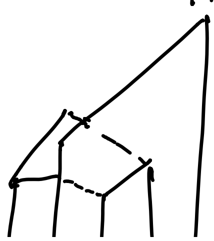


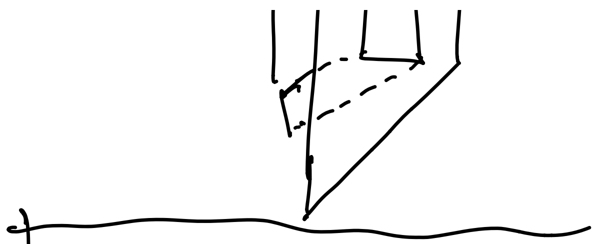
Open problem : is it true

$$U_{u,v}(u) \cap \mathrm{Cone}(D_{u,v}(v)) \neq \emptyset \, ?$$



$H : f = a$





$$\boxed{\sum a_i \alpha_{r_i} = \sum b_j \alpha_{t_j}} \qquad a_i, b_j > 0$$

$$r_i(x(p)) = x(p) + c_i \alpha_{r_i} \qquad c_i > 0$$

$$t_j(y(p)) = y(p) - d_j \alpha_{t_j} \qquad d_j > 0$$

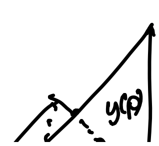
$$f(c_i \alpha_{r_i}) = c_i f(\alpha_{r_i}) \leq 0 \Rightarrow f(\alpha_{r_i}) \leq 0$$

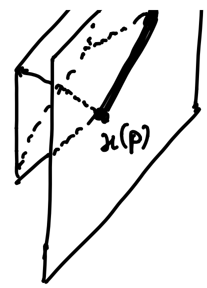
similarly $f(\alpha_{t_j}) \geq 0$

$$\Rightarrow f(\alpha_{r_i}) = f(\alpha_{t_j}) = 0 \quad \ldots$$

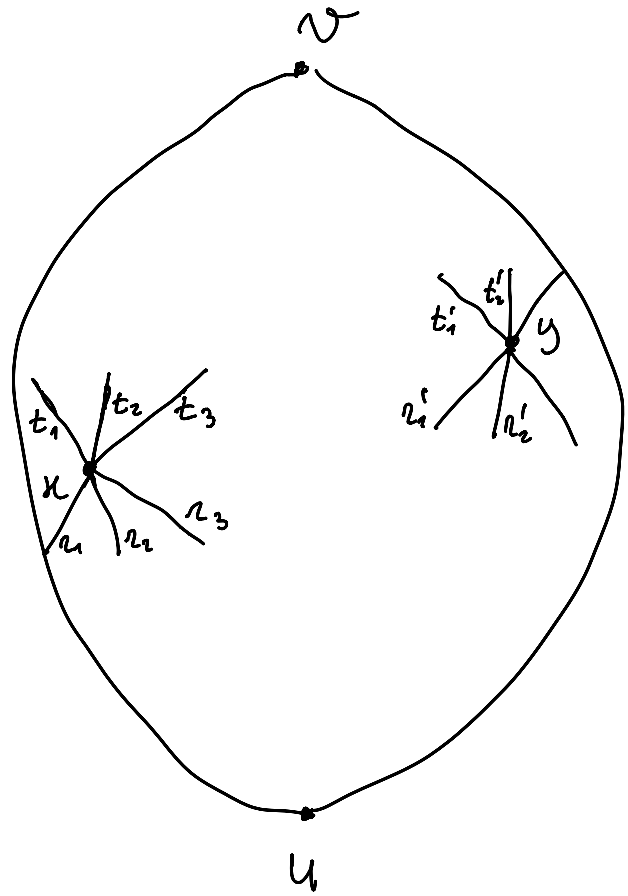
<u>Theorem:</u> edges of $\Delta_{u,v}$ are parallel to roots and so $[u,v]$ is a Coxeter matroid.

<u>Pf.:</u> $x(p) - y(p)$ be an edge





Show that x & y are comparable in Bruhat order. Otherwise



Can find a hyperplane which separates all roots α_{t_i} & $\alpha_{t_i'}$ from α_{r_i} & $\alpha_{r_i'}$ (exercise)

$\Rightarrow$ there is a reflection ordering s.t. all r_i & r_i' are smaller than t_i & t_i'

Concatenation of increasing chains from u to x and from x to v provides an increasing chain from u to v. Similarly for y $\Rightarrow$ x & y are comparable by uniqueness.

If $x < y$ all elements $z(p)$: $x \leq z \leq y$ must belong to the edge.

$\Rightarrow$ $x \lhd y$ $\Rightarrow$ $y = tx$ $\Rightarrow$ the edge $x(p)$ $y(p)$ is parallel to α_t.

Open problems

- Theory for affine Coxeter groups.
- W any Coxeter group $u, v, w \in W$.

is it true that $w[u,v]$ has a maximum.

- If W is a Weyl group: let $\mathcal{F}$ be

a representable Coxeter matroid.

Is it true that $\mathcal{F} = w[u,v]$?