De Concini - Procesi models of arrangements and symmetric group actions

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Contents

Chapter 0

Introduction

In this thesis we deal with the models of subspace arrangements which De Concini and Procesi introduced in [4]. In particular we study their integer cohomology rings, which are torsion free $\mathbb{Z}$-modules of which we find $\mathbb{Z}$-bases.

When the considered arrangement is the braid hyperplane arrangement, this leads to the study of the integer cohomology rings of the moduli spaces $M_{0,n}$ of n-pointed curves of genus 0 and of their Mumford-Deligne compactifications $\overline{M}_{0,n}$. In fact we prove that $\overline{M}_{0,n}$ is isomorphic to a particular De Concini - Procesi model of the braid arrangement. In these cases, we deal with the action of the symmetric group on the cohomology rings: we give explicit formulas for the associated generalized Poincarè series, and provide recursive formulas for the characters. We also extend part of our results to the root arrangements of types $B_n(= C_n)$ and D_n. Let us describe more in detail the content of this thesis.

Models of arrangements and moduli spaces

The first part of this thesis is devoted to recalling some of the definitions and results established by De Concini and Procesi in [4]. Given a subspace arrangement $\mathcal{G}^* \subset \mathbb{C}^n$, we describe it by considering, in the dual $(\mathbb{C}^n)^*$, the subset $\mathcal{G}$ of the subspaces orthogonal (with respect to the standard pairing between $(\mathbb{C}^n)$ and $(\mathbb{C}^n)^*$) to the ones in $\mathcal{G}^*$, and call by $\mathcal{M}_\mathcal{G}$ the complement of $\mathcal{G}^*$ in $\mathbb{C}^n$. A model for $\mathcal{M}_\mathcal{G}$ is a smooth irreducible variety $Y_\mathcal{G}$ equipped with a proper map $\pi : Y_\mathcal{G} \mapsto \mathbb{C}^n$ which is an isomorphism on the preimage of $\mathcal{M}_\mathcal{G}$ and such that the complement of this preimage is a divisor with normal crossings.

Let us sketch the explicit construction of the De Concini-Procesi models. We start by considering the map

$$i : \mathcal{M}_\mathcal{G} \mapsto \mathbb{C}^n \times \prod_{D\in\mathcal{G}} \mathbf{P}(\mathbb{C}^n/D^\perp)$$

where $D^\perp$ is the dual space of D, $\mathbf{P}(\mathbb{C}^n/D^\perp)$ denotes the projective space of lines in $\mathbb{C}^n/D^\perp$ and the map from $\mathcal{M}_\mathcal{G}$ to $\mathbf{P}(\mathbb{C}^n/D^\perp)$ is the restriction of the canonical projection $\mathbb{C}^n - D^\perp \mapsto \mathbf{P}(\mathbb{C}^n/D^\perp)$. Then we form $Y_\mathcal{G}$ as the closure of the graph of i. It turns out that, when the set $\mathcal{G}$ has suitable properties (when it is "building", see Chapter 1, Section 1), $Y_\mathcal{G}$ is a "wonderful model" in the sense explained by the following theorem.

Theorem 0.0.1 *(see [4] and Chapter 1, Section 1)*
Let $\mathcal{G}$ be a building set of subspaces. Then
(1) *$Y_\mathcal{G}$ is a model for $\mathcal{M}_\mathcal{G}$.*
(2) *The complement $\mathcal{D}$ of $\mathcal{M}_\mathcal{G}$ in $Y_\mathcal{G}$ is the union of smooth irreducible divisors D_G indexed by the elements $G \in \mathcal{G}$.*
(3) *Let G be a minimal (with respect to inclusion) element in $\mathcal{G}$. If we put $\mathcal{G}' = \mathcal{G} - \{G\}$, and denote by $\overline{\mathcal{G}}$ the family in $(\mathbb{C}^n)^* /G$ given by the elements $\{A + G/G \,:\, A \in \mathcal{G}'\}$, we have that $\mathcal{G}'$ and $\overline{\mathcal{G}}$ are building and that $Y_\mathcal{G}$ can be obtained from $Y_{\mathcal{G}'}$ by blowing up a subvariety isomorphic to $Y_{\overline{\mathcal{G}}}$.*

The blow-up property (3) is the key to understand the geometry of $Y_\mathcal{G}$: we will also use it to study the integer cohomology rings of $Y_\mathcal{G}$ and of the subvarieties $D_S = \bigcap_i D_{A_i}$ ($S = \{A_1, \ldots, A_r\} \subset \mathcal{G}$) which are intersection of the divisors in the boundary. De Concini and Procesi provided a presentation of these cohomology rings as quotients of polynomial algebras.

Theorem 0.0.2 *(see [4] and Chapter 1, Section 3)*
The ring $H^*(D_S, \mathbb{Z})$ is the quotient of $\mathbb{Z}[c_A]$ by an ideal I_S, where the variables c_A are indexed by the elements in $\mathcal{G}$ and, for $A \in \mathcal{G}$, c_A is interpreted as the cohomology class $[D_A]$, associated to the divisor D_A, restricted to D_S (and therefore it has degree 2). In particular, keeping the above definitions also in the case $S = \emptyset$, we obtain that $H^*(Y_\mathcal{G}, \mathbb{Z})$ is a quotient of $\mathbb{Z}[c_A]$.

We note that in [4] arrangements of linear subspaces in $\mathbf{P}(\mathbb{C}^n)$ have also been studied. As a result, a theory has been obtained, which gives compact models and is quite similar to the above described one. In fact, the connection between the two settings is given by the following assertion.

Theorem 0.0.3 *(see [4] and Chapter 1, Section 2)*
Let $\mathcal{G}$ be a building set containing $(\mathbb{C}^n)^*$ and let $\widehat{Y}_\mathcal{G}$ be the compact model associated to the linear subspace arrangement induced by $\mathcal{G}$ in $\mathbf{P}(\mathbb{C}^n)$. Then $Y_\mathcal{G}$ is the total space of a line bundle on $D_{(\mathbb{C}^n)^*}$ and $\widehat{Y}_\mathcal{G}$ is isomorphic to $D_{(\mathbb{C}^n)^*}$.

An interesting example of compact model is provided by the moduli space $\overline{M}_{0,n+1}$ of stable curves of genus zero, with $n+1$ labeled pairwise distinct points. It corresponds in our setting to the case of the braid arrangement:

in fact, in Chapter 2 we will show that $\overline{M}_{0,n+1}$ is isomorphic to a particular De Concini- Procesi model, denoted by $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, of the projective hyperplane arrangement induced by a root system of type A_{n-1}.

Cohomology bases for the models and Poincarè polynomials

A special case of what we have described above occurs when the building set $\mathcal{G}$ is such that every element of $\mathcal{C}_{\mathcal{G}}$ is generated by some lines belonging to $\mathcal{G}$ or, in a dual way, when every element of $\mathcal{G}^*$ can be obtained as an intersection of certain hyperplanes $H_1, \ldots, H_N$ belonging to $\mathcal{G}^*$. We will refer to such a $\mathcal{G}$ as a "refinement" of the hyperplane arrangement $\{H_1, \ldots, H_N\}$ (or we will say that $\mathcal{G}$ "refines" $\{H_1, \ldots, H_N\}$).

In his recent paper [25], Yuzvinsky has found bases for the $\mathbb{Z}$-module $H^*(Y_{\mathcal{F}}, \mathbb{Z})$, when $\mathcal{F}$ is the minimal building set which refines an hyperplane arrangement (i.e. when $\mathcal{F}$ is the building set of irreducibles, even if his method could be extended to the case of any subspace arrangement, as he observes in Remark 3.11 of [25]). In Chapter 3 we provide bases for the $\mathbb{Z}$-modules $H^*(D_S, \mathbb{Z})$ and $H^*(Y_{\mathcal{G}}, \mathbb{Z})$, when $\mathcal{G}$ is any subspace arrangement and S a $\mathcal{G}$-nested set, thus generalizing the mentioned result. The methods we use in the proof are different from those in [25]: they are based on the geometric structure of the model, in particular on the blow-up property (part 3 of Theorem 0.0.1) which gives rise to some useful recurrence relations.

Then we provide formulas for the Poincarè polynomials of $Y_{\mathcal{F}}$ when $\mathcal{F}$ is the building set of irreducibles which refines a root hyperplane arrangement of type A_n, $B_n\,(= C_n)$, D_n. Furthermore, given a building set $\mathcal{G}$, we study a particular class of subspace arrangements, which we call " induced by $\mathcal{G}$" since they are obtained by tensoring the elements of $\mathcal{G}$ by $\mathbb{C}^h$. In particular, given $\mathcal{F}$ as above, we study the "induced root arrangements" of types A_n, $B_n\,(= C_n)$, D_n and give formulas for the Poincarè polynomials of the associated models.

Finally, the last section of Chapter 3 is devoted to finding squarefree bases for $H^*(D_S, \mathbb{Z})$ (and for $H^*(Y_{\mathcal{G}}, \mathbb{Z})$), when $\mathcal{G}$ is a building set which refines a hyperplane arrangement (squarefree bases may not exist in the general case of subspace arrangements). They are interesting because of their geometrical meaning, since they are in correspondence with irreducible varieties obtained by intersecting, without multiplicities, the irreducible divisors in $Y_{\mathcal{G}}$.

Symmetric group representations: a recursive relation

In the last part (Chapters 4 and 5) of this thesis we specialize all the above mentioned results to the study of the symmetric group representations connected with some root hyperplane arrangements and their De Concini -

Procesi models.

Let us consider the root hyperplane arrangement $\mathcal{A}_{n-1}^* \subset (\mathbb{C}^n)$ of type A_{n-1} and its orthogonal arrangement $\mathcal{A}_{n-1} \subset (\mathbb{C}^n)^*$: we can realize the symmetric group $\mathcal{S}_n$ as the Weyl group generated by reflections in the hyperplanes orthogonal to the roots; the symmetric group action on $\mathbb{C}^{n-1}$ restricts to an $\mathcal{S}_n$-action on the complement $\mathcal{M}_{\mathcal{A}_{n-1}} = \mathbb{C}^{n-1} - \bigcup_{A \in \mathcal{A}_{n-1}^*} A$.
This action induces on the cohomology ring $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$ a linear $\mathcal{S}_n$ action that has been studied by several authors.
In particular Lehrer (see [17]) and Lehrer-Solomon (see [19]) determined, using combinatorial techniques, the structure of the $\mathcal{S}_n$-module $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$, showing that it is, degree by degree, the sum of certain induced representations. In [19] it has been proved that, as $\mathcal{S}_n$-modules,

$$H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C}) \cong 2Ind_{<r>}^{\mathcal{S}_n}(1) \tag{1}$$

where r is a reflection and 1 is the trivial one dimensional representation. In Chapter 4 we construct an extension to $\mathcal{S}_{n+1}$ of the natural $\mathcal{S}_n$ action described above. As a first step we shift the problem for the arrangement $\mathcal{A}_{n-1}$ to the same one for its projectivization which has complement $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$. Next we use the fact, proved in Chapter 2, that $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ is isomorphic to $M_{0,n+1}$: as a consequence, we get on $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ an $\mathcal{S}_{n+1}$-action that can be lifted to an $\mathcal{S}_{n+1}$- representation, compatible with the natural $\mathcal{S}_n$ one, on the cohomology ring $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$.

Then we find an interesting recursive relation which connects the extended action with the already known one. Let us express it in terms of characters, introducing the following notation: let $\chi_{n+1}(i,n)$ be the character of the natural $\mathcal{S}_{n+1}$ action on $H^i(\mathcal{M}_{\mathcal{A}_n}, \mathbb{C})$, $\chi_{n+2}(i,n)$ be the character of the extended $\mathcal{S}_{n+2}$ action on $H^i(\mathcal{M}_{\mathcal{A}_n}, \mathbb{C})$ and p_{n+1} be the character of the standard representation of $\mathcal{S}_{n+1}$. Then, for $n \geq 2$, we have :

$$\chi_{n+1}(i,n) = \chi_{n+1}(i,n-1) + p_{n+1}\chi_{n+1}(i-1,n-1) \tag{2}$$

As a first application, we will use relation (2) to obtain a polynomial formula which involves the characters of the $\mathcal{S}_{n+1}$ action on the homogeneous components of the graded ring $H^*(\mathcal{M}_{\mathcal{A}_n}, \mathbb{C})$. Secondly, we will provide a quick proof of the important Lehrer's result (1). Finally we will compute the Euler characteristic $\chi(M_{0,n}/\mathcal{S}_j)$ of the quotient spaces $M_{0,n}/\mathcal{S}_j$ ($n \geq 3$ and $2 \leq j \leq n$, that is to say, $\mathcal{S}_j$ is identified with a subgroup of $\mathcal{S}_n$). The interest of this computation lies in the fact that $\chi(M_{0,n}/\mathcal{S}_j)$ plays a crucial role in the computation of the Euler characteristic of the moduli spaces $M_{1,n}$, $M_{2,n}$ of n-pointed curves of genus 1 and 2 and of their compactifications $\overline{M}_{1,n}$, $\overline{M}_{2,n}$, as it is shown in [3].

Generalized Poincarè polynomials

The last sections of Chapter 4 and most of Chapter 5 are devoted to studying the $\mathcal{S}_{n+1}$ action on the cohomology ring of the De Concini - Procesi model $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$. Since $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is isomorphic to $\overline{M}_{0,n+1}$, the importance of this action (which we will call the "extended" action) was first pointed out in the context of moduli spaces. It has been studied by E. Getzler in [13] and it turns out to be compatible with the $\mathcal{S}_n$ action coming from the geometry of $\mathcal{A}_{n-1}$ (which we will call the "non extended" or "natural" action). In [13] Getzler calculates the characteristic of the cyclic $\mathcal{S}$-module $Poly$ which corresponds to the cohomology of the moduli spaces $\overline{M}_{0,n}$. This characteristic takes values in the ring of symmetric functions and encodes all the information about the characters of the "extended" action (therefore, a fortiori, of the "non extended" one).

Assuming the point of view of models of arrangements, we can study in a new and elementary way these modules. For instance we will give very direct formulas for the trace of operators in the "non extended"case and we will explicitly describe the $\mathcal{S}_{n+1}$-module $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ and its decomposition into irreducibles. The reason why our approach is elementary lies in the combinatorial properties of the Yuzvinsky basis of $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$. In fact it turns out that the elements of this basis are permuted by the symmetric group $\mathcal{S}_n$ (unfortunately, not by $\mathcal{S}_{n+1}$): this allows us to compute the trace of operators in a direct combinatorial way.

Our interest is thus devoted to the generalized Poincarè polynomial with respect to an element $w \in \mathcal{S}_n$, i.e. to the polynomial

$$P_{w,\mathcal{A}_{n-1}}(q) = \sum_i (tr\, w_{|\, H^{2i}(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})})q^i$$

(note that the variable q has to be considered of degree 2: in fact the odd-degree cohomology groups are trivial).

Now we can view $w \in \mathcal{S}_n$ as an element of $\mathcal{S}_m$ for every $m > n$, by the obvious immersion $\mathcal{S}_n \mapsto \mathcal{S}_m$; this makes w to act on all the rings $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{m-1}}}, \mathbb{Z})$ for $m > n$. It turns out that, in order to determine the generalized Poincarè polynomials $P_{w,\mathcal{A}_{m-1}}(q)$ as m varies, it is convenient to compute directly the two variables "generalized Poincarè series"

$$P_{w,A}(q,t) = \sum_{m=n}^{\infty} P_{w,\mathcal{A}_{m-1}}(q)\frac{t^m}{m!}$$

In Section 2 of Chapter 5 we give some formulas for $P_{w,A}(q,t)$ for any $w \in \mathcal{S}_n$. These formulas are particularly explicit when w is a cyclic permutation (see Theorem 5.2.1); when w is the identity, we recover the well known formula for the ordinary Poincarè series.

We then note that a substantial simplification to the equations provided in Section 2 can be obtained if we consider, instead of the series $P_{w,A}(q,t)$, a "universal" graded series $\mathcal{H}$ in some formal graded variables S_j and $P_k^{(d)}$ ($j \geq 1$, $k \geq 1$, $d \geq 0$). It encodes all the information about the series $P_{w,A}(q,t)$, for every w, in the way explained by the following steps:

1. Given $w \in \mathcal{S}_n$ with decomposition $w = c_1 \cdots c_l$, where $c_1, \ldots, c_l$ are non trivial disjoint cycles of length $\lambda_1 \geq \ldots \geq \lambda_l > 1$ respectively, consider the polynomial $\mathcal{H}_l$ which is the homogeneous component of degree l of $\mathcal{H}$.

2. Then substitute in $\mathcal{H}_l$ the formal variables S_j (and $P_k^{(d)}$) with some special functions from $\mathbb{Z}^j$ ($\mathbb{Z}^k$ respectively) to $\mathbb{Z}[[q,t]]$.

3. Finally put the numbers $\lambda_1, \ldots, \lambda_l$ as inputs of these special functions and sum over all the possible permutations of $\lambda_1, \ldots, \lambda_l$.

4. The universal series $\mathcal{H}$ is constructed in such a way that, after the steps 1,2 and 3, $P_{w,A}^{(\sum_{i=1}^{l} \lambda_i)}(q,t)$ is obtained (here the superscript (n) means "n-th derivative with respect to t").

Note that, although the series $\mathcal{H}$ and the non cyclic characteristic ch_t of the rings $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ (which can be computed by differentiating Getzler's formula for the cyclic characteristic in [13]) encode the same information, they are different combinatorial objects. Section 3 is devoted to finding a nice and compact formula for the universal formal series $\mathcal{H}$ (see Theorem 5.3.4). This formula is obtained by studying certain sums over rooted trees: the remarkable combinatorial properties of these sums are summarized by the Theorems 5.3.2 and 5.3.3 which are of independent interest.

Finally, in Section 4 of Chapter 5, we focus on the complex Coxeter arrangements $\mathcal{B}_n$ of type B_n and on the associated De Concini - Procesi models of irreducibles: we provide a formal series $\mathcal{H}_B$ which, by the same methods as above, gives us the generalized Poincarè series with respect to the elements of some subgroups (isomorphic to $\mathcal{S}_n$) of the Weyl group of type B_n.

Chapter 1

Models of subspace arrangements

1.1 Construction of the De Concini-Procesi models

Let us consider a subspace arrangement $\mathcal{G}^*$ in $\mathbb{C}^n$; given $K \in \mathcal{G}^*$, we can construct its orthogonal subspace $K^\perp$ in the dual $(\mathbb{C}^n)^*$ and therefore we can describe the arrangement $\mathcal{G}^*$ by means of the arrangement $\mathcal{G} \in (\mathbb{C}^n)^*$, $\mathcal{G} = \{K^\perp \mid K \in \mathcal{G}^*\}$.

Let now $\mathcal{A}_\mathcal{G} \subset \mathbb{C}^n$ be the union of the subspaces of $\mathcal{G}^*$:

$$\mathcal{A}_\mathcal{G} = \bigcup_{A \in \mathcal{G}^*} A = \bigcup_{B \in \mathcal{G}} B^\perp$$

and let $\mathcal{M}_\mathcal{G}$ be the complement of $\mathcal{A}_\mathcal{G}$ in $\mathbb{C}^n$.

By construction, given $A \in \mathcal{G}$, the rational map

$$\pi_A \, : \, \mathbb{C}^n \mapsto \mathbb{C}^n / A^\perp$$

is defined outside $A^\perp$ and thus there is a morphism

$$\phi_\mathcal{G} \, : \, \mathcal{M}_\mathcal{G} \mapsto \prod_{A \in \mathcal{G}} \mathbf{P} \left(\mathbb{C}^n / A^\perp \right)$$

The graph of $\phi_\mathcal{G}$ is a closed subset of

$$\mathcal{M}_\mathcal{G} \times \prod_{A \in \mathcal{G}} \mathbf{P} \left(\mathbb{C}^n / A^\perp \right)$$

which embeds as open set into

$$\mathbb{C}^n \times \prod_{A \in \mathcal{G}} \mathbf{P} \left(\mathbb{C}^n / A^\perp \right)$$

Finally we have an embedding

$$\widehat{\phi}_{\mathcal{G}} \,:\, \mathcal{M}_{\mathcal{G}} \mapsto \mathbb{C}^n \times \prod_{A\in\mathcal{G}} \mathbf{P}\left(\mathbb{C}^n/A^{\perp}\right)$$

as a locally closed subset. This construction allows us to give the definition:

Definition 1.1.1 *We denote by $Y_{\mathcal{G}}$ the closure of $\widehat{\phi}_{\mathcal{G}}(\mathcal{M}_{\mathcal{G}})$ in*

$$\mathbb{C}^n \times \prod_{A\in\mathcal{G}} \mathbf{P}\left(\mathbb{C}^n/A^{\perp}\right)$$

The variety $Y_{\mathcal{G}}$ will be the main object of our study. Its properties essentially depend on the combinatorial properties of $\mathcal{G}$, therefore we will shortly recall some definitions and facts concerning the combinatorics of a subspace arrangement.

First, we call by $\mathcal{C}_{\mathcal{G}}$ the closure, under the sum, of $\mathcal{G}$, that is to say, the set of subspaces in $(\mathbb{C}^n)^*$ which are sums of subspaces in $\mathcal{G}$.

Definition 1.1.2 *Given a subspace $U \in \mathcal{C}_{\mathcal{G}}$, a decomposition of U is a collection of non zero subspaces $U_1, U_2, \ldots, U_k \in \mathcal{C}_{\mathcal{G}}$ $(k > 1)$ which satisfy the following properties:*

1. *$U = U_1 \oplus U_2 \oplus \ldots \oplus U_k$*
2. *for every subspace $A \subset U$ in $\mathcal{C}_{\mathcal{G}}$, we have that $A\cap U_1,\ A\cap U_2, \ldots, A\cap U_k$ lie in $\mathcal{C}_{\mathcal{G}}$ and $A = (A\cap U_1) \oplus (A \cap U_2) \oplus \ldots \oplus (A\cap U_k)$*

Definition 1.1.3 *If a subspace in $\mathcal{C}_{\mathcal{G}}$ does not admit a decomposition, it is called “irreducible”. The set of all irreducible subspaces is denoted $\mathcal{F}_{\mathcal{G}}$.*

The above mentioned definitions lead immediately to the following expected proposition:

Proposition 1.1.1 *Every subspace $U \in \mathcal{C}_{\mathcal{G}}$ has a unique decomposition $U = \oplus_{i=1}^{k} U_i$ into irreducible subspaces. This is called “the irreducible decomposition” of U. If $A \subset U$ is irreducible, then $A \subset U_i$ for some i.*

In the sequel, we will deal with a further property of the arrangement $\mathcal{G}$:

Definition 1.1.4 *A collection of subspaces $\mathcal{G} \subset (\mathbb{C}^n)^*$ is called ”building set” if every element C of $\mathcal{C}_{\mathcal{G}}$ is the direct sum $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ of the set of the maximal elements $G_1, G_2, \ldots, G_k$ of $\mathcal{G}$ contained in C. We say in this case that $\{G_1, \ldots, G_k\}$ is “the decomposition of C in (the building set) $\mathcal{G}$”.*

Remarks

1) One can easily see that the "decomposition of C in $\mathcal{G}$" is a decomposition in the previous sense.

2) The sets $\mathcal{C}_\mathcal{G}$ and $\mathcal{F}_\mathcal{G}$ defined above are building sets. Furthermore, for every building set $\mathcal{G}$, we have $\mathcal{F}_\mathcal{G} \subset \mathcal{G} \subset \mathcal{C}_\mathcal{G}$. Let in fact $A \in \mathcal{C}_\mathcal{G}$ be irreducible. Then A can be decomposed in $\mathcal{G}$, but then $A \in \mathcal{G}$ since A is irreducible. This proves the first inclusion, the second being trivial.

The notion of "building set", introduced by De Concini and Procesi in [4], plays a fundamental role in the theory, since it turns out that, in the case when the arrangement $\mathcal{G}$ is building, the variety $Y_\mathcal{G}$ is a "wonderful model" for $\mathcal{M}_\mathcal{G}$, in the sense specified by Theorem 0.0.1 in the Introduction.

De Concini and Procesi proved this in [4] by using the explicit description of an open affine covering of $Y_\mathcal{G}$. We will give here a sketch of the proof provided in [4]. As a first step it is worth recalling the construction of the above mentioned open charts. For this purpose we need to introduce the notion of "nested set" (see [4]) by means of the following definitions. This notion is close to the one introduced by Fulton and MacPherson in their paper [9] on models of configuration spaces.

Definition 1.1.5 *A set S of subspaces in $(\mathbb{C}^n)^*$ is called nested if, given any its subset $\{U_1, \dots, U_k\}$ of pairwise non comparable elements, one has $U = U_1 \oplus \cdots \oplus U_k$ and $U \notin S$.*

Definition 1.1.6 *Let $\mathcal{K}$ be a building set of subspaces in $(\mathbb{C}^n)^*$. A subset $S \subset \mathcal{K}$ is called "nested relative to $\mathcal{K}$", or $\mathcal{K}$-nested, if*

1. *S is nested*
2. *given a subset $\{A_1, \dots, A_h\}$ of pairwise non comparable elements in S, then $C = A_1 \oplus \cdots \oplus A_h$ is the decomposition of C in $\mathcal{K}$.*

Let now S be a nested set of subspaces in $(\mathbb{C}^n)^*$. For every set $A \subset (\mathbb{C}^n)^*$, $A \neq \{0\}$, the set

$$S^A = \{(\mathbb{C}^n)^*\} \cup \{B \in S \mid A \subset B\}$$

is linearly ordered (with respect to inclusion) and non empty. We let $p_S(A)$ to be the minimum of S^A. We will write $p_S(v)$ instead of $p_S(\mathbb{C}v)$ if v is a vector in $(\mathbb{C}^n)^*$.

Definition 1.1.7 *A basis b of $(\mathbb{C}^n)^*$ is called "adapted" to S if, for all $A \in S$, the set*

$$b_A := b \cap A = \{v \in b | p_S(v) \subset A\}$$

is a basis of A. A "marking" of a basis b adapted to S is a choice, for all $A \in S$, of an element $x_A \in b$ with $p_S(x_A) = A$.

One can easily observe that, given a nested set S, one can always find a basis b adapted to S and a marking for b.

Consider now a space of functions $\mathbb{C}^b$ with coordinates u_x indexed by the elements of b and, given $A \in S$, set $u_A := u_{x_A}$ where $x_A \in b$ is the marked element associated to A. We can define a map:

$$\rho_S \,:\, \mathbb{C}^b \mapsto \mathbb{C}^b$$

by means of the following relation:

$$v = u_v \prod_{B \supset A} u_B \quad \text{if } \ A = p(v) \text{ and } \ v \ \text{ is not marked} \tag{1.1}$$

$$v = \prod_{B \supset A} u_B \qquad \text{if } v = x_A \tag{1.2}$$

where the elements of b have been chosen as coordinates on the target space. This map is easily seen to be birational and, since b is a basis of $(\mathbb{C}^n)^*$, we can consider it as a map

$$\rho_S \,:\, \mathbb{C}^b \mapsto \mathbb{C}^n$$

Proposition 1.1.2 *The map ρ_S restricts to an isomorphism between the open set where all the coordinates u_A $(A \in S\ )$ are different from 0 and the open set where the coordinates $x_A \in b$ are different from 0 $(\ A \in S)$, and maps the hyperplane defined by $u_A = 0$ in the subspace $A^\perp$.*

If we now consider the variety

$$Y_S \subset \mathbb{C}^n \times \prod_{A \in S} \mathbf{P}\left(\mathbb{C}^n / A^\perp\right)$$

constructed according to Definition 1.1.1, we have that

Proposition 1.1.3 *(see [4]) The map ρ_S lifts to an open embedding of $\mathbb{C}^b$ into Y_S.*

Proof.

This essentially follows from the fact that the composition of ρ_S with the rational map

$$\pi_A \,:\, \mathbb{C}^n \mapsto \mathbf{P}\left(\mathbb{C}^n / A^\perp\right) \quad (A \in S)$$

is given by the formulas (1.1), (1.2), if we choose on $\mathbf{P}\left(\mathbb{C}^n / A^\perp\right)$ the projective coordinates coming from the basis b_A of A. Thus as monomials in the u_x, these coordinates are all divisible by the monomial expressing x_A; we deduce that the map $\pi_A\, \rho_S$ is a morphism to the affine part $\mathbf{P}^0\left(\mathbb{C}^n / A^\perp\right) \subset \mathbf{P}\left(\mathbb{C}^n / A^\perp\right)$ where $x_A = 1$. We can then form a morphism (again denoted by ρ_S)

$$\rho_S \,:\, \mathbb{C}^n \mapsto \mathbb{C}^n \times \prod_{A \in S} \mathbf{P}\left(\mathbb{C}^n / A^\perp\right)$$

the image of which is easily seen to be equal to the intersection between Y_S and $\mathbb{C}^n \times \prod_{A\in S} \mathbf{P}^0\left(\mathbb{C}^n/A^\perp\right)$. ■

We will denote by $\mathcal{U}_S^b$ the open set in Y_S provided by the previous proposition and identify with ρ_S the restriction to $\mathcal{U}_S^b$ of the projection from Y_S to $\mathbb{C}^n$. Moreover we observe (see [4], page 465), that ρ_S depends only on the marked elements of the basis b.

Now, let us describe one possible way to select adapted marked bases for S. Choose for every $B \in S$ a basis $b(B)$ of B made by vectors not contained in any $C \subset B$, $C \in S$. Choose a vector $x_B \in b(B)$ for every $B \in S$. Then these vectors are linearly independent and thus can be completed to a basis b which is adapted to S and in which they are the marked vectors.

If we fix the bases $b(B)$ $(B \in S)$ and perform the above algorithm in all the possible ways (that is to say, if we choose the marked vectors in all the possible ways), we get a family Θ of adapted marked bases. Since the open sets $\mathcal{U}_S^b$ depend only on the marking of the basis b, this gives rise to a finite family $\mathcal{V} = \{\mathcal{U}_S^{b'} \mid b' \in \Theta\}$ of open sets.

Proposition 1.1.4 *(see [4])*

1. *The variety Y_S is covered by the open sets $\mathcal{U}_S^b$ in the family $\mathcal{V}$.*

2. *Given a minimal element $A \in S$ and put $S' = S - \{A\}$, Y_S is the blow-up of $Y_{S'}$ along the proper transform Z_A of the subspace $A^\perp$ which is a smooth subvariety. Furthermore Z_A is canonically isomorphic to Y_Λ where $\Lambda := \{(B + A)/A \in (\mathbb{C}^n)^*/A \mid B \in S'\}$.*

3. *Consider $\mathcal{M}_S = \mathbb{C}^n - \cup_{A\in S} A^\perp$ embedded as an open set in Y_S. Then $Y_S - \mathcal{M}_S$ is a divisor with normal crossing with smooth irreducible components D_A^S parametrized by the elements $A \in S$.*

4. *All the intersections of the divisors D_A^S are irreducible.*

Proof.
1),2).
We observe that S' is still nested and we want to study the two varieties Y_S, $Y_{S'}$. By their very construction, there is a birational morphism $p : Y_S \mapsto Y_{S'}$.

Let us consider a basis b adapted to S and marked. Then b is also adapted to S' and, up to forget the marking on the element $x_A \in b$, is marked for S'. It follows that the map ρ_S equals the composition of p restricted to $\mathcal{U}_S^b$ with $\rho_{S'}$.

We want to explicit the relations between the coordinate charts $\mathcal{U}_S^b$ and $\mathcal{U}_{S'}^b$; in order to do this we will denote by u_v the coordinates in $\mathcal{U}_S^b$ and by

u'_v the coordinates in $\mathcal{U}^b_{S'}$. We observe that we have $u_v = u'_v$ if $p_S(v) \neq A$ or $v = x_A$ and $u'_v = u_v u_A$ if $p_S(v) = A$ and $v \neq x_A$.

These are exactly the explicit maps of the blow up of $\mathcal{U}^b_{S'}$ along the subvariety $u'_A = 0, u'_v = 0$ ($p_S(v) = A$ and $v \neq x_A$) in the charts

$$p : \mathcal{U}^b_S \mapsto \mathcal{U}^b_{S'}$$

In particular, in the case of the claim 2), since A is a minimal element in S, if we start from an adapted basis b of S and we mark it for S' in such a way that no marked vector belongs to A, we can complete the marking to S in $m = dim\, A$ different ways. Let us call b_i ($i = 1, \dots, m$) the marked bases we get; the associated charts $\mathcal{U}^{b_i}_S$ cover the blow up of $\mathcal{U}^b_{S'}$ along the subspace defined by the equations $u'_v = 0$ ($v \in A$), hence the induced map $\cup_i \mathcal{U}^{b_i}_S \mapsto \mathcal{U}^b_{S'}$ is a proper map.

Now, using the formulas (1.1) and (1.2), we can conclude that the variety we blow up in $\mathcal{U}^b_{S'}$ is exactly the proper transform of the subspace $A^\perp$. In fact we have

$$v = u'_v \prod_{B \supsetneq A,\, B \in S'} u'_B$$

for $v \in b_A$ and thus the claim follows dividing by

$$\prod_{B \supsetneq A,\, B \in S'} u'_B$$

These observations allow us to prove the claims 1) and 2) by induction on the cardinality of S. As for the claim concerning Z_A, we notice that $A^\perp$ meets $\mathcal{M}_{S'}$ and

$$A^\perp \cap \mathcal{M}_{S'} = A^\perp - \cup_{B \in S'} (B^\perp \cap A^\perp)$$

But if $(B + A)/A = (B' + A)/A$ in Λ, then $B^\perp \cap A^\perp = (B')^\perp \cap A^\perp$, so, after identifying $(\mathbb{C}^n)^*/A$ with $(A^\perp)^*$, we can rewrite

$$A^\perp \cap \mathcal{M}_{S'} = A^\perp - \cup_{\overline{C} \in \Lambda} (\overline{C})^\perp$$

Finally, under the map

$$\mathcal{M}_{S'} \mapsto \mathbb{C}^n \times \prod_{B \in S'} \mathbf{P}\left(\mathbb{C}^n / B^\perp\right)$$

$A^\perp \cap \mathcal{M}_{S'}$ maps to

$$A^\perp \times \prod_{(B+A)/A \in \Lambda} \mathbf{P}\left(A^\perp / B^\perp \cap A^\perp\right)$$

3), 4).

First we note that for every open set $\mathcal{U}_S^b$, $p^{-1}(A^\perp) \cap \mathcal{U}_S^b$ is the divisor of equation $\prod_{B \supset A} u_B$. Thus, if we set D_A equal to the closure of $p^{-1}(A^\perp - \cup_{B \supsetneq A} B^\perp)$ we obtain a smooth divisor whose intersection with $\mathcal{U}_S^b$ is the hyperplane of equation $u_A = 0$. Furthermore $Y_S - \mathcal{M}_S = \cup_{A \in S} D_A$. This proves 3) and 4) except for the irreducibility of D_A. Let us prove that D_A is irreducible by induction on the cardinality of S. We assume the statement for $S' = S - \{A\}$ (A minimal) and we consider, by 2), Y_S as the blow up of $Y_{S'}$ along Z_A. Then D_A is the exceptional divisor and its irreducibility follows from the one of Z_A. This also proves the base of the induction (that is to say, $|S| = 1$). For $B \neq A$, D_B is the proper transform of the divisor corresponding to B in $Y_{S'}$, so it is irreducible by the inductive hypothesis. At the same way, using the local description, we can prove 4). ■

Let us now focus on the variety $Y_{\mathcal{G}}$ in the case when $\mathcal{G}$ is any building set. Let us take a $\mathcal{G}$-nested set S and a marked basis b adapted to it.

Lemma 1.1.5 *Given any $x \in (\mathbb{C}^n)^* - \{0\}$, suppose $A = p_S(x) \in S$. Then $x = x_A P_x(u)$, where $P_x(u)$ is a polynomial depending only on the variables u_v with v such that $p_S(v) \subseteq A$ and $v \neq x_A$.*

Proof.

Since b_A is a basis of A, we have an expression

$$x = \sum_{v \in b_A} a_v v = x_A (a_{x_A} + \sum_{\substack{v \in b_A \\ v \neq x_A}} a_v \frac{v}{x_A})$$

If we substitute for the v's their expression in terms of the u's provided by the relations (1.1), (1.2), we get the requested polynomial $P_x(u)$. ■

Now, given $G \in \mathcal{G}$, the previous lemma allows us to define polynomials $P_x^G(u)$, $x \in G$, by the formula $x = x_A P_x^G(u)$.

Let us denote by Z_G the subvariety in $(\mathbb{C})^b$ defined by the vanishing of these polynomials. Then we observe that the map

$$\mathbb{C}^b \mapsto \mathbb{C}^n \mapsto \mathbb{C}^n / G^\perp$$

given by the coordinate functions $x \in G$, can be composed in $\mathbb{C}^b - Z_G$ with the rational map $\mathbb{C}^n/G^\perp \mapsto \mathbf{P}(\mathbb{C}^n/G^\perp)$, giving a regular morphism.

Definition 1.1.8 *Given a $\mathcal{G}$-nested set S, we define the open set $\mathcal{U}_S^b$ or $\mathcal{U}_S^b(\mathcal{G})$ as the complement in $\mathbb{C}^b$ of the union of all the varieties Z_G, $G \in \mathcal{G}$.*

The open set $\mathcal{U}_S^b$ has been defined in such a way that all the rational morphisms

$$\mathcal{U}_S^b \mapsto \mathbf{P}(\mathbb{C}^n/G^\perp)$$

are well-defined; therefore we obtain an embedding j_S^b of $\mathcal{U}_S^b$ in $Y_\mathcal{G}$. By construction, and by the formula $x_A = \prod_{A\subset B} u_B$, we have that the complement in $\mathcal{U}_S^b$ of the divisors $u_A = 0$ ($A \in S$), maps to the open set $\mathcal{M}_\mathcal{G}$ injectively, while the divisor $u_A = 0$ maps to $A^\perp$.

The fact that the maps j_S^b are open embeddings (as S, b vary) easily follows from the diagram

$$\begin{tikzcd} \mathcal{U}_S^b \arrow[r, "j_S^b"] \arrow[d, "i"'] & Y_\mathcal{G} \arrow[d, "\pi'"'] \\ (\mathbb{C})^b \arrow[r, "i_S^b"] & Y_S \end{tikzcd}$$

since i_S^b is the open embedding of Proposition 1.1.3 and π' is a birational map. From now on we will identify $\mathcal{U}_S^b$ with its image $j_S^b(\mathcal{U}_S^b)$.

We have recalled the construction of the open sets $\mathcal{U}_S^b$ in $Y_\mathcal{G}$ since they play in the theory the role of local coordinates. This is assured by the following

Theorem 1.1.6 *(for the proof see [4], Theorem 3.1.1)*

1. *$Y_\mathcal{G} = \cup_S \mathcal{U}_S^b$. In particular $Y_\mathcal{G}$ is smooth.*

2. *Set D_S^b equal to the divisor in $\mathcal{U}_S^b$ defined by $\prod_{A\in S} u_A = 0$. Set $D = \cup_S D_S^b$. Then, considering the projection*

$$\pi\ :\ Y_\mathcal{G} \mapsto \mathbb{C}^n$$

we have that $\pi^{-1}(\mathcal{M}_\mathcal{G}) = Y_\mathcal{G} - D$ and D is a divisor with normal crossings.

From the geometrical point of view, one of the more interesting property of the variety $Y_\mathcal{G}$ is that it can be constructed by means of a sequence of blow-ups along varieties which are of the same kind, that is to say, models associated to building sets of smaller cardinality.

This explicit construction allows us to study the models $Y_\mathcal{G}$ recursively: for instance, the proof of Theorem 3.1.1 of the present thesis (see Chapter 3), which provides a $\mathbb{Z}$-basis for the cohomology ring of $Y_\mathcal{G}$, is based on the geometry of these blow-ups.

Theorem 1.1.7 (1) *The complement D of $\pi^{-1}(\mathcal{M}_\mathcal{G})$ in $Y_\mathcal{G}$ is the union of smooth irreducible divisors D_G indexed by the elements $G \in \mathcal{G}$, where D_G is the unique irreducible component in D such that $\pi(D_G) = G^\perp$.*

(2) *Given divisors $D_{A_1}, \ldots D_{A_n}$, they have non empty intersection if and only if the set $S = \{A_1, \ldots, A_n\}$ is nested in $\mathcal{G}$. In this case the intersection is transversal and we obtain a smooth irreducible variety $D_S = \bigcap_i D_{A_i}$.*

(3) *Let G be a minimal (with respect to inclusion) element in $\mathcal{G}$. Let us put $\mathcal{G}' = \mathcal{G} - \{G\}$, and let $\overline{\mathcal{G}}$ be the family in $(\mathbb{C}^n)^* /G$ given by the elements $\{(A+G)/G \ : \ A \in \mathcal{G}'\}$. We have that $\mathcal{G}'$ and $\overline{\mathcal{G}}$ are building and that $Y_{\mathcal{G}}$ can be obtained from $Y_{\mathcal{G}'}$ by blowing up the proper transform T_G of $G^\perp$. Furthermore, T_G is isomorphic to the variety $Y_{\overline{\mathcal{G}}}$.*

Proof.
We proceed by induction on the cardinality of $\mathcal{G}$, the case $|\mathcal{G}| = 1$ being easily verified. By induction we assume that 1) and 2) hold for $\mathcal{G}'$ and show 3).

The proper transform T_G of $G^\perp$ in $Y_{\mathcal{G}'}$ is isomorphic to $Y_{\overline{\mathcal{G}}}$ since for any $H \in \mathcal{G}'$ the restriction to $G^\perp$ of the rational morphism $\mathbb{C}^n \mapsto \mathbf{P}(\mathbb{C}^n/H^\perp)$ factors through the rational morphisms

$$G^\perp \mapsto \mathbf{P}(G^\perp/G^\perp \cap H^\perp) = \mathbf{P}(G^\perp/(G+H)^\perp)$$

Notice that we can identify $(\mathbb{C}^n)^*/G$ with $G^\perp$ and $((G+H)/G)^\perp$ with $(G^\perp \cap H^\perp) = (G+H)^\perp$.

We have then to check that the subscheme where the rational morphism is not defined coincides with T_G. Consider a maximal nested set T relative to $\mathcal{G}'$, a marked basis adapted to T, and take the open set $\mathcal{U}_T^b(\mathcal{G}') \subset Y_{\mathcal{G}'}$. There are two possibilities: either T is also maximal in $\mathcal{G}$, or $T \cup \{G\}$ is maximal nested in $\mathcal{G}$.

In the first case the projection map $Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$ induces an open embedding $\mathcal{U}_T^b(\mathcal{G}) \subset \mathcal{U}_T^b(\mathcal{G}')$. Otherwise we can embed $\mathcal{U}_T^b(\mathcal{G}')$ into Y_T. Then Proposition 1.1.4 implies that on the open set $\mathcal{U}_T^b(\mathcal{G}')$ the map $Y_{\mathcal{G}} \mapsto Y_{\mathcal{G}'}$ is a blow up along the intersection with T_G.

Having proved 3) under our inductive assumption, we deduce 1) and 2).

Let us set D_G to be equal to the exceptional divisor of the blow up $Y_{\mathcal{G}} \mapsto Y_{\mathcal{G}'}$ and let D_H ($H \in \mathcal{G}'$) be equal to the proper transform of the corresponding divisor D'_H in $Y_{\mathcal{G}'}$. The first claim now follows after observing that the intersection of D'_H with T_G is a smooth irreducible divisor isomorphic to $D_{\overline{H}}$, where $\overline{H} = (H+G)/G$ in $\overline{\mathcal{G}}$.

We then note that the divisor D_A meets the open chart $\mathcal{U}_T^b$ if and only if $A \in T$. Furthermore, if $A \in T$, then $D_A \cap \mathcal{U}_T^b$ is the divisor of equation $u_A = 0$. Now all the statements in 2) are immediate, except for the irreducibility of $Z = D_{A_1} \cap \ldots \cap D_{A_n}$ in the case when Z is non empty, that is to say, when the set $\{A_1, \ldots, A_n\}$ is $\mathcal{G}$-nested.

But if $G \notin \{A_1, \ldots, A_n\}$, then Z is the proper transform of the corresponding variety in $Y_{\mathcal{G}'}$. If instead $G \in \{A_1, \ldots, A_n\}$, Z is the preimage of the subvariety $D_{\overline{A}_1} \cap \ldots \cap D_{\overline{A}_n}$ in T_G which is irreducible by the inductive hypothesis (here $\overline{A}_j = (A_j + G)/G$ in $\overline{\mathcal{G}}$). ∎

Remark. We note that in [4] a slightly different version of the above Theorem 1.1.7 has been stated (see Theorem 3.2 in [4]), that is to say, the cases in which G is a minimal element in $\mathcal{G} - \mathcal{F}_{\mathcal{G}}$ and G is a minimal element in $G = \mathcal{F}_{\mathcal{G}}$ have been distinguished. We do not stress this distinction in the present thesis; anyway, the proof essentially doesn't change.

1.2 Projective arrangements and compact models

In their paper [4], De Concini and Procesi extend the constructions provided in Section 1 to the case of a configuration of linear subspaces in $\mathbf{P}(\mathbb{C}^n)$.

What they get is a family of compact models the geometry of which is strictly connected with the one of the varieties described until now.

Let us recall the construction of these compact models. Let $\mathcal{G}$ be a non empty family (not necessarily building, at the moment) of non zero subspaces in $(\mathbb{C}^n)^*$ and let $\mathbf{P}(A^\perp)$ $(A \in \mathcal{G})$ be the elements of a configuration of subspaces in $\mathbf{P}(\mathbb{C}^n)$.

As before, we set

$$\widehat{\mathcal{A}}_{\mathcal{G}} = \bigcup_{A \in \mathcal{G}} \mathbf{P}(A^\perp)$$

$$\widehat{\mathcal{M}}_{\mathcal{G}} = \mathbf{P}(\mathbb{C}^n) - \widehat{\mathcal{A}}_{\mathcal{G}}$$

We note that the multiplicative group $\widehat{\mathbb{C}} = \mathbb{C} - \{0\}$ acts on $\mathcal{M}_{\mathcal{G}}$ and $\widehat{\mathcal{M}}_{\mathcal{G}} = \widehat{\mathbb{C}} \setminus \mathcal{M}_{\mathcal{G}}$. The regular morphism

$$\mathcal{M}_{\mathcal{G}} \mapsto \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^\perp)$$

is constant on the $\widehat{\mathbb{C}}$ orbits, therefore we have a morphism

$$\widehat{\mathcal{M}}_{\mathcal{G}} \mapsto \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^\perp)$$

the graph of which is a closed subset of $\widehat{\mathcal{M}}_{\mathcal{G}} \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^\perp)$ which embeds as open set into $\mathbf{P}(\mathbb{C}^n) \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^\perp)$.

Finally we obtain an embedding

$$\widehat{\phi}_{\mathcal{G}} \,:\, \widehat{\mathcal{M}}_{\mathcal{G}} \mapsto \mathbf{P}(\mathbb{C}^n) \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^\perp)$$

as a locally closed subset.

Definition 1.2.1 *We let $\widehat{Y}_{\mathcal{G}}$ to be the closure of the image of $\widehat{\mathcal{M}_{\mathcal{G}}}$ under $\widehat{\phi}_{\mathcal{G}}$.*

We can make $\widehat{\mathbb{C}}$ to act naturally on $Y_{\mathcal{G}}$ in such a way that the projection $\pi : Y_{\mathcal{G}} \mapsto \mathbb{C}^n$ is equivariant. This is obtained by observing that the regular morphism

$$\mathcal{M}_{\mathcal{G}} \mapsto \mathbb{C}^n \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^{\perp})$$

is $\widehat{\mathbb{C}}$ equivariant if $\widehat{\mathbb{C}}$ acts on $\mathcal{M}_{\mathcal{G}}$ and $\mathbb{C}^n$ by multiplication and trivially on the product $\prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^{\perp})$.

Let us now study the closure $Y_{\mathcal{G}}^0$ of $\mathcal{M}_{\mathcal{G}}$ in the embedding

$$\mathcal{M}_{\mathcal{G}} \mapsto (\mathbb{C}^n - \{0\}) \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^{\perp}).$$

We observe that this closure is $Y_{\mathcal{G}}^0 = Y_{\mathcal{G}} - \pi^{-1}(0)$ and it is open in $Y_{\mathcal{G}}$ and $\widehat{\mathbb{C}}$ stable. This leads to the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}_{\mathcal{G}} & \xrightarrow{i} & Y_{\mathcal{G}}^0 & \xrightarrow{j} & (\mathbb{C}^n - \{0\}) \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^{\perp}) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{M}_{\mathcal{G}}} & \xrightarrow{\widehat{i}} & \widehat{Y}_{\mathcal{G}} & \xrightarrow{\widehat{j}} & \mathbf{P}(\mathbb{C}^n) \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^{\perp}) \end{array} \tag{1.3}$$

where i, $\widehat{i}$ are open embeddings and j, $\widehat{j}$ closed embeddings. Furthermore, since

$$\widehat{\mathbb{C}} \setminus \left[(\mathbb{C}^n - \{0\}) \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^{\perp}) \right] = \mathbf{P}(\mathbb{C}^n) \times \prod_{A \in \mathcal{G}} \mathbf{P}(\mathbb{C}^n/A^{\perp})$$

we have that $\widehat{Y}_{\mathcal{G}} = \widehat{\mathbb{C}} \setminus Y_{\mathcal{G}}^0$. As a consequence, since the canonical rational map to $\mathbf{P}(\mathbb{C}^n) = \mathbf{P}\left(\mathbb{C}^n/((\mathbb{C}^n)^*)^{\perp}\right)$ is well defined on $Y_{\mathcal{G}}^0$, we can assume, without restricting our hypothesis, that $(\mathbb{C}^n)^*$ is an element of $\mathcal{G}$.

If we now call by $E_{\mathbb{C}^n}$ the total space of the tautological bundle of $\mathbf{P}(\mathbb{C}^n)$ we obtain from the diagram (1.3) a fiber product diagram

$$\begin{array}{ccc} Y_{\mathcal{G}} & \xrightarrow{j} & E_{\mathbb{C}^n} \\ \downarrow & & \downarrow \\ \widehat{Y}_{\mathcal{G}} & \xrightarrow{\widehat{j}} & \mathbf{P}(\mathbb{C}^n) \end{array}$$

from which it follows that $Y_{\mathcal{G}}$ is the pullback, under the canonical map $\widehat{Y}_{\mathcal{G}} \mapsto \mathbf{P}(\mathbb{C}^n)$, of the tautological line bundle.

This allows us to deduce the geometric properties of the compact model $\widehat{Y}_{\mathcal{G}}$ from the properties of the divisors in $Y_{\mathcal{G}}$. In fact we have the following theorem:

Theorem 1.2.1 *(see [4])*
Let $\mathcal{G}$ be a building set of subspaces in $(\mathbb{C}^n)^*$ which contains $(\mathbb{C}^n)^*$ itself. Then we have:

1. $\widehat{Y}_{\mathcal{G}}$ *is a smooth variety.*
2. $Y_{\mathcal{G}}$ *is the total space of a line bundle on* $\pi^{-1}(0) = D_{(\mathbb{C}^n)^*}$ *and* $\widehat{Y}_{\mathcal{G}}$ *is isomorphic to* $D_{(\mathbb{C}^n)^*}$.

Proof
The claim follows from the observations above, since $\widehat{Y}_{\mathcal{G}}$ is isomorphic to the 0 section of a line bundle the total space of which is $Y_{\mathcal{G}}$. The smoothness of this section is assured by Theorems 1.1.6 and 1.1.7. ∎

This implies that $\widehat{Y}_{\mathcal{G}}$ is an irreducible projective variety. Furthermore, we have that the map $\widehat{\pi} : \widehat{Y}_{\mathcal{G}} \mapsto \mathbf{P}(\mathbb{C}^n)$ is surjective and restricts to an isomorphism on $\widehat{\mathcal{M}_{\mathcal{G}}}$. As in the non projective case, the boundary of $\widehat{Y}_{\mathcal{G}}$ is a divisor with normal crossings whose irreducible components are smooth and in bijective correspondence with the elements of $\mathcal{G} - \{(\mathbb{C}^n)^*\}$.

We also note that the intersection $\widehat{D}_S$ of a collection $\widehat{D}_{A_1}, \ldots \widehat{D}_{A_n}$ of irreducible components of the boundary is non empty if and only if $S = \{A_1, \ldots A_n\}$ is a $\mathcal{G}$-nested set. The corresponding transversal irreducible variety is identified with $\bigcap_{A \in S \cup \{(\mathbb{C}^n)^*\}} D_A$ in $Y_{\mathcal{G}}$.

Remark. All the above mentioned facts are consequences (actually particular cases) of Theorem 1.1.7.

In [4], De Concini and Procesi pointed out some further deeper facts about the geometry of the compact model $\widehat{Y}_{\mathcal{G}}$, namely about the structure of the subvarieties $\widehat{D}_S$ which are intersection of irreducible divisors in $\widehat{Y}_{\mathcal{G}}$. Before recalling these results, we need to introduce some notation.

Let $\mathcal{G}$ be a building set of subspaces (containing $(\mathbb{C}^n)^*$) and let S be a $\mathcal{G}$-nested set. Given $A \in S$, consider the (non zero) vector space

$$W_A = A/(\sum_{\substack{B \in S \\ B \subsetneq A}} B)$$

and call by $\pi_A : A \mapsto W_A$ the canonical projection. Let us put

$$\mathcal{G}_A^S := \{D \subset W_A \mid \text{there exists } B \in \mathcal{G}, B \subset A \text{ with } \pi_A(B) = D\}$$

We note that $(C_{\mathcal{G}})_A^S$ is closed under sum and $\mathcal{G}_A^S \subset (C_{\mathcal{G}})_A^S$ is building.

Theorem 1.2.2 *(see Theorem 4.3 of [4]) Let $\mathcal{G}$ be as before, and S be a $\mathcal{G}$-nested set such that $(\mathbb{C}^n)^* \notin S$. Let $\widehat{D}_S \subset \widehat{Y}_{\mathcal{G}}$ be the subvariety $\widehat{D}_S = \bigcap_{A \in S} \widehat{D}_A$. Then we have a natural isomorphism*

$$\widehat{D}_S \cong \prod_{A \in S} \widehat{Y}_{\mathcal{G}_A^S}$$

where $\widehat{Y}_{\mathcal{G}_A^S}$ is a compact projective De Concini - Procesi model.

1.3 The cohomology

A large part of this thesis is devoted to the study of the integer cohomology rings of the varieties $Y_{\mathcal{G}}$ and of their subvarieties D_S (this includes, as mentioned in the preceding section, the case of compact models).

In [4] a presentation of these cohomology rings as quotients of polynomial algebras has been provided. Let us recall the results in [4] and give a sketch of the proof.

Suppose we have fixed a $\mathcal{G}$-nested set $S \subset \mathcal{G}$. Let us take a subset $\mathcal{H} \subset \mathcal{G}$ such that there is an element $B \in \mathcal{G}$ with the property that $A \subsetneq B$ for all $A \in \mathcal{H}$. Set $S_B = \{A \in S \,:\, A \subsetneq B\}$. As in [4], we define the non negative integer $d^S_{\mathcal{H},B}$.

Definition 1.3.1

$$d^S_{\mathcal{H},B} = dimB - dim\left(\sum_{A \in \mathcal{H} \cup S_B} A\right)$$

Then we consider the polynomial ring $\mathbb{Z}[c_A]$ where the variables c_A are indexed by the elements of $\mathcal{G}$; in $\mathbb{Z}[c_A]$ we can define the following polynomials:

Definition 1.3.2

$$P^S_{\mathcal{H},B} = \left(\prod_{A \in \mathcal{H}} c_A\right)\left(\sum_{B \subset C} c_C\right)^{d^S_{\mathcal{H},B}}$$

Let us now call by I_S the ideal in $\mathbb{Z}[c_A]$ generated by these polynomials, for fixed S and varying $\mathcal{H}, B$.

Theorem 1.3.1 *(see [4]) The natural map $\phi : \mathbb{Z}[c_A] \mapsto H^*(D_S, \mathbb{Z})$, defined by sending c_A to the cohomology class $[D_A]$ associated to the divisor D_A (restricted to D_S), induces an isomorphism between $\mathbb{Z}[c_A]/I_S$ and $H^*(D_S, \mathbb{Z})$. Therefore each variable c_A has degree 2. In particular, keeping the above definitions in the case $S = \emptyset$, we obtain*

$$\mathbb{Z}[c_A]/I_\emptyset \simeq H^*(Y_{\mathcal{G}}, \mathbb{Z})$$

Remark. A consequence of Theorem 1.2.1 is that the cohomology rings of $Y_{\mathcal{G}}$ and $\widehat{Y}_{\mathcal{G}}$ are isomorphic. The description of the cohomology rings of $H^*(D_S, \mathbb{Z})$ provided by Theorem 1.3.1 explicitly shows that we have $H^*(Y_{\mathcal{G}}, \mathbb{Z}) = H^*(\widehat{Y}_{\mathcal{G}}, \mathbb{Z})$.

Before giving a proof of the theorem, we focus on the following lemma since it singles out a property of the ideals I_S which we will refer to in the sequel.

Lemma 1.3.2 *Let $\mathcal{H} \subset \mathcal{G}$ be such that $\mathcal{H} \cup S$ is not $\mathcal{G}$-nested. Then*

$$\prod_{A \in \mathcal{H}} c_A \ \in I_S$$

Proof.
First we prove that there is an element $C \in \mathcal{G}$ which is a sum of a family of subspaces $\mathcal{H}' \cup S'$ ($\mathcal{H}' \subset \mathcal{H}$, $S' \subset S$) and which properly contains all these summands.

Let us put $\mathcal{L}_0 = \mathcal{H} \cup S$. Let $\mathcal{L}_1$ be the set of non maximal elements in $\mathcal{L}_0$ and let recursively $\mathcal{L}_k$ be $(\mathcal{L}_{k-1})_1$. We set $< \mathcal{L}_j >= \sum_{A \in \mathcal{L}_j} A$, then we call by $< \mathcal{L}_j >= A_1^j \oplus \cdots \oplus A_{r_j}^j$ the decomposition of $< \mathcal{L}_j >$ in $\mathcal{G}$.

We immediately observe that since by definition, for every j, the A_r^j are the maximal elements of $\mathcal{G}$ contained in $< \mathcal{L}_j >$, the set $\mathcal{T} = \{A_k^j\}$ constructed in this way is $\mathcal{G}$-nested.

Given that $\mathcal{L}_0$ is not $\mathcal{G}$-nested, there must be an element $B \in \mathcal{L}_0$ such that $B \notin \mathcal{T}$. Let h be the maximal index such that $B \subset < \mathcal{L}_h >$. Then by the maximality of the A_m^h's, we have $B \subset A_s^h$ for a certain s.

By construction A_s^h is a sum of subspaces in $\mathcal{L}_0$. Now B is maximal among the subspaces of $\mathcal{L}_0$ contained in A_s^h otherwise it would be $B \subset < \mathcal{L}_{h+1} >$. Furthermore, $B \neq A_s^h$ since $B \notin \mathcal{T}$, thus $A_s^h \notin \mathcal{L}_0$. Then $A_s^h = C$ has the requested property, that is to say, we can write $C = \sum_{D \in \mathcal{H}' \cup S'} D$.

Now, since S is $\mathcal{G}$ nested, $\mathcal{H}'$ must be not empty and $S' \subset S_C$. It follows that $d^S_{\mathcal{H}',C} = 0$ and therefore $P^S_{\mathcal{H}',C} = \prod_{A \in \mathcal{H}'} c_A$. Since $\prod_{A \in \mathcal{H}} c_A$ is divided by $P^S_{\mathcal{H}',C}$, it lies in I_S. ∎

The preceding lemma allows us to say that I_S is generated by the polynomials $\prod_{A \in \mathcal{H}} c_A$ (when $\mathcal{H} \cup S$ is not $\mathcal{G}$-nested) and by the polynomials $P^S_{\mathcal{H},B}$ (when $\mathcal{H} \cup S$ is $\mathcal{G}$-nested).

Proof of the Theorem.

The proof is by induction on the cardinality of the building set $\mathcal{G}$. The first step $|\mathcal{G}| = 0$ is trivial. Then we will make use of the blow-up property described in Theorem 1.1.7 and of the following lemma stated by Keel in [16].

Lemma 1.3.3 *1. Let Z be a smooth variety and $W \subset Z$ a smooth subvariety. Assume that the restriction map $H^*(Z, \mathbb{Z}) \to H^*(W, \mathbb{Z})$ is surjective with kernel J. Let Z' denote the blow up of Z along W. Denote by E both the exceptional divisor and its class in $H^2(Z', \mathbb{Z})$. Then*

$$H^*(Z', \mathbb{Z}) \equiv H^*(Z, \mathbb{Z})[E]/(J \cdot E, P_{Z/W}(-E)),$$

where $P_{Z/W}(x) \in H^*(Z, \mathbb{Z})[x]$ is any polynomial whose restriction to $H^*(W, \mathbb{Z})[x]$ gives the Chern polynomial of the normal bundle of W in Z.

2. $H^*(E, \mathbb{Z}) \equiv H^*(W, \mathbb{Z})[E]/(P_{Z/W}(-E))$, where $P_{Z/W}(x) \in H^*(W, \mathbb{Z})[x]$ denotes the Chern polynomial of the normal bundle of W in Z.

Let us assume that the claim is true for building sets of cardinality strictly lesser than $p > 0$ and let $\mathcal{F}$ be a building set of cardinality p. Furthermore, let A be a minimal element in $\mathcal{F}$. Then by Theorem 1.1.7 we know that $Y_{\mathcal{F}}$ is obtained from $Y_{\mathcal{G}}$, where $\mathcal{G} := \mathcal{F} - \{A\}$, by blowing up the proper transform Z_A of $A^{\perp}$. We take a $\mathcal{G}$-nested set S in $\mathcal{F}$ and want to compute the cohomology of D_S. Let us denote by $\overline{\mathcal{G}}$ the building family of subspaces $\overline{B} = B + A/A$ ($B \in \mathcal{G}$) which defines the variety $Z_A = Y_{\overline{\mathcal{G}}}$. In general, if $\mathcal{H} = \{H_1, \ldots, H_r\} \subset \mathcal{G}$, we shall set $\overline{\mathcal{H}} = \{\overline{H_1}, \ldots, \overline{H_r}\} \subset \overline{\mathcal{G}}$.

As a matter of notation, if $S \subset \mathcal{G}$ we wish to distinguish the variety $D_S \subset Y_{\mathcal{F}}$ by the variety defined by the same set in $Y_{\mathcal{G}}$, which we will denote by D'_S (similarly for I_S and $I'_{\mathcal{S}}$, $P^S_{\mathcal{H},\mathcal{B}}$, ${P'}^S_{\mathcal{H},\mathcal{B}}$).

Let us now present the cohomology rings of $Y_{\mathcal{G}}$ and its subvarieties under consideration as quotients of the polynomial ring $\mathbb{Z}[c'_B]$ for $B \in \mathcal{G}$, where the generators c'_B correspond to the cohomology classes of the divisors D'_B.

Remark that, under the blowing up map π, we have

$$\pi^*([D'_B]) = [D_B]$$

From the above considerations we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[c'_B]_{B \in \mathcal{G}} & \xrightarrow{\phi'} & H^*(Y_{\mathcal{G}}, \mathbb{Z}) \\ \gamma \downarrow & & \pi^* \downarrow \\ \mathbb{Z}[c_B]_{B \in \mathcal{F}} & \xrightarrow{\phi} & H^*(Y_{\mathcal{F}}, \mathbb{Z}) \end{array}$$

where $\gamma(c'_B) = c_B$.

Moreover for the inclusion $i : Z_A = Y_{\overline{\mathcal{G}}} \to Y_{\mathcal{G}}$ we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[c'_B]_{B\in\mathcal{G}} & \xrightarrow{\phi'} & H^*(Y_{\mathcal{G}},\mathbb{Z}) \\ \overline{\gamma}\downarrow & & i^*\downarrow \\ \mathbb{Z}[c_{\overline{B}}]_{B\in\mathcal{G}} & \xrightarrow{\overline{\phi}} & H^*(Y_{\overline{\mathcal{G}}},\mathbb{Z}) \end{array} \tag{1.4}$$

where $\overline{\gamma}(c'_B)$ is equal to $c_{\overline{B}}$ if $\{B, A\}$ is $\mathcal{G}$-nested and to 0 otherwise. Notice that in particular, since A is minimal, if $\mathcal{H} \cup S \subset \mathcal{G}$, and if $B \in \mathcal{G}$ properly contains all the elements in $\mathcal{H}$, we have

$$P^S_{\mathcal{H},B} = \gamma P'^S_{\mathcal{H},B} \tag{1.5}$$

Case 1. $S \cup \{A\}$ is not $\mathcal{F}$-nested.

In this case the variety Z_A does not meet the variety $D'_{\mathcal{S}}$ and we have $D'_S = D_S$. Also, by Lemma 1.3.2, $c_A \in I_S$. By the relation 1.5, we immediately deduce that I_S is generated by $\gamma(I'_S)$ and c_A as desired.

Case 2. S does not contain A and $S \cup A$ is $\mathcal{F}$-nested.

In this case the variety D'_S intersects transversally Z_A and D_S is the blow up of D'_S along $Z^S_A := Z_A \cap D'_S$.

The divisor D'_C, for $\{C, A\}$ $\mathcal{G}$-nested, intersects Z_A in the divisor $D_{\overline{C}}$. We thus have that Z^S_A is the subvariety $D_{\overline{\mathcal{S}}}$ in $Y_{\overline{\mathcal{G}}}$ and, by the inductive assumptions, the cohomology of Z^S_A is generated by the image of the cohomology of D'_S and we can apply Lemma 1.3.3. As for the normal bundle of Z^S_A in D'_S, we have, by transversality, that it is the restriction to Z^S_A of the normal bundle of Z_A in $Y_{\overline{\mathcal{G}}}$. With the notations of Lemma 1.3.3, in this case $P_{Z/W}(-E) = (-\sum_{B\supsetneq A}[D_B])^{dim(A)}$ since in [4] it is easily shown that the Chern polynomial of the normal bundle of Z_A in $Y_{\mathcal{F}}$ is

$$(t - \sum_{B\supsetneq A}[D_B])^{dim(A)}.$$

Let I^S_A denote the kernel of the map

$$\mathbb{Z}[c'_B] \xrightarrow{\phi_S} H^*(D_S,\mathbb{Z}) \xrightarrow{j^*} H^*(Z^S_A,\mathbb{Z})$$

where j is the inclusion. Using the diagram (1.4) and the inductive assumptions we immediately deduce that $j^* \circ \phi_S$ is surjective and that I^S_A coincides with the kernel of $\phi_{\overline{S}} \circ \overline{\gamma}$.

We must thus show by Lemma 1.3.3 that

$$I_S = ((\sum_{B\supseteq A} c_B)^{dim(A)}, c_A\gamma(I^S_A), \gamma(I'_{\mathcal{S}})) := J_S.$$

Let us first show that $I_S \supset J_S$. By 1.5 we get that $\gamma(I'_S) \subset I_S$, but we also have $(\sum_{B\supseteq A} c_B)^{dim(A)} = P^S_{\emptyset,A} \in I_S$.

Finally we need to show that $c_A\gamma(I_A^S) \subset I_{\mathcal{S}}$. Recall that $I_A^S = \overline{\gamma}^{-1}(I_{\overline{S}})$ and that $I_{\overline{S}}$ is generated by the polynomials $P^{\overline{S}}_{\overline{\mathcal{H}},\overline{B}}$. Notice that, since $ker\,\overline{\gamma}$ is generated by the c'_B such that $\{A, B\}$ is not $\mathcal{F}$-nested, clearly $c_A\gamma(ker\overline{\gamma}) \subset I_S$. So we are reduced to showing that for any polynomial $P^{\overline{S}}_{\overline{\mathcal{H}},\overline{B}}$ there is a representative $P' \in \mathbb{Z}[c'_B]$ such that $c_A\gamma(P') \in I_S$.

Write $\overline{\mathcal{G}} = \overline{\mathcal{G}}_1 \cup \overline{\mathcal{G}}_2$, where $\overline{\mathcal{G}}_1$ is the set of elements in $\overline{\mathcal{G}}$ whose preimage under the quotient homomorphism $q : V^* \to V/A$ (this preimage is well defined by the building property) is in $\mathcal{G}$, and $\overline{\mathcal{G}}_2 = \overline{\mathcal{G}} - \overline{\mathcal{G}}_1$. We can now define a unique lifting of $\overline{\mathcal{G}}$ to $\mathcal{G}$ as follows. If $\overline{B} \in \overline{\mathcal{G}}_1$ we lift it to $q^{-1}(B)$, otherwise we note that, for the building property, $q^{-1}(B) \notin \mathcal{G}$ can be expressed as the direct sum $A \oplus B'$ (with $B' \in \mathcal{G}$) and we lift $\overline{B}$ to B'.

Now take a polynomial $P^{\overline{S}}_{\overline{\mathcal{H}},\overline{B}}$. Using our lifting to lift $\overline{\mathcal{H}}$ to $\mathcal{H}$ and $\overline{B}$ to B we can consider the polynomial $P'^S_{\mathcal{H},B}$. We easily see that unless $\overline{\mathcal{H}} \cup \overline{\mathcal{S}}_{\overline{B}} \subset \overline{\mathcal{G}}_2$ and $\overline{B} \in \overline{\mathcal{G}}_1$ we have $\overline{\gamma}(P'^S_{\mathcal{H},B}) = P^{\overline{S}}_{\overline{\mathcal{H}},\overline{B}}$. If on the other hand $\overline{\mathcal{H}} \cup \overline{\mathcal{S}}_{\overline{B}} \subset \overline{\mathcal{G}}_2$ and $\overline{B} \in \overline{\mathcal{G}}_1$ then $P'^S_{\mathcal{H},B}$ is divisible by $(\sum_{C \supseteq B} c_C)^{dimA}$ and, if we set $Q'^S_{\mathcal{H},B} = P'^S_{\mathcal{H},B}/(\sum_{C \supseteq B} c_C)^{dimA}$, we obtain that $\overline{\gamma}(Q'^S_{\mathcal{H},B}) = P^{\overline{S}}_{\overline{\mathcal{H}},\overline{B}}$.

Now in the first case $P'^S_{\mathcal{H},B} \in I'_S$ so our claim is obvious. In the second $c_A\gamma(Q'^S_{\mathcal{H},B}) = P^S_{\mathcal{H}\cup\{A\},B} \in I_S$.

Then we must show $I_S \subset J_S$. From the previous analysis $\gamma(P'^S_{\mathcal{H},B}) = P^S_{\mathcal{H},B}$ so we only have to consider the case in which A appears in $\mathcal{H}$ or $A = B$. Now $A = B$ can occur only when $\mathcal{H} = \emptyset$ and we have already treated this case. Let now $A \in \mathcal{H}$. We immediately deduce from our previous analysis that $P^S_{\mathcal{H},B} = c_A\gamma(P')$ with $P' \in I_A^S$.

Case 3. $S = S' \cup \{A\}$.

In this case D_S is the exceptional divisor in $D_{\mathcal{S}'}$ or, equivalently, the preimage in $Y_{\mathcal{F}}$ of Z_A^S. Our claim then follows by a completely similar argument to the above described one, using Lemma 1.3.3. ∎

Remark. This proof is essentially analogous to the one provided in [4]: it is shorter since it uses Theorem 1.1.7 whose statement is slightly different from the corresponding one in [4].

Chapter 2

Root hyperplane arrangements

2.1 De Concini-Procesi models for root arrangements

An important application of the theory exposed in the preceding chapter is the study of the De Concini -Procesi models of root hyperplane arrangements.

Here for "root hyperplane arrangement" we mean a complexified arrangement $\mathcal{A}_\Phi^* \subset \mathbb{C}^n$ provided by the hyperplanes orthogonal to the roots of a complexified irreducible root system Φ of dimension n in $(\mathbb{C}^n)^*$.

In particular, we will deal with the De Concini-Procesi compact models $\widehat{Y}_{\mathcal{F}}$, where $\mathcal{F}$ is the building set of irreducibles which refines $\mathcal{A}_\Phi^*$. This means that $\mathcal{F}$ consists of the irreducible subspaces in $\mathcal{C}_{\mathcal{A}_\Phi}$ (note that $\mathcal{F}$ in particular contains all the linear subspaces $\mathbb{C}\theta$, where θ is a root of Φ). The following proposition gives a characterization of $\mathcal{F}$:

Proposition 2.1.1 *A subspace $A \subset (\mathbb{C}^n)^*$, spanned by some of the roots in Φ, belongs to $\mathcal{F}$ if and only if $\Phi_A = \Phi \cap A$ is an irreducible root subsystem of Φ. In particular, $(\mathbb{C}^n)^*$ belongs to $\mathcal{F}$.*

Proof.

Let us consider a subspace A which is spanned by roots and such that $\Phi_A = \Phi \cap A$ is an irreducible root subsystem of Φ. We will prove that A belongs to $\mathcal{F}$ (note that the case $n = 1$ is obvious, hence we suppose dim $A \geq 2$).

If $A \notin \mathcal{F}$ we could find a decomposition (in the sense of Definition 1.1.2) into irreducibles $A_1 \oplus A_2 \oplus \cdots \oplus A_k = A$ $(k > 1)$. Let us call by Φ_{A_i} the intersections $\Phi \cap A_i$ $(i = 1, \ldots, k)$. Since Φ_A is an irreducible root system, we can find a root $\alpha \in \Phi_{A_1}$ and a root $\beta \in \Phi_{A_j}$ for a certain j such that $(\alpha, \beta) \neq 0$ (here $(\, , \,)$ is the standard symmetric bilinear form on $(\mathbb{C}^n)^*$). Therefore, by the elementary properties of root systems (see [15]), $\alpha + \beta$ (if $(\alpha, \beta) < 0$) or $\alpha - \beta$ (if $(\alpha, \beta) > 0$) belongs to Φ_A. Let us suppose that

$(\alpha+\beta) \in \Phi$, the other case being analogous; now, $\alpha+\beta$ does not belong to A_i for any $i = 1, \ldots, k$ since the sum $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ is direct. This contradicts the fact that $A_1 \oplus A_2 \oplus \cdots \oplus A_k = A$ is a decomposition, given that $\mathbb{C}(\alpha+\beta) \neq \{0\} = \bigoplus_{i=1}^{k} (A_i \cap \mathbb{C}(\alpha+\beta))$.

Let us now consider a subspace $A \in \mathcal{F}$ and prove that $\Phi_A = \Phi \cap A$ is an irreducible root system. If this was not the case, we could find some non empty irreducible pairwise orthogonal root systems $\Phi_1, \ldots, \Phi_s$ $(s > 1)$ such that $\bigcup_{j=1}^{s} \Phi_j = \Phi_A$. Now, calling by B_i the span of Φ_i $(i = 1, \ldots, s)$, we have that $B_1 \oplus B_2 \cdots \oplus B_s = A$. Furthermore, taking a subspace $C \subset A$ spanned by roots and calling by Φ_C the intersection $\Phi \cap C$, we have that Φ_C is the disjoint union of the pairwise orthogonal subsystems $\Phi_C \cap \Phi_1, \Phi_C \cap \Phi_2, \ldots, \Phi_C \cap \Phi_s$ (some of them may be empty). Therefore C is equal to $(C \cap B_1) \oplus \cdots \oplus (C \cap B_2)$. This means that $B_1 \oplus \cdots \oplus B_s$ is a (non trivial) decomposition of A, contradicting the assumption $A \in \mathcal{F}$. ■

A remarkable property of the root arrangement $\mathcal{A}_\Phi^*$ is that the De Concini -Procesi model $\widehat{Y}_\mathcal{F}$ can be obtained via an embedding of $\mathcal{M}_\mathcal{F}$ into a product of projective lines $\mathbf{P}^1$ $(= \mathbf{P}(\mathbb{C}^2))$.

In other words, one does not need to embed $\mathcal{M}_\mathcal{F}$ in the product of the various $\mathbf{P}(\mathbb{C}^n/A^\perp)$ $(A \in \mathcal{F})$, but it suffices to consider only the $\mathbf{P}(\mathbb{C}^n/B^\perp)$ with $B \in \mathcal{F}$ and $dim\, B = 2$, as it is shown by the following theorem.

Theorem 2.1.2 *Let $\mathcal{F}$ be the building set of irreducibles which refines a root hyperplane arrangement $\mathcal{A}_\Phi^*$. Then the restriction to $\widehat{Y}_\mathcal{F}$ of the projection*

$$\prod_{A \in \mathcal{F}} \mathbf{P}(\mathbb{C}^n/A^\perp) \mapsto \prod_{\substack{A \in \mathcal{F} \\ dim\, A = 2}} \mathbf{P}(\mathbb{C}^n/A^\perp)$$

induces a closed embedding

$$\zeta \;:\; \widehat{Y}_\mathcal{F} \mapsto \prod_{\substack{A \in \mathcal{F} \\ dim\, A = 2}} \mathbf{P}(\mathbb{C}^n/A^\perp)$$

Proof.

We can prove the theorem using local coordinates. Let us first choose a suitable collection of open charts.

Lemma 2.1.3 *We can choose an open covering $\mathcal{U}$ of $\widehat{Y}_\mathcal{F}$ made by open charts U_S^b of the following kind: S is a maximal $\mathcal{F}$-nested set which contains $(\mathbb{C}^n)^*$; b is a marked basis, consisting of roots (i.e., $b \subset \Phi$), adapted to S.*

Proof of the Lemma.

We begin by observing that the cardinality of a maximal $\mathcal{F}$-nested set S is equal to n and that $\sum_{A\in S} A = (\mathbb{C}^n)^*$ (see Proposition 1.1 (2) in [5]). Therefore S contains $(\mathbb{C}^n)^*$; otherwise, calling by $B_1, B_2, \ldots, B_r$ ($r \geq 2$) the maximal elements in S, we should have $B_1 \oplus B_2 \oplus \cdots \oplus B_r = (\mathbb{C}^n)^*$. But this is impossible since $(\mathbb{C}^n)^* \in \mathcal{F}$, S is $\mathcal{F}$-nested and $B_1 \oplus B_2 \oplus \cdots \oplus B_r$ is not the decomposition of $(\mathbb{C}^n)^*$ in $\mathcal{F}$.

Let us now show that we can choose marked bases, made by roots, adapted to S. First we observe that, since $|S| = n$, every element of a basis adapted to S is marked. Then we recall the following algorithm (see Chapter 1, Section 1) which selects a suitable collection of adapted (marked) bases for S. Choose for every $B \in S$ a basis $b(B)$ of B made by vectors not contained in any $C \subsetneq B$, $C \in S$. Choose a vector $x_B \in b(B)$ for every $B \in S$. Then these vectors are linearly independent and give a basis of $(\mathbb{C}^n)^*$ adapted to S. If we fix the bases $b(B)$ ($B \in S$) and perform the above algorithm in all the possible ways (that is to say, if we choose the vectors x_B in all the possible ways), we get a family $\Theta(S)$ of adapted (marked) bases. Then the collection of open charts U_S^b, where S ranges over the maximal nested sets in $\mathcal{F}$ and b ranges over $\Theta(S)$, is a covering of $\widehat{Y}_{\mathcal{F}}$. Therefore we have to prove that, for every element $B \in S$, we can choose a basis of B consisting of roots which belong to $B - \bigcup_{A\in S_B} A$, where $S_B = \{C \in S \mid C \subsetneq B\}$.

We will construct such a basis using the following two-steps algorithm. Let $A_1, \ldots, A_k$ be the maximal elements in S_B. As a preliminary step, we note that we can find a root α in $B - \bigoplus_{i=1}^k A_i$ since B is generated by roots.

Step 1. For every $1 \leq i \leq k$, let us choose a basis Δ_i for the irreducible root system $\Phi_{A_i} = \Phi \cap A_i$. Then, since $\Phi_B = \Phi \cap B$ is irreducible, there exists at least an i and a root β_i of Δ_i such that $(\alpha, \beta_i) \neq 0$. Therefore $\alpha + \beta$ (if $(\alpha, \beta_i) < 0$) or $\alpha - \beta$ (if $(\alpha, \beta_i) > 0$) is a root in $\Phi_B \cap \big(B - \bigcup_{A\in S_B} A\big)$. Now let us consider all the other roots of the basis Δ_i. If $\gamma \in \Delta_i$ and $(\alpha, \gamma) \neq 0$, we find, as before, $\alpha - \gamma$ or $\alpha + \gamma$ in $\Phi_B \cap \big(B - \bigcup_{A\in S_B} A\big)$. If $\delta \in \Delta_i$ is such that $(\alpha, \delta) = 0$, we can consider the set Γ_δ of all the roots θ in Δ_i such that $(\alpha, \theta) \neq 0$. Then we can form the subset $\Gamma_{\delta'}$ of Γ_δ made by all the roots of Γ_δ which have minimal distance from δ (where the distance is the graph distance in the Dynkin diagram). Finally we can choose in a canonical way a root ε among all the roots of $\Gamma_{\delta'}$ (for example, we can take ε to be the minimal one with respect to a previously fixed total order in Φ). Now, if $\delta, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s, \varepsilon$ are all the roots of Δ_i which lie in the path of the Dynkin diagram of Φ_{A_i} which connects δ to ε, it turns out that $\delta' = \delta + \varepsilon + \sum_{i=1}^s \varepsilon_i$ is a root in Φ_{A_i}. Moreover, we have that $(\delta', \alpha) \neq 0$ and $\alpha - \delta'$ or $\alpha + \delta'$ is in $\Phi_B \cap \big(B - \bigcup_{A\in S_B} A\big)$.

Repeating this construction for every root in Δ_i we find $|\Delta_i| + 1$ linearly independent elements in $\Phi_B \cap \big(B - \bigcup_{A\in S_B} A\big)$; one of these elements is α and the other elements are of type $\alpha - \nu_i$ ($i = 1, \ldots, |\Delta_i|$) where ν_i are linearly independents roots of Φ_{A_i}.

Step 2. Let us consider the root α and search for a $j \neq i$ such that a root β_j of the basis Δ_j of Φ_{A_j} satisfies $(\alpha, \beta_j) \neq 0$. If such a j does not exist we search again considering $\alpha - \nu_1$ instead of α. If we fail again we search considering $\alpha - \nu_2$ instead of $\alpha - \nu_1$ and so further. Since Φ is irreducible, one among the roots $\alpha, \alpha - \nu_1, \ldots, \alpha - \nu_{|\Delta_i|}$ (say $\alpha - \nu_{\bar{i}}$) has non zero scalar product with a root β_j of Δ_j for a certain $j \neq i$, therefore after some steps our search succeeds.

Conclusion. Then we can repeat the construction of Step 1 with the root $\alpha - \nu_{\bar{i}}$ and the basis Δ_j. Thus we find $|\Delta_j|$ linearly independent elements in $\Phi_B \cap \left(B - \bigcup_{A \in S_B} A\right)$ which, together with the $|\Delta_i| + 1$ we found before, give $|\Delta_i| + |\Delta_j| + 1$ linearly independent roots in $B - \bigcup_{A \in S_B} A$. One of these roots is α and the others are of type $\alpha - \mu$ with μ a root in the span of $\Phi_i \cup \Phi_j$. After k applications of our two-steps algorithm we find a basis of B made by roots which belong to $B - \bigcup_{A \in S_B} A$, which is what we were searching for. ■

In what follows we will use an open covering $\mathcal{U}$ with the properties described in the lemma. Let us consider $U_S^b \in \mathcal{U}$: we will prove that ζ restricted to U_S^b is an open embedding by showing that there is a local inverse

$$\eta_S^b \,:\, \zeta(U_S^b) \mapsto \widehat{Y}_{\mathcal{F}}$$

For every $E \in S$, let us call by γ_E the element of b which belongs to $E - \bigcup_{C \in S_E} C$. Moreover, for every $D \in \mathcal{F}$, let us call by π_D the projection

$$\pi_D \,:\, \widehat{Y}_{\mathcal{F}} \mapsto \mathbf{P}\left(\mathbb{C}^n / D^{\perp}\right)$$

Therefore we can define η_S^b by defining, for every $D \in \mathcal{F}$, the morphism $\pi_D \, o \, \eta_S^b$ (note that the case dim $D = 2$ is obvious).

Let us first consider the case $D \in S$. Reasoning in the same way as in the proof of the Lemma 2.1.3, we can find a basis $\gamma_D, \gamma_D - \mu_1, \ldots, \gamma_D - \mu_{\dim D - 1}$ of D where the μ_i and the $\gamma_D - \mu_i$ are roots and $(\gamma_D, \mu_i) \neq 0$ for every $i = 1, \ldots, \dim D - 1$. This means that the two dimensional subspaces $< \gamma_D, \mu_i >$ spanned by γ_D and μ_i ($i = 1, \ldots, \dim D - 1$) belong to $\mathcal{F}$.

Furthermore, if we take a point $p \in \zeta(U_S^b)$ and denote by $[p_{\gamma_D}, p_{\mu_i}]$ its homogeneous coordinates in $\mathbf{P}\left(\mathbb{C}^n / < \gamma_D, \mu_i >^{\perp}\right)$ with respect to the basis dual to γ_D, μ_i, we can consider $p_{\gamma_D} = 1$ by the definition of U_S^b.

Now, given $C \in S_D$, we can write

$$\gamma_C = \sum_{r=1}^{\dim D - 1} a_r(C) \mu_r$$

for certain scalars $a_r(C)$. Therefore we can define $\pi_D \, o \, \eta_S^b(p)$ giving its projective coordinates in terms of the basis dual to γ_D, γ_C ($C \in S_D$): we put

$$\gamma_D \left(\pi_D \, o \, \eta_S^b(p)\right) = 1$$

and

$$\gamma_C\left(\pi_D\, o\, \eta_S^b(p)\right) = \sum_{r=1}^{\dim D-1} a_r(C)p_{\mu_r}$$

Now it remains the case when $D \notin S$. Keeping the notation of Chapter 1, Section 1, we call by $p_S(D)$ the minimal (with respect to inclusion) subspace in S which includes D. As before, we find a basis $\gamma_{p_S(D)}, \mu_1, \dots, \mu_{\dim p_S(D)-1}$, consisting of roots, of $p_S(D)$ such that $(\gamma_{p_S(D)}, \mu_l) \neq 0$ for every $l = 1, .., \dim p_S(D) - 1$. After choosing a basis $e_1, \dots, e_{\dim D}$ of D, we can write, for every $j = 1, \dots, \dim D$

$$e_j = a_0(j)\gamma_{p_S(D)} + \sum_{r=1}^{\dim p_S(D)-1} a_r(j)\mu_r$$

Therefore we can define the projective coordinates of $\pi_D\, o\, \eta_S^b(p)$ (in terms of the basis dual to $e_1, \dots, e_{\dim D}$) in the following way:

$$e_j\left(\pi_D\, o\, \eta_S^b(p)\right) = a_0(j) + \sum_{r=1}^{\dim p_S(D)-1} a_r(j)p_{\mu_r}$$

(here $[p_{\gamma_{p_S(D)}}, p_{\mu_r}]$ are the homogeneous coordinates in $\mathbf{P}\left(\mathbb{C}^n/ < \gamma_{p_S(D)}, \mu_r >^{\perp}\right)$ with respect to the basis dual to $\gamma_{p_S(D)}, \mu_r$, and we take $p_{\gamma_{p_S(D)}} = 1$). By construction, the above defined map η_S^b is a morphism and it is the inverse of ζ restricted to U_S^b. It follows that ζ restricted to any U_S^b is an isomorphism with its image; therefore $\zeta(\widehat{Y}_{\mathcal{F}})$ is smooth and ζ is an open embedding unless it has an exceptional subvariety, i.e. a subvariety $Z \subset \widehat{Y}_{\mathcal{F}}$ such that codim $Z = 1$ but codim $\zeta(Z) \geq 2$. But since $\widehat{Y}_{\mathcal{F}}$ is covered by a finite number of coordinate charts such a subvariety cannot exist. ■

In this thesis we will deal with the root arrangements in $\mathbb{C}^n$ associated to the root systems of types A_n, B_n, D_n, which will be denoted respectively by $\mathcal{A}_n^*$, $\mathcal{B}_n^*$ and $\mathcal{D}_n^*$ respectively (the root arrangement $\mathcal{C}_n^*$ associated with the root system of type C_n will be shown to coincide with $\mathcal{B}_n^*$) and with the associated root systems of irreducibles $\mathcal{F}_{\mathcal{A}_n}$, $\mathcal{F}_{\mathcal{B}_n}$, and $\mathcal{F}_{\mathcal{D}_n}$. In the next section we will focus on the consequences of Theorem 2.1.2 in the A_n case, showing that, for every integer $n \geq 3$, $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and the moduli space $\overline{M}_{0,n+1}$ of $n+1$-pointed stable curves of genus 0 are isomorphic.

2.2 The braid arrangement and the moduli space of pointed curves of genus 0

Let us start from a realization of the moduli space $M_{0,n+1}$ of $n+1$-pointed curves of genus 0.

Definition 2.2.1

$$M_{0,n+1} = SL(2) \setminus \left\{ (p_0, \ldots, p_n) \in \underbrace{\mathbf{P}^1 \times \ldots \times \mathbf{P}^1}_{n+1 \; times} \mid p_i \neq p_j \; \forall \, i \neq j \right\}$$

where $SL(2)$ acts componentwise.

Given an element $p \in M_{0,n+1}$, after making $SL(2)$ to act, we can canonically write

$$p = [(0,1),\ (1,0),\ (1,1),\ (x_1, y_1), \ldots,\ (x_{n-2}, y_{n-2})]$$

As a matter of notation, here, and everywhere we deal with orbits, the brackets mean: "equivalence class of".

It follows that $M_{0,n+1}$ is in bijective correspondence with the set

$$\widehat{M}_{0,n+1} = \left\{ (q_1, \ldots, q_{n-2}) \in \underbrace{\mathbf{P}^1 \times \ldots \times \mathbf{P}^1}_{n-2 \; times} \mid q_i \neq q_j,\ q_i \neq 1, 0, \infty \right\}$$

A relevant point is that we can also give another description of $\widehat{M}_{0,n+1}$, showing that it can be identified with the complement $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ of the projective arrangement of type A_{n-1}.

In fact, let us consider $\mathbb{C}^n$ and the braid arrangement, that is to say, the hyperplane arrangement given by the hyperplanes $z_{ij} : x_j - x_i = 0$, where $x_i \in (\mathbb{C}^n)^*$ $(i = 1, \ldots, n)$ are the coordinate functions. We note that the intersection of all the hyperplanes is the subspace $N = \mathbb{C}(\underbrace{1, \ldots, 1}_{n \; times})$. We can thus consider the quotient $V = \mathbb{C}^n / N$ equipped with the arrangement $\mathcal{A}^*_{n-1}$ provided by the images of the hyperplanes z_{ij} via the quotient map $\pi : \mathbb{C}^n \mapsto V$. We can immediately see that $\mathcal{A}^*_{n-1}$ is a root arrangement of type A_{n-1}. We will call by t_{hk} the functionals in V^* the zeroes of which form the hyperplane $\pi(z_{hk})$ in V and such that $(t_{hk}, t_{hk}) = 2$ (where $(\ ,\ )$ is the scalar product in V^*).

Then $\{t_{hk} \mid h, k = 1, \ldots, n\} \cup \{-t_{hk} \mid h, k = 1, \ldots, n\}$ is a root system (which we will denote by $\Phi_{\mathcal{A}_{n-1}}$) of type A_{n-1} and we observe that the set $\{t_{12}, t_{23}, \ldots, t_{(n-1)n}\}$ can be taken as a basis.

Let us now call by ψ the projection map $\psi : V \mapsto \mathbf{P}(V)$ and consider the projectivization of $\mathcal{A}^*_{n-1}$. According to our notation, we call by $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ the complement in $\mathbf{P}(V)$ of the union of the images $\psi(D)$ $(D \in \mathcal{A}^*_{n-1})$.

Theorem 2.2.1 *There is a bijective map between $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $\widehat{M}_{0,n+1}$ that gives rise to an isomorphism between $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $M_{0,n+1}$.*

Proof.

Let us choose in V the basis $\{v_2, \ldots, v_n\}$ dual to the basis $\{t_{12}, t_{13}, \ldots, t_{1n}\}$ of V^*. We note that a set of representatives for the v_j's can be chosen as follows: $v_j = \pi((0, \ldots, 0, 1, 0, \ldots, 0))$ where the only non zero entry is the j-th.

Then we can define a map $\phi : \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \mapsto \widehat{M}_{0,n+1}$:

$$\phi(\gamma_1, \ldots, \gamma_{n-1}) = ((\gamma_1, \gamma_2), (\gamma_1, \gamma_3), \ldots, (\gamma_1, \gamma_{n-1}))$$

Note that if $\gamma_j = 0$ for a certain j, then $(\gamma_1, \ldots, \gamma_{n-1}) \in H_{1(j+1)}$ and if $\gamma_i = \gamma_j$ $(i < j)$ then $(\gamma_1, \ldots, \gamma_{n-1}) \in H_{(i+1)(j+1)}$. This implies that ϕ is well defined. The injectivity is trivial, while the surjectivity is a consequence of the above remarks, a right inverse being given by the map θ such that $\theta((1, r_1), \ldots, (1, r_{n-2})) = (1, r_1, \ldots, r_{n-2})$. ■

The above theorem is the reason of the connections between the theory of hyperplane arrangements and the theory of moduli spaces of pointed curves of genus 0. In order to examine closely these connections, let us now focus on the arrangement $\mathcal{A}^*_{n-1}$, our aim being to describe the De Concini-Procesi model $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$.

The elements of $\mathcal{F}_{\mathcal{A}_{n-1}}$ are the subspaces of V^* spanned by the irreducible root subsystems of $\Phi_{\mathcal{A}_{n-1}}$ (see Proposition 2.1.1). These subspaces can be described by means of a collection of subsets of $\{1, 2, \ldots, n\}$; in fact, given a subset $\overline{\Delta} = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ with $|\overline{\Delta}| \geq 2$, the subspace $\Delta \subset V^*$, generated by all the functionals t_{ij} such that $\{i, j\} \subset \overline{\Delta}$, is irreducible (a basis of $\Delta \cap \Phi_{\mathcal{A}_{n-1}}$ is given by $t_{i_1 i_2}, t_{i_2 i_3}, \ldots, t_{i_{p-1} i_p}$). Furthermore, the following proposition shows that all the elements of $\mathcal{F}_{\mathcal{A}_{n-1}}$ can be obtained in this way.

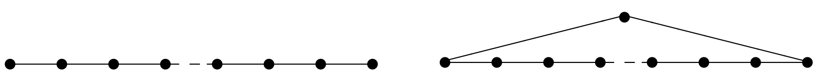
Proposition 2.2.2 *The above described correspondence between the elements of $\mathcal{F}_{\mathcal{A}_{n-1}}$ and the subsets of $\{1, \ldots, n\}$ with cardinality greater than or equal to 2 is bijective.*

Proof.

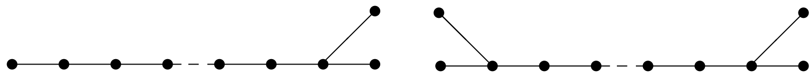
We have to prove that every element of $\mathcal{F}_{\mathcal{A}_{n-1}}$ can be determined by providing a subset of $\{1, \ldots, n\}$.

We need first to recall a well-known algorithm (see [6]) that, given a root system Φ, allows us to find the bases of all the root subsystems of Φ, starting from the bases of Φ.

Let us consider a basis Δ of Φ, the Dynkin diagram D_Φ of Φ and its affine extension $\widehat{D}_\Phi$: the latter is obtained by adding to D_Φ a vertex representing the negative of the longest root of Φ. For instance, the diagrams in the A_n, B_n and D_n cases are the following ones:

A_n 

B_n 

D_n 

We can now describe the steps of the algorithm: consider $\widehat{D}_\Phi$ and cancel a vertex v representing a simple root and the open part of all the edges which contain v. Then what remains is a Dynkin diagram D_{Φ_1} associated with a certain root system Φ_1. Take a connected part of D_{Φ_1}, consider its affine extension and continue. The connected parts of the Dynkin diagrams obtained via this algorithm provide the bases of all the irreducible root subsystems of Φ.

Let us now return to the root system $\Phi_{\mathcal{A}_{n-1}}$ where the roots are given by the functionals t_{ij}. Looking at the Dynkin diagrams $D_{\Phi_{\mathcal{A}_{n-1}}}$ and $\widehat{D}_{\Phi_{\mathcal{A}_{n-1}}}$ it follows that an irreducible root subsystem of $\Phi_{\mathcal{A}_{n-1}}$ is determined by a connected Dynkin diagram of type A_m whose vertices represent roots of the following kind: $t_{i_1 i_2}, t_{i_2 i_3}, \ldots, t_{i_m i_{m+1}}$. It is now immediate to check that this root subsystem is made by all the roots t_{ij} with $\{i,j\} \subset \{i_1, \ldots, i_{m+1}\}$. $\blacksquare$

As a matter of notation, given an element $A \in \mathcal{F}_{\mathcal{A}_{n-1}}$, we will call by $\overline{A}$ the associated subset $\overline{A} \subset \{1, \ldots, n\}$. Note that the dimension of A over $\mathbb{C}$ is equal to $|\overline{A}| - 1$.

Let us now study the nested sets in $\mathcal{F}_{\mathcal{A}_{n-1}}$. First we observe that, since $\mathcal{F}_{\mathcal{A}_{n-1}}$ is the building set of irreducibles, by definition a $\mathcal{F}_{\mathcal{A}_{n-1}}$-nested set is nothing else than a nested set. Therefore one can easily see that a collection S of subsets of $\{1, \ldots, n\}$ corresponds to a nested set in $\mathcal{F}_{\mathcal{A}_{n-1}}$ if and only if, for every $\overline{A}, \overline{B} \in S$, either $\overline{A} \cap \overline{B} = \emptyset$ or one of these subsets is included into the other.

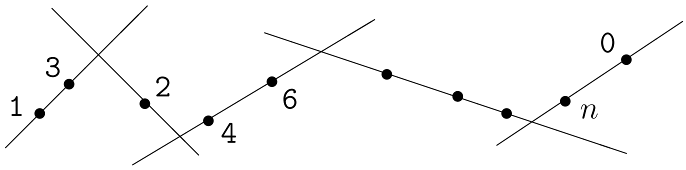
We now recall that Yuzvinsky, in his paper [25], associated an oriented graph to every nested set $T \subset \mathcal{F}_{\mathcal{A}_{n-1}}$ in the following way. Take as vertices the elements of T and the numbers $1, \ldots, n$. Let then A and B be elements in T such that A is maximal in $T_B = \{C \in T \mid C \subsetneq B\}$: then draw an edge which joins the vertices A and B and is oriented from B to A.

If, given $A \in T$, the set $T_A = \{C \in T \mid C \subsetneq A\}$ is empty, for each number $k \in \overline{A} \subset \{1, \ldots, n\}$, draw an edge which joins the vertices A and the number k and is oriented from A to k.

The resulting graph is a forest on n leaves (which are identified with the vertices $1, \ldots, n$), the connected components of which are rooted oriented trees. We will focus on these graphs in the next chapters, given that they play a crucial role in the study of the cohomology ring $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$.

We can now return to the moduli spaces and consider the compactification $\overline{M}_{0,n+1}$ of $M_{0,n+1}$, that is to say, the moduli spaces of stable $n+1$-pointed curves of genus 0. What we want to point out is that the isomorphism of Theorem 2.2.1 between the open subvarieties $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \subset \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and $M_{0,n+1} \subset \overline{M}_{0,n+1}$ can be extended to the boundary, that is to say, we have an isomorphism between $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and $\overline{M}_{0,n+1}$.

To prove this, we start by giving a description of the elements of $\overline{M}_{0,n+1}$ as connected tree-like stable $n+1$-pointed curves. This means that we are considering elements of this kind



Here each line represents an irreducible curve of genus 0 (i.e, $\mathbf{P}^1$), every double point represents a point of transversal intersection between the irreducible curves, the other special points, i.e. the punctures, are numbered from 0 to n and the stability is given by the request that the special points (punctures or double points) on each irreducible component are at least 3.

It is well known that there is a morphism

$$\mu_{n+1} \; : \; \overline{M}_{0,n+1} \mapsto \overline{M}_{0,n}$$

obtained by forgetting the point labeled with n and, if it is the case, collapsing some irreducible components. At the same way we can construct the maps μ_i which "forget" the point labeled with i ($i = 1, \ldots, n-1$).

Let us then call $\overline{M}_{0,ijk}$ ($1 \le i < j < k \le n$) the moduli space $\overline{M}_{0,4}$ in which the points are labeled using the numbers i, j, k. A composition of some of the maps μ_i gives a morphism

$$\overline{M}_{0,n+1} \mapsto \overline{M}_{0,ijk}$$

Now the morphism we are interested in is

$$\nu \; : \; \overline{M}_{0,n+1} \mapsto \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \overline{M}_{0,ijk}$$

which is given by the above described projections to each component.

Proposition 2.2.3 *The morphism ν is injective.*

Proof.

First we note that we can reduce ourselves to prove that, for every n, the map

$$\prod_{i=1}^{n} \mu_i \; : \; \overline{M}_{0,n+1} \mapsto \prod_{1 \leq i \leq n} \overline{M}_{0,n}$$

is injective. Therefore we have to check that an element $p \in \overline{M}_{0,n+1}$ is uniquely determined by its image $\prod_i \mu_i(p)$. For this we notice that if there is an irreducible component of p which has at least two marked points (say "i" and "j") different from "0" and at least four special points, p can be determined by knowing $\mu_i(p)$ and $\mu_j(p)$.

Let us now assume that $n \geq 6$ and that p has not irreducible components with the above mentioned properties. Then in every irreducible component of p there are at most two marked points and thus, being $n+1 \geq 7$, there are at least four irreducible components. In particular there are two irreducible components c_1, c_2 of p, which some marked points different from "0" belong to, and such that $c_1 \cap c_2 = \emptyset$. Now, if the point "i" belongs to c_1 and the point "j" belongs to c_2 $(i, j \neq 0)$, p is determined by $\mu_i(p)$ and $\mu_j(p)$. Then our claim is proved after a case-by-case check for $3 \leq n \leq 5$. ∎

Now we note that in the theory of De Concini - Procesi models we came across a map similar to $\prod_i \mu_i$, namely the map ζ of Theorem 2.1.2, specialized to the $\mathcal{A}_{n-1}$ case. In fact, given

$$\zeta \; : \; \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} \mapsto \prod_{\substack{A \in \mathcal{F}_{\mathcal{A}_{n-1}} \\ dim\, A = 2}} \mathbf{P}(V/A^{\perp})$$

we observe that the irreducible two dimensional subspaces $A \in \mathcal{F}_{\mathcal{A}_{n-1}}$ can be parametrized, according to the conventions established above, by the triples of integers i, j, k with $1 \leq i < j < k \leq n$. As a matter of notation, we will call $\mathbf{P}_{ijk}$ the projective space $\mathbf{P}(V/A^{\perp})$ when $\overline{A} = \{i, j, k\} \subset \{1, \ldots, n\}$.

Then we want to define in a suitable way an isomorphism

$$\gamma \; : \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \overline{M}_{0,ijk} \mapsto \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \mathbf{P}_{ijk}$$

Our request is that γ should be compatible with the isomorphism between the subvarieties $\zeta(\widehat{\mathcal{M}}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$ and $M_{0,n+1}$ (notice that $\widehat{\mathcal{M}}_{\mathcal{F}_{\mathcal{A}_{n-1}}} = \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ since the building set $\mathcal{F}_{\mathcal{A}_{n-1}}$ refines $\mathcal{A}^*_{n-1}$). Such a γ can be obtained by identifying $\mathbf{P}_{ijk}$ with $\overline{M}_{0,ijk}$ in the following way. Let $p = [(0,1), (1,0), (1,1), (x_3, y_3), \ldots, (x_n, y_n)]$ be a point of $M_{0,n+1}$ and let us put, for convenience of notation, $(x_2, y_2) = (1,1)$ and $(x_1, y_1) = (1, 0)$.

Then the projection of p to $\overline{M}_{0,ijk}$ is given by $[(0,1),\ (x_i,y_i),\ (x_j,y_j),\ (x_k,y_k)]$ and it can be written in canonical way if we use $SL(2)$ to send (x_i,y_i) to $(1,0)$ and (x_j,y_j) to $(1,1)$ (keeping fixed $(0,1)$). The matrix of $SL(2)$ we use is (up to scalar)

$$\begin{pmatrix} \frac{y_j}{x_j} - \frac{y_i}{x_i} & 0 \\ -\frac{y_i}{x_i} & 1 \end{pmatrix}$$

(note that, for every $i = 1, \ldots, n$, $x_i \neq 0$).

Thus we obtain $\left[(0,1),\ (1,0),\ (1,1),\ (\dfrac{y_j}{x_j} - \dfrac{y_i}{x_i}, \dfrac{y_k}{x_k} - \dfrac{y_i}{x_i})\right]$. If we consider the isomorphism ϕ of Theorem 2.2.1 between $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $M_{0,n+1}$, we have that $\phi^{-1}(p) = (1, \frac{y_3}{x_3}, \ldots, \frac{y_n}{x_n})$.

Let us now study the projection of $\phi^{-1}(p)$ to $\mathbf{P}_{ijk}$. We recall that if $\overline{A} = \{i,j,k\} \subset \{1,\ldots,n\}$ then $A^{\perp}$ is the $n-3$-dimensional subspace the elements of which have the i-th, j-th and k-th components equal. This means that the projection of $\phi^{-1}(p)$ to $\mathbf{P}_{ijk}$ is $(\frac{y_j}{x_j} - \frac{y_i}{x_i}, \frac{y_k}{x_k} - \frac{y_i}{x_i})$ in the projective coordinates given by the vectors v_j and v_k (note that $k > j \geq 2$). As a consequence, we can identify $\mathbf{P}_{ijk}$ with $\overline{M}_{0,ijk}$ via γ by choosing in $\mathbf{P}_{ijk}$ the projective coordinates given by v_j and v_k.

Now let us consider the diagram

$$\begin{tikzcd} \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} \arrow[d, "\zeta"'] & \overline{M}_{0,n+1} \arrow[d, "\nu"'] \\ \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \mathbf{P}_{ijk} & \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \overline{M}_{0,ijk} \arrow[l, "\approx"', "\gamma"] \end{tikzcd}$$

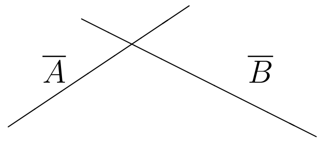
Theorem 2.2.4 *The above diagram can be completed with an isomorphism* $\Gamma : \overline{M}_{0,n+1} \mapsto \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$.

Proof.
First we note that $\gamma(\nu(M_{0,n+1})) = \zeta(\widehat{\mathcal{M}}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$ and then, since $\gamma(\nu(\overline{M}_{0,n+1}))$ is closed, the closure of $\zeta(\widehat{\mathcal{M}}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$ is included in $\gamma(\nu(\overline{M}_{0,n+1}))$. But this closure is equal to $\zeta(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$. Since $\zeta(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$ and $\gamma(\nu(\overline{M}_{0,n+1}))$ are closed and contain the same open dense subvariety, they must coincide. Then we observe that the map $\zeta^{-1} \circ \gamma \circ \nu$ is a well defined birational morphism between $\overline{M}_{0,n+1}$ and $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ which is also bijective, since we have proven that it is onto and furthermore ν is injective, γ is bijective and ζ is injective. Since the two varieties are smooth, this implies that $\zeta^{-1} \circ \gamma \circ \nu$ is an isomorphism. ∎

2.3 Divisors in $\overline{M}_{0,n+1}$.

Let us now focus on the map Γ and in particular on the image of the subvarieties in the boundary of $\overline{M}_{0,n+1}$. Recall that an irreducible divisor D in the boundary of $\overline{M}_{0,n+1}$ can be represented (see [16]) by the picture

$D =$



where $\overline{A} \subset \{0, \ldots, n\}$ and $\overline{B} = \{0, \ldots, n\} - \overline{A}$ satisfy $|\overline{A}| \geq 2$, $|\overline{B}| \geq 2$. The divisor D is the one which contains as an open set the set of all the elements δ of $\overline{M}_{0,n+1}$ which satisfy the following property: δ has two irreducible components such that the labels of the special points of each component are the elements of $\overline{A}$ and $\overline{B}$ respectively.

Now, given the model $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, let us call by $\hat{\pi}$ its projection to $\mathbf{P}(V)$.

Proposition 2.3.1 *Given D, $\overline{A}$ and $\overline{B}$ as before, let us suppose that $0 \in \overline{B}$. Furthermore, keeping the notation of the preceding section, let us indicate by A the irreducible subspace in V^* associated to $\overline{A}$. Then we have that $\Gamma(D) = D_A$.*

Proof.

Let us consider a chart U_S^b in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, where S is a $\mathcal{F}_{\mathcal{A}_{n-1}}$-nested set not containing V^* and $A \in S$. The intersection between D_A and U_S^b is given by the equation $u_A = 0$ (recall that if $A \notin S$ the intersection is empty). Therefore, given an element p in $D_A \cap U_S^b$, it satisfies the following property:

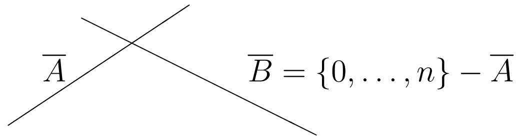
1. Given any triple (i, j, k) with $1 \leq i < j < k \leq n$ and $|\{i, j, k\} \cap \overline{A}| = 2$, the projection p_{ijk} of $\zeta(p)$ to $\mathbf{P}_{ijk}$ is 1,0,∞ when respectively $i \notin \overline{A}$, $j \notin \overline{A}$, $k \notin \overline{A}$.

This follows by construction of the chart U_S^b; let us consider for example the case $i \notin \overline{A}$. The projective coordinates on $\mathbf{P}_{ijk}$ are the ones provided by the basis v_j, v_k of $V/B^{\perp}$, where $\overline{B} = \{i, j, k\}$. Thus the projection to $V/B^{\perp}$ of an element $v = x_2v_2 + \cdots + x_nv_n$ in V is given in coordinates by $(x_j - x_i, x_k - x_i)$ (here we put $x_1 = 0$) and the corresponding projective coordinates in $\mathbf{P}_{ijk}$ are $[x_j - x_i, x_k - x_i]$. But $t_{jk} \in B$ and its expression in terms of the coordinates of U_S^b is a multiple of u_A. Since $B \notin S$ and $D_A \cap U_S^b = \{u_A = 0\}$, given a point $p \in D_A \cap U_S^b$, the projective coordinates $[x_2, \ldots, x_n]$ of $\hat{\pi}(p)$ satisfy $t_{jk}((x_2, \ldots, x_n)) = 0$, that is to say, $x_j = x_k$.

Therefore, since $p_{ijk} = [x_j - x_i, x_k - x_i] \neq [0, 0]$ by construction of U_S^b, we have $p_{ijk} = [1, 1]$. At the same way we can treat the cases $j \notin \overline{A}$, $k \notin \overline{A}$.

Let us now consider the points of the divisor

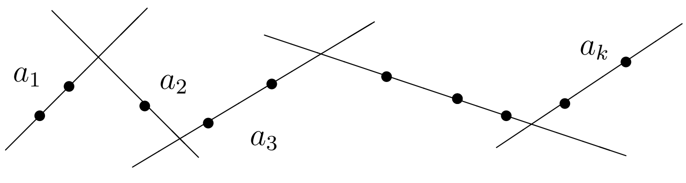
$D =$



Let $q \in D$ and let us take a triple (i, j, k) with $1 \leq i < j < k \leq n$, $i \notin \overline{A}$ and $\{j, k\} \subset \overline{A}$. Then the projection $\gamma\, o\, \nu(q)_{ijk}$ of $\gamma\, o\, \nu(q)$ to $\mathbf{P}_{ijk}$ is provided by the cross-ratio (p_0, p_i, p_j, p_k) of the special points p_0, p_i, p_j, p_k where we have collapsed the points p_j, p_k to the double point of D. Then this cross ratio is equal to 1. Reasoning in the same way when $j \notin \overline{A}$ and $k \notin \overline{A}$ we can conclude that $\Gamma(q)$ satisfies the property 1.

Furthermore, looking at the tree like representation of an element $z \in \overline{M}_{0,n+1}$ and at the cross ratios (p_0, p_i, p_j, p_k) we immediately see that $\Gamma(z)$ satisfies the property 1 if and only if $z \in D$. Since Γ is an isomorphism this means that the set of elements in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ which satisfy property 1 is exactly $\Gamma(D)$. Therefore $D_A \cap U_S^b \subset \Gamma(D)$ and, since D_A and $\Gamma(D)$ are irreducible divisors in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, it follows that $\Gamma(D) = D_A$. ■

As a consequence of Proposition 2.3.1, if we consider an irreducible subvariety $T \subset \overline{M}_{0,n+1}$ which is intersection of irreducible divisors, we can give a simple rule which allows us to recover $\Gamma(T)$. Recall that, on one hand, in the language of models, the subvariety $\Gamma(T)$ is described by a nested set $S \subset \mathcal{F}_{\mathcal{A}_{n-1}} - V^*$, that is to say, $\Gamma(T) = D_S = \bigcap_{A \in S} D_A$ (here we are considering the divisors D_A with $A \neq V^*$ since $D_{V^*} = \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$). On the other hand, T can be described in $\overline{M}_{0,n+1}$ by a picture of this kind



where $a_1, \ldots, a_k$ are the irreducible components and furthermore a partition $\{A_1, \ldots, A_k\}$ of $\{1, \ldots, n\}$ is provided, such that on each component a_j the labels of the points belong to A_j.

We can pass from the tree-like description of T to the nested set S associated to $\Gamma(T)$ using the following algorithm which construct the graph of S. Before starting the algorithm, as a preliminary step, we find the curve a_j which contains the point labeled with 0 and we delete it except for its points of intersection with the other curves. Then we label all these points by 0. We can now describe recursively the algorithm which we will apply on each of the remaining connected components of the tree. Let $\mathcal{C}$ be one of this components. Suppose that is made by the curves $a_1, a_2, \ldots, a_s, \ldots, a_q$ and that $0 \in A_s = \{0, d_1, \ldots, d_t\}$. Then draw a vertex v_s with as many outgoing edges as the number of special points (labeled points plus double points) lying on a_s minus 1. These edges satisfy the following properties: $t = |A_s| - 1$ of them connect v_s with the leaves labeled by $d_1, \ldots, d_t$, while the other edges connect v_s with some new vertices v_i, for every i such that $a_i \cap a_s \neq \emptyset$. Finally, delete a_s except for its points of intersection with the other curves (if any), label these points by 0, and start again the algorithm for every remaining connected component (i.e., given a_i that intersected a_s, we start by considering the already drawn vertex v_i and draw as many outgoing vertices from it as the number of special points on a_i minus 1.... and so further). It is easy to check that what we get at the end is the Yuzvinsky graph (according to the construction described in Section 2) of the nested set S.

2.4 Combinatorics of the root arrangements of types B_n, C_n, and D_n.

Let $\mathcal{B}_n^* \subset \mathbb{C}^n$ (resp. $\mathcal{C}_n^* \subset \mathbb{C}^n$) be the root hyperplane arrangement of type B_n (resp. C_n). We observe that $\mathcal{B}_n^* = \mathcal{C}_n^*$. In fact, we recall that we can choose an orthonormal basis $\{\omega_1, \omega_2, \ldots, \omega_n\}$ of the euclidean space $\mathbb{R}^n$ such that the roots of type B_n are $\pm\omega_i$ $(i = 1, \ldots, n)$, $\omega_i - \omega_j$ $(i \neq j)$, $\pm(\omega_i + \omega_j)$ $(i < j)$. Therefore they differ only for scalar multiplication from the roots of type C_n, which are $\pm 2\omega_i$ $(i = 1, \ldots, n)$, $\omega_i - \omega_j$ $(i \neq j)$, $\pm(\omega_i + \omega_j)$ $(i < j)$. Thus the hyperplanes orthogonal to the roots coincide in these two cases; as a consequence, in the sequel we will only refer to the arrangement $\mathcal{B}_n^* \subset \mathbb{C}^n$ made by the hyperplanes of equations $x_i - x_j = 0$, $x_i + x_j = 0$ $(1 \leq i < j \leq n)$ and $x_i = 0$ $(i = 1, \ldots, n)$. Let us call by t_{ij}, t_{ij}^+ and t_i respectively the roots in $(\mathbb{C}^n)^*$orthogonal to these hypperplanes and denote by $\Phi_{\mathcal{B}_n} \subset (\mathbb{C}^n)^*$ the associated root system.

As in the case of $\mathcal{A}_n^*$, we look for a combinatorial description of the building set $\mathcal{F}_{\mathcal{B}_n}$, that is to say, a combinatorial description of the subspaces in $(\mathbb{C}^n)^*$ that are spanned by the irreducible root subsystems of $\Phi_{\mathcal{B}_n}$. Let us describe a collection of subspaces which will be proved to be irreducible; consider first the subspaces $S_I^\perp$ orthogonal to the intersections S_I of some of the hyperplanes $x_i = 0$:

$$S_I = \{p = (p_1, \ldots, p_n) \in \mathbb{C}^n \mid p_i = 0 \, \forall i \in I \subset \{1, \ldots, n\}\}$$

We can identify S_I^* with the subset $I \subset \{1, \ldots, n\}$ and we then say that I is a "strong" subset of $\{1, \ldots, n\}$.

Secondly we consider the subspaces $L^{\perp}_{A_1,A_2}$ ($A_1, A_2 \subset \{1,\ldots,n\}$) orthogonal to the subspaces L_{A_1,A_2} of $\mathbb{C}^n$ which are made by the points $p = (p_1,\ldots,p_n)$ such that $p_i = p_j$ when $(i,j) \in A_1 \times A_1$ or $(i,j) \in A_2 \times A_2$ and $p_i = -p_k$ when $(i,k) \in A_1 \times A_2$ or $(i,k) \in A_2 \times A_1$.

The subspaces $L^{\perp}_{A_1,A_2}$ can therefore be identified with the unordered pairs $(A_1,\, A_2)$ of disjoint subsets of $\{1,\ldots,n\}$ such that $|A_1 \cup A_2| \geq 2$. This is equivalent to saying that the elements $L^{\perp}_{A_1,A_2}$ are in bijective correspondence with the subsets $U \subset \{1,\ldots,n\}$ which are equipped with a (possibly trivial) partition $U = U_1 \cup U_2$ ($U_1 \cap U_2 = \emptyset$). The above mentioned subsets U will be called "weak" subsets of $\{1,\ldots,n\}$.

We note that the dimension of $S^{\perp}_I$ is equal to $|I|$ and the dimension of $L^{\perp}_{A_1,A_2}$ is equal to $|A_1 \cup A_2| - 1$.

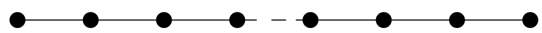
Proposition 2.4.1 *The subspaces $S^{\perp}_I$ and $L^{\perp}_{A_1,A_2}$ form the building set of irreducibles $\mathcal{F}_{\mathcal{B}_n}$.*

Proof.

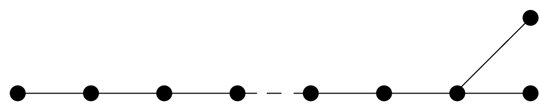
We will refer to the algorithm (and to the pictures of the Dynkin diagrams of type B_n) described in the proof of Proposition 2.2.2.

In this case, the Dynkin diagrams associated to the irreducible proper root subsystems can be of the three following types:

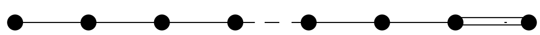
1)



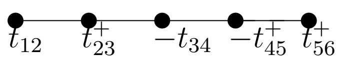
2)



3)



Now, in the first case, the vertices correspond to "long" roots, i.e., to roots of types t_{ij} and t^{+}_{ij}; this means that the associated irreducible subspace is of type $L^{\perp}_{A_1,A_2}$. As an example, let us consider the graph



It determines the subset $\{1,\ldots,6\} \subset \{1,\ldots,n\}$ equipped with the partition $\{1,2,5\} \cup \{3,4,6\}$. It is immediate to see that, changing the choices of the roots, we can obtain all the irreducibles $L^{\perp}_{A_1,A_2}$; in a completely similar (and easy) way we can check that the other two cases provide all the subspaces of type $S^{\perp}_I$. ∎

Let us now pass to the characterization of the nested sets; following [25], we observe that the nested sets in $\mathcal{F}_{\mathcal{B}_n}$ can be identified with the collections $\mathcal{S}$ of elements $S^{\perp}_I$ and $L^{\perp}_{A_1,A_2}$ which satisfy the following two conditions. First, every two sets in $\mathcal{S}$ are either disjoint or one is embedded into the other. Here the embedding among weak sets is understood as embedding of their partition also. Second, no two strong sets in $\mathcal{S}$ are incomparable, that is to say, they are totally ordered by inclusion.

We can now associate a graph (namely a forest) to every nested set in $\mathcal{F}_{B_n}$. This can be done as in the A_n case except that now the vertices of each tree are divided in two classes, weak vertices and strong vertices (the leaves will be considered as weak vertices). In each tree, for any strong vertex v, all the vertices closer to the root than v are strong and the edges among them form a topological line interval. Furthermore, a forest can have at most one connected component with strong vertices.

We note that when we associate a weak vertex to a weak subset U of $\{1,\dots,n\}$ we "forget" the partition $U = U_1 \cup U_2$. The following lemma will allow us to take into account this fact in our computations.

Lemma 2.4.2 *(see [25]) Let ς be a nested set in $\mathcal{F}_{B_n}$, and let τ be the function which associates forests to the nested sets. Let π be the number of the nested sets γ such that $\tau(\gamma) = \tau(\varsigma)$. Then we have $\pi = 2^{\sum rk(v)}$, where v runs through the closest to the roots weak vertices of $\tau(\varsigma)$ (the roots themselves included, if it is the case), and $rk(v)$ is the rank of the irreducible subspace associated to v.*

Proof
We observe that the closest to the roots weak vertices correspond to the maximal weak subsets in ς. Therefore the partitions of these sets will uniquely define partitions on all the weak sets of ς. Let us then focus on a maximal weak set S in ς: the number of unordered partitions of S with at most two parts is $2^{|S|-1}$, that is to say, $2^{rk(v_S)}$, where v_S is the vertex associated to S. Multiplying with respect to all the maximal weak sets in S we get the result. ∎

Let us now pass to the complex arrangement $\mathcal{D}^*_n$ of type D_n: it is defined in $\mathbb{C}^n$ by the hyperplanes $x_i + x_j = 0$ and $x_i - x_j = 0$ ($1 \le i < j \le n$). The building set of irreducibles $\mathcal{F}_{\mathcal{D}_n}$ can be described in the same way as $\mathcal{F}_{\mathcal{B}_n}$, with the only difference that every strong subset of $\{1,\dots,n\}$ should have at least 3 elements. In fact, referring to the picture of the affine Dynkin diagram of type D_n and to the notation introduced in the B_n case, we note that the diagrams associated to the irreducibles are of types 1) and 2) (there are not irreducibles with diagrams of type 3)). This means that we can obtain all the irreducible subspaces of type $L^{\perp}_{A_1,A_2}$ (associated to diagrams of type 1)) and all the irreducibles of type $S^{\perp}_{I}$ which have dimension strictly greater than 2 (associated to the diagrams of type 2) which in fact have at least three vertices). As an example, consider the strong subset $\{1,2\}$, corresponding to the orthogonal of the subspace $\{x_1 - x_2 = 0\} \cap \{x_1 + x_2 = 0\}$; in $\mathcal{F}_{\mathcal{D}_n}$ $S^{\perp}_{\{1,2\}}$ is not irreducible since t_{12}, t^{+}_{12} is a decomposition.

As a consequence, the forests that we associate (see [25]) to the nested sets in $\mathcal{F}_{\mathcal{D}_n}$ are of the same kind of the forests associated to the nested sets in $\mathcal{F}_{\mathcal{B}_n}$, except that any strong vertex should now be connected by directed paths to at least three leaves.

We will focus again on the combinatorics of the building sets $\mathcal{F}_{\mathcal{A}_n}$, $\mathcal{F}_{\mathcal{B}_n}$ and $\mathcal{F}_{\mathcal{D}_n}$ in the next chapters, when we will deal with the cohomology rings of the corresponding De Concini - Procesi compact models.

Chapter 3

Cohomology bases for the models and Poincarè polynomials

3.1 Bases for cohomology rings

Let $\mathcal{G}$ be a building set, and let $S \subset \mathcal{G}$ be a $\mathcal{G}$-nested set. This section is devoted to finding a $\mathbb{Z}$-basis for the integer cohomology rings $H^*(Y_\mathcal{G}, \mathbb{Z})$ and $H^*(D_S, \mathbb{Z})$. We note that we can regard $Y_\mathcal{G}$ as a variety D_S with $S = \emptyset$.

The first step is provided by the following

Definition 3.1.1 *Given $\mathcal{G}$ and S as above, a function $f : \mathcal{G} \mapsto \mathbb{N}$ is called $\mathcal{G}, S$-admissible if it is $f = 0$ or if $f \neq 0$, $supp\, f \cup S$ is $\mathcal{G}$-nested and, for every $A \in supp\, f$,*

$$f(A) < d^S_{(supp\, f)_A, A} = dim A - dim \left(\sum_{B \in (supp\, f)_A \cup S_A} B \right)$$

where $S_A = \{E \in S \ : \ E \subsetneq A\}$ and $(supp\, f)_A = \{E \in supp\, f \ : \ E \subsetneq A\}$.

Note that, since $supp\, f \cup S$ is $\mathcal{G}$-nested, $d^S_{(supp\, f)_A, A} > 0$ for every $A \in supp\, f$.

Now, given a $\mathcal{G}, S$-admissible function f, we can construct in $H^*(D_S, \mathbb{Z}) \simeq \mathbb{Z}[c_A]/I_S$ the monomial $m_f = \prod_{A \in \mathcal{G}} c_A^{f(A)}$. We will call "$\mathcal{G}, S$-admissible" such monomials.

Theorem 3.1.1 *Let $\mathcal{G}$ be a building set and let $S \subset \mathcal{G}$ be a $\mathcal{G}$-nested set. Then the set $\mathcal{B}_{\mathcal{G},S}$ of $\mathcal{G}, S$-admissible monomials is a $\mathbb{Z}$-basis for $H^*(D_S, \mathbb{Z})$.*

Proof.
First we prove that the elements in $\mathcal{B}_{\mathcal{G},S}$ span $H^*(D_S, \mathbb{Z})$.
Let $m_g = \prod_{A \in \mathcal{G}} c_A^{g(A)}$, for a certain function $g : \mathcal{G} \mapsto \mathbb{N}$, be a (non zero) monomial in $H^*(D_S, \mathbb{Z})$: because of Lemma 1.3.2, $supp\, g \cup S$ must be $\mathcal{G}$-nested. Let us now suppose that g is not $\mathcal{G}, S$-admissible, i.e. there is an $A \in supp\, g$ such that $g(A) > d^S_{(supp\, g)_A, A}$. We call such an A a "bad component" for the monomial if it is minimal with this property, and we prove the claim by reverse induction on the rank of bad components.

In fact if a bad component A of m_g is a maximal element in $\mathcal{G}$ then the polynomial $P^S_{(supp\, g)_A, A}$ divides m_g, therefore $m_g = 0$ in $H^*(D_S, \mathbb{Z})$. Otherwise, given a bad component A, we note that the polynomial

$$\left(\prod_{B \in (supp\, g)_A} c_B \right) (c_A)^{d^S_{(supp\, g)_A, A}}$$

divides m_g so, using the relation provided by $P^S_{(supp\, g)_A, A}$, (and repeating this for all the bad components) we can express m_g as sum of monomials that are in $\mathcal{B}_{\mathcal{G},S}$ or have bad components strictly greater than the ones of m_g. Therefore we can conclude using the inductive hypothesis.
It remains to prove the linear independence of monomials in $\mathcal{B}_{\mathcal{G},S}$; we will do it first in the case $S = \emptyset$ for simplicity.
Now, given a minimal element $G \in \mathcal{G}$ and keeping the same notation as in Theorem 1.1.7, we know that $Y_{\mathcal{G}}$ can be obtained by blowing up $Y_{\mathcal{G}'}$ along a subvariety isomorphic to $Y_{\overline{\mathcal{G}}}$. This implies that, calling by p the blowing up map $p : Y_{\mathcal{G}} \mapsto Y_{\mathcal{G}'}$, we have

$$H^*(Y_{\mathcal{G}}, \mathbb{Z}) \cong p^* H^*(Y_{\mathcal{G}'}, \mathbb{Z}) \oplus \left(H^*(E, \mathbb{Z}) / p^* H^*(Y_{\overline{\mathcal{G}}}, \mathbb{Z}) \right)$$

The exceptional divisor E is isomorphic to the projectivization of the normal bundle of $Y_{\overline{\mathcal{G}}}$ in $Y_{\mathcal{G}'}$. Then it is well known (see for instance [7] or [14]) that $H^*(E, \mathbb{Z})$ is generated, as $p^* H^*(Y_{\overline{\mathcal{G}}}, \mathbb{Z})$-algebra, by the Chern class $\zeta = c_1(T)$ of the tautological line bundle $T \mapsto E$. Furthermore the class ζ has in $H^*(E, \mathbb{Z})$ the unique relation provided by the Chern polynomial of the normal bundle $N_{Y_{\overline{\mathcal{G}}}/Y_{\mathcal{G}'}}$. This, if we let $\chi(Y_{\mathcal{G}})$ denote the Euler-Poincarè characteristic of $Y_{\mathcal{G}}$, allows us to write (recall that the odd degree components of $H^*(Y_{\mathcal{G}}, \mathbb{Z})$ are zero)

$$\chi(Y_{\mathcal{G}}) = \chi(Y_{\mathcal{G}'}) + (dim\, G - 1) \chi(Y_{\overline{\mathcal{G}}})$$

Therefore, in order to see that the elements in $\mathcal{B}_{\mathcal{G}}$ (here we write $\mathcal{B}_{\mathcal{G}}$ instead of $\mathcal{B}_{\mathcal{G},\emptyset}$ for brevity) give a basis, it suffices to show that

$$|\mathcal{B}_{\mathcal{G}}| = \chi(Y_{\mathcal{G}})$$

Let us proceed by induction on the cardinality of $\mathcal{G}$, the case $|\mathcal{G}| = 1$ being obvious.

Given $\mathcal{G}$ and G as before, we can divide admissible functions in two sets: $Z_1 = \{f \text{ admissible } : f(G) = 0\}$ and $Z_2 = \{f \text{ admissible } : f(G) > 0\}$.

We note that there is a bijective correspondence between M_{Z_1} , the set of monomials associated to admissible functions in Z_1, and $\mathcal{B}_{\mathcal{G}'}$, hence we have $|Z_1| = |\mathcal{B}_{\mathcal{G}'}|$.

Let us now recall that in Chapter 1 (see Theorem 1.1.7) we denoted by $\overline{\mathcal{G}}$ the building set $\{\overline{D} = (D + G)/G \ : \ D \in \mathcal{G}'\}$. If $f \in Z_2$ satisfies $f(B) > 0$ for $B \neq G$, we have that either $B \cap G = \{0\}$ or $G \subset B$. This implies that the function $\bar{f} \ : \ \overline{\mathcal{G}} \mapsto \mathbb{N}$, associated to f and constructed by putting $\bar{f}(\bar{D}) = f(D)$ if $f(D) > 0$ and 0 otherwise, is $\overline{\mathcal{G}}$-admissible. We then observe that the so established correspondence between Z_2 and the set of $\overline{\mathcal{G}}$-admissible functions is surjective and $\dim G - 1$ to 1, so

$$|Z_2| = |\mathcal{B}_{\overline{\mathcal{G}}}|(\dim G - 1)$$

We have then proved that

$$|\mathcal{B}_{\mathcal{G}}| = |Z_1| + |Z_2| = |\mathcal{B}_{\mathcal{G}'}| + |\mathcal{B}_{\overline{\mathcal{G}}}|(\dim G - 1)$$

that is to say, $|\mathcal{B}_{\mathcal{G}}|$ satisfies the same recurrence relation as $\chi(Y_{\mathcal{G}})$. Thus by induction the claim (in the case $S = \emptyset$) follows.

Let now $S \neq \emptyset$ be a $\mathcal{G}$-nested set: as before we can proceed by induction on the cardinality of $\mathcal{G}$ (the case $|\mathcal{G}| = 1$ is obvious) but we have to study separately three cases (which are essentially the same cases as in the proof of Theorem 1.3.1).

Case 1. $S \cup \{G\}$ is not $\mathcal{G}$-nested

In this case S is $\mathcal{G}'$ nested and the restriction to D_S of the natural projection $p \ : \ Y_{\mathcal{G}} \mapsto Y_{\mathcal{G}'}$ is an isomorphism onto its image, which is D'_S, the variety associated to S in $Y_{\mathcal{G}'}$. The theorem is then true, by induction, for $H^*(D'_S, \mathbb{Z})$, and, since a function $f \ : \ \mathcal{G} \mapsto \mathbb{N}$ is S-admissible if and only if *supp* $f \subset \mathcal{G}'$ and $f_{|\mathcal{G}'}$ is S-admissible, it is also true for $H^*(D_S, \mathbb{Z})$.

Case 2. $S \cup \{G\}$ is $\mathcal{G}$-nested but $G \notin S$.

In this case S is $\mathcal{G}'$-nested. Furthermore, we can consider the set $\bar{S} = \{\bar{A} \ : \ A \in S\} \subset \overline{\mathcal{G}}$ which has the same cardinality as S and turns out to be $\overline{\mathcal{G}}$-nested.

The geometric picture of Theorem 1.1.7 implies that, if D'_S is the subvariety associated to S in $Y_{\mathcal{G}'}$, then D_S can be obtained by blowing up D'_S along a subvariety isomorphic to the subvariety $D_{\bar{S}}$ in $Y_{\overline{\mathcal{G}}}$.

Now our setting is quite similar to the one of the case $S = \emptyset$ and the proof is analogous. We start by noticing that, since the codimension of $D_{\bar{S}}$ in D'_S is equal to dim G, we have

$$\chi(D_S) = \chi(D'_S) + (dim\, G - 1)\chi(D_{\bar{S}})$$

and then it suffices to show that

$$|\mathcal{B}_{\mathcal{G},S}| = \chi(D_S)$$

As before, we can divide admissible functions in two sets: $Z_{1,S} = \{f\ \mathcal{G}, S-$admissible $: f(G) = 0\}$ and $Z_{2,S} = \{f\ \mathcal{G}, S-\text{admissible} : f(G) > 0\}$.

We note that there is a bijective correspondence between $M_{Z_{1,S}}$, the set of monomials associated to admissible functions in $Z_{1,S}$, and $\mathcal{B}_{\mathcal{G}',S}$; hence $|Z_{1,S}| = |\mathcal{B}_{\mathcal{G}',S}|$.

Furthermore, if $f \in Z_{2,S}$ satisfies $f(B) > 0$ for $B \neq G$, we have that either $B \cap G = \{0\}$ or $G \subset B$. This implies that the function $\bar{f} : \overline{\mathcal{G}} \mapsto \mathbb{N}$, constructed by putting $\bar{f}(\bar{D}) = f(D)$ if $f(D) > 0$ ($\bar{D} = D + G/G$) and 0 otherwise, is $\overline{\mathcal{G}}, \bar{S}$-admissible. We then observe that the so established correspondence between $Z_{2,S}$ and the set of $\overline{\mathcal{G}}$-admissible functions is surjective and $\dim G - 1$ to 1, so

$$|Z_{2,S}| = |\mathcal{B}_{\overline{\mathcal{G}},\bar{S}}|(\dim G - 1)$$

The claim then follows by induction since we have proved that

$$|\mathcal{B}_{\mathcal{G},S}| = |Z_{1,S}| + |Z_{2,S}| = |\mathcal{B}_{\mathcal{G}',S}| + |\mathcal{B}_{\overline{\mathcal{G}},\bar{S}}|(\dim G - 1).$$

Case 3. $G \in S$.

In this case, let $\tilde{S} = S - G$. Let also $\overline{\tilde{S}} \subset \overline{\mathcal{G}}$ be the projection of $\tilde{S}$ in $\overline{\mathcal{G}}$: it turns out to be $\overline{\mathcal{G}}$-nested. In this case D_S is the exceptional divisor in $D_{\tilde{S}}$, that is to say, it is the preimage in $Y_{\mathcal{G}}$ of $D_{\overline{\tilde{S}}}$. Then it is a $\mathbf{P}^{\dim G-1}$ bundle over $D_{\overline{\tilde{S}}}$, so

$$\dim_{\mathbb{Z}} H^*(D_S, \mathbb{Z}) = (\dim_{\mathbb{C}} G)(\dim_{\mathbb{Z}} H^*(D_{\overline{\tilde{S}}}, \mathbb{Z}))$$

But now, given an S-admissible function $f : \mathcal{G} \mapsto \mathbb{N}$, we see that we can associate to it the $\overline{\tilde{S}}$-admissible function $\overline{f} : \overline{\mathcal{G}} \mapsto \mathbb{N}$ defined as follows:

$$\overline{f}(\bar{D}) = f(D) \quad \forall\ D \in \mathcal{G},\ D \neq G$$

This map is easily seen to be surjective and $(\dim G)$ to 1. Therefore

$$|\mathcal{B}_{\mathcal{G},S}| = (\dim G)|\mathcal{B}_{\overline{\mathcal{G}},\overline{\tilde{S}}}|$$

and the theorem is proved by induction.

∎

Remark. We point out that the bases $\mathcal{B}_{\mathcal{G},\emptyset}$ coincide, if $\mathcal{G} = \mathcal{F}$ is the building set of irreducibles which refines an hyperplane arrangement, with Yuzvinsky's bases (see [25]).

In the next sections we will see some remarkable examples and applications of Theorem 3.1.1.

3.2 Poincarè polynomials for hyperplane arrangements of type A_n, B_n, D_n

As a first application of the result of the preceding section we can describe explicitely, following [25], the integer bases (which will be called " Yuzvinsky bases") for the $\mathbb{Z}$-modules $H^*(Y_{\mathcal{F}}, \mathbb{Z})$ in the case when $\mathcal{F} = \mathcal{F}_{A_{n-1}}, \mathcal{F}_{B_n}, \mathcal{F}_{D_n}$ is one of the building set of irreducibles introduced in the preceding chapter.

A consequence of this explicit description is that we will be able to give, by means of a simple combinatorial argument (different from Yuzvinsky's one), formulas for the Poincarè polynomials of the varieties $\widehat{Y}_{\mathcal{F}_{A_n}}, \widehat{Y}_{\mathcal{F}_{B_n}}, \widehat{Y}_{\mathcal{F}_{D_n}}$. We observe that, in view of the remark of Chapter 1, Section 3, these polynomials will also be the Poincarè polynomials of the varieties $Y_{\mathcal{F}_{A_n}}, Y_{\mathcal{F}_{B_n}}, Y_{\mathcal{F}_{D_n}}$.

Type A_n.

Since a monomial m_f of the Yuzvinsky basis for $H^*(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$ is non zero only if $supp\, f$ is a $\mathcal{F}_{A_{n-1}}$-nested set, we will refer to the bijective correspondence between nested sets in $\mathcal{F}_{A_{n-1}}$ and forests, which was described in Chapter 2, Section 2.

Therefore the supports $supp\, f$ of the monomials m_f can be represented by forests, on n leaves, the connected components of which are rooted oriented trees. Now we can take into account the exponents which appear in m_f by adding labels to the vertices of these forests, i.e., we can associate to each vertex v_A ($A \in supp\, f$) the label $f(A)$. The leaves remain unlabeled since they do not correspond to any element in $supp\, f$.

We note that, given $A \in supp\, f$, we have $d^{\emptyset}_{(supp\, f)_A, A} = |out\, v_A| - 1$, where $|out\, v_A|$ is equal to the number of outgoing edges from v_A.

This means that the label $f(A)$ satisfies $1 \le f(A) < |out\, v_A| - 1$. Thus there is a bijective correspondence between the elements of the Yuzvinsky basis for $H^*(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$ and the forests on n numbered leaves the connected components of which are rooted, oriented labeled trees which satisfy this further condition: the label of a certain vertex v (which is not a leaf) is a positive integer $m(v)$ such that $1 \le m(v) < |out(v) - 1|$.

Finally we observe that the necessary and sufficient condition for the existence of such a labeling on a rooted oriented tree T is that $|out\, v| \ge 3$ for every vertex $v \in T$ which is not a leaf.

Type B_n.

The elements of the Yuzvinsky basis for $H^*(Y_{\mathcal{F}_{B_n}}, \mathbb{Z})$ can be put, in a similar way as above, in surjective correspondence with a family of forests (on n numbered leaves) the connected components of which are rooted, oriented trees with labeled vertices. If we forget the labels, the involved forests are the ones that we associated to the nested sets in Chapter 2, Section 4; therefore we have two classes of vertices: weak and strong. Let then m_f be a monomial of the Yuzvinsky basis for $H^*(Y_{\mathcal{F}_{B_n}}, \mathbb{Z})$. If $A \in supp\, f$ corresponds to a "strong" subset of $\{1, \dots, n\}$, the associated vertex v_A is "strong". Then $d^{\emptyset}_{(supp\, f)_A, A} = |wout\ v_A|$, where $|wout\ v_A|$ is equal to the number of weak outgoing edges from v_A. This implies that the label $f(A)$ of v_A satisfies $1 \leq f(A) < |wout\, v_A|$.

If instead $B \in supp\, f$ corresponds to a "weak" subset of $\{1, \dots, n\}$, the associated vertex v_B is "weak" and $d^{\emptyset}_{(supp\, f)_B, B} = |out\, v_B| - 1$. Thus we have $1 \leq f(B) < |out\, v_B| - 1$.

We then observe that the necessary and sufficient condition for the existence of such a labeling on the trees with weak and strong vertices is that for any weak vertex w we have $|out\, w| \geq 3$ and for any strong vertex v we have $|wout\, v| \geq 2$.

Finally we recall that (see Chapter 2, Section 4) under this surjective correspondence, the preimage of a forest τ has $2^{(\sum \mathrm{rk}(v))}$ elements, where v runs through the closest to the roots weak vertices (the roots themselves included, if it is the case) of τ, and

$$\mathrm{rk}(v) = |\{\text{leaves connected by a directed path with } v\}| - 1$$

Type D_n.

The same as for type B_n with the further condition that any strong vertex v is connected by directed paths to at least three leaves (see Chapter 2, Section 4).

We are now ready to compute formulas for the Poincarè polynomials of these reflection arrangements. Our method is different from the one in [25], since it avoids the use of the "Feynmann integral" method and it is connected with the geometric picture of blow-ups of the De Concini-Procesi models.

Type A_{n-1}.

The series we want to compute is the following

$$\Phi_A(q, t) = \Phi(q, t) = t + \sum_{n=2}^{\infty} P_{A_{n-1}}(q) \frac{t^n}{n!}$$

where $P_{A_{n-1}}(q)$ is the Poincarè polynomial of the model $Y_{\mathcal{F}_{A_{n-1}}}$ (here and from now on the variable q has degree 2).

Let us call by $\lambda_A(q, t) = \lambda(q, t)$ the contribution provided to $\Phi(q, t)$ by the elements represented by connected graphs (i.e. trees), including the degenerate graph given by a single leaf. Then we have $\Phi = e^{\lambda} - 1$ by elementary combinatorial arguments. We can thus reduce our problem to that of finding a recurrence formula for λ. As a matter of notation we agree that from now on the superscript (n) in formulas will mean "n-th derivative with respect to t".

Theorem 3.2.1 *Let λ be defined as above, then we have the following recurrence relation:*

$$\lambda^{(1)} = 1 + \frac{\lambda^{(1)}}{q-1}[e^{q\lambda} - qe^{\lambda} + q - 1] \tag{3.1}$$

Proof.

For every $n \geq 2$ we will regard the Yuzvinsky basis for $H^*(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$ as the set of marked forests described above, with the n leaves identified with the numbers from 1 to n. We will directly search for a relation for the generating function Φ instead of studying separately the polynomials $P_{A_{n-1}}(q)$.

As a first step, we single out the leaf 1, and divide the elements of the bases of the various $H^*(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$ $(n \geq 2)$ in two parts: the ones containing the leaf 1 as a singleton (called I-elements), and the ones such that an edge ends in that leaf (called II-elements).

Let us look at the contribution to Φ of I-elements. Let θ be a I-element: we can associate to it the element $\overline{\theta}$ obtained by cutting out the leaf 1. This gives a bijective correspondence between I-elements and elements of Yuzvinsky type whose graphs are constructed on the leaves associated to numbers greater than or equal to 2. Here we associate the Yuzvinsky element 1 to the degenerate graph given by the leaf 1 alone. Therefore, summing the contributions provided by the elements $\overline{\theta}$, we obtain $1 + \Phi$. By simple combinatorial arguments, we have

$$\Phi = \int (1 + \Phi) dt + \text{contribution of II-elements}$$

Let us now work on II-elements: given a II-element ϱ, let us consider its associated graph and let us call by E the edge the end of which is the leaf 1. We will call "singular" the vertex from which E stems.

Therefore, given ϱ as above, we can construct the following two new Yuzvinsky-type elements, ϱ' and ϱ'': the graph of ϱ' is obtained from the one of ϱ by collapsing to the singular vertex v, which becomes a leaf, the subtree ρ_v that stems out of v. The graph of ϱ'' is ρ_v (we note that we are considering $v \in \rho_v$).

We observe that a II-element can be uniquely determined by giving its associated couple (ϱ' , ϱ'').

Therefore, in order to obtain the contribution to Φ of II-elements, we must multiply the series originated respectively by elements of type ϱ' and ϱ''. The second one is easily shown to be

$$\sum_{r \geq 1} \frac{q^r - q}{q-1} \frac{\lambda^r}{r!}$$

In fact the number of edges that go out from the singular one is $r + 1$, if r is the number of connected components obtained from the graph of ϱ'' when we cut the (closed) edge E connecting the leaf 1 and the singular vertex. Note that we are giving the value “t” to every leaf except for leaf 1.

The contribution to the series due to the elements of type ϱ' is $\Phi^{(1)}$. In this case the first derivative is needed since the elements of type ϱ' have an artificial leaf (the singular vertex). Summing up we have:

$$\Phi = \int (1+\Phi)dt + \int \Phi^{(1)} \left(\sum_{r\geq 1} \frac{q^r - q}{q-1} \frac{\lambda^r}{r!} \right) dt$$

that is to say:

$$\Phi^{(1)} = (1+\Phi) + \Phi^{(1)} \left(\sum_{r\geq 1} \frac{q^r - q}{q-1} \frac{\lambda^r}{r!} \right)$$

From this formula, since $\Phi = e^{\lambda} - 1$, we get:

$$\lambda^{(1)} e^{\lambda} = e^{\lambda} + \frac{\lambda^{(1)} e^{\lambda}}{q-1} \left(e^{q\lambda} - q e^{\lambda} + q - 1 \right)$$

$$\lambda^{(1)} = 1 + \frac{\lambda^{(1)}}{q-1} \left[e^{q\lambda} - q e^{\lambda} + q - 1 \right]$$

which is a recurrence relation since the series in brackets satisfies:

$$\left[e^{q\lambda} - q e^{\lambda} + q - 1 \right] = 0 + 0t + (q^2 - q)t^2 + \cdots$$

■

We note that formula (3.1) is equivalent to the one found by Yuzvinsky in [25] since it can be obtained from it by differentiating with respect to t. Furthermore we remark that it was also found, in the context of moduli spaces, by several authors (see for instance [9], [13], [20]).

Type B_n.

Here the Poincarè series which we are interested in is

$$\Phi_B(q,t) = t + \sum_{n=2}^{\infty} P_{B_n}(q) \frac{t^n}{2^n\, n!}$$

where $P_{B_n}(q)$ is the Poincarè polynomial of the model $Y_{\mathcal{F}_{B_n}}$ and $2^n\, n!$ is the order of the Weyl group of type B_n.

Let us call λ_B and μ_B the contribution provided to Φ_B respectively by trees with only weak vertices and by trees containing strong vertices. Then

$$\Phi_B = e^{\lambda_B}(\mu_B + 1) - 1$$

and we can reduce ourselves to find formulas for λ_B and μ_B. As for λ_B, we note that, according to the description of the forests, $\lambda_B = \frac{1}{2}\lambda_A = \frac{1}{2}\lambda$, therefore, from Theorem 3.2.1 we deduce the formula

$$2\lambda_B^{(1)} = 1 + \frac{2\lambda_B^{(1)}}{q-1}[e^{2q\lambda_B} - qe^{2\lambda_B} + q - 1]$$

In order to compute μ_B, we first observe (see [25]) that, if we call by γ_B the contribution to μ_B provided by trees with only one strong vertex (i.e. with a strong root) then $\mu_B = \frac{1}{1-\gamma_B} - 1$.

So we only have to give a formula for γ_B, which is, because of the above description of the marked trees:

$$\gamma_B = \sum_{r\geq 2} \frac{q^r - q}{q-1}\frac{\lambda_B^r}{r!} = \frac{e^{q\lambda_B} - qe^{\lambda_B}}{q-1} + 1$$

At the end, we can write the following formula for Φ_B in terms of λ_B:

$$\Phi_B = \frac{e^{q\lambda_B} - e^{\lambda_B}}{qe^{\lambda_B} - e^{q\lambda_B}}$$

Type D_n.

Simple combinatorial reasons allow us to recover the Poincarè polynomial $P_{D_n}(q)$ from $P_{B_n}(q)$ by means of the following formula:

$$P_{D_n}(q) = P_{B_n}(q) - \left(\begin{array}{c} n \\ 2 \end{array} \right) qP_{B_{n-2}}(q)$$

3.3 Induced subspace arrangements

In this section we will deal with a class of interesting subspace arrangements, namely the "induced subspace arrangements". They can be constructed by observing that the tensor product provides us a way to get some new subspace arrangements $\mathcal{G}_h^*$ starting from a given arrangement $\mathcal{G}^*$. As an application of Theorem 3.1.1 we will explicitly describe the bases for the cohomology rings of the De Concini-Procesi models of $\mathcal{G}_h^*$ when the starting arrangement $\mathcal{G}^*$ corresponds to the building set of irreducibles associated to root arrangements. In these cases we will also provide formulas for the Poincarè polynomials of the models.

Definition 3.3.1 *Let $\mathcal{G}^*$ be a subspace arrangement in $\mathbb{C}^n$ such that $\mathcal{G}$ is building. We will call "induced by $\mathcal{G}^*$" the subspace arrangement $\mathcal{G}_h^*$ in $\mathbb{C}^n \otimes \mathbb{C}^h$ given by the subspaces $A \otimes \mathbb{C}^h$, as A varies in $\mathcal{G}^*$.*

First we note that, since the subspaces of $\mathcal{G}_h^*$ are in bijective correspondence with the subspaces of $\mathcal{G}^*$, we can give a bijective correspondence between $\mathcal{G}_h$, the set of subspaces in $(\mathbb{C}^n \otimes \mathbb{C}^h)^*$ orthogonal to the ones in $\mathcal{G}_h^*$, and $\mathcal{G}$. This correspondence is given as follows: if $B \in \mathcal{G}$, we associate to it $B \otimes (\mathbb{C}^h)^*$, which is included in $(\mathbb{C}^n \otimes \mathbb{C}^h)^*$ via the isomorphism $(\mathbb{C}^n)^* \otimes (\mathbb{C}^h)^* \cong (\mathbb{C}^n \otimes \mathbb{C}^h)^*$ and belongs to $\mathcal{G}_h$. Note that this construction implies that $\mathcal{G}_h$ is building.
Therefore we can express a $\mathcal{G}_h, \emptyset$-admissible function $\tilde{f} : \mathcal{G}_h \mapsto \mathbb{N}$ as a function $f : \mathcal{G} \mapsto \mathbb{N}$, which turns out to have $\mathcal{G}$-nested support and that satisfies the following relation:

$$0 < f(B) < h\, d^{\emptyset}_{(supp\, f)_B, B} = d^{\emptyset}_{(supp\, \tilde{f})_{B \otimes \left(\mathbb{C}^h\right)^*}, B \otimes \left(\mathbb{C}^h\right)^*}$$

if $B \in supp\, f$.

Let us now consider a building set $\mathcal{G}$ and a $\mathcal{G}$-nested set S, and describe a way to associate an oriented labeled graph to every element of the basis $\mathcal{B}_{\mathcal{G},S}$ (note that the following is the obvious generalization of the Yuzvinsky graphs that we used in the cases of root arrangements).

Definition 3.3.2 *Given a monomial $m_f \in \mathcal{B}_{\mathcal{G},S}$, we call by G_f the oriented labeled graph whose vertices are identified with the elements of $supp\, f$ and that is constructed in the following way. Let A and B be two elements in $supp\, f$ such that A is a maximal element (with respect to inclusion) in $(supp\, f)_B$; then we draw an edge which joins the vertices A and B and is oriented from B to A. Furthermore, if C belongs to $supp\, f$, the vertex C is labeled with $f(C)$.*

Then, in terms of the above defined graphs, we have:

Proposition 3.3.1 *The graphs which represent the monomial basis for $H^*(Y_{\mathcal{G}}, \mathbb{Z})$ and $H^*(Y_{\mathcal{G}_h}, \mathbb{Z})$ have the same shape but the upper bounds for the labels in the $\mathcal{G}_h$ case are obtained by multiplying by h the upper bounds in the $\mathcal{G}$ case.*

This allows us to provide a generalization of the computations in Section 2 of the present chapter; let us in fact consider the "induced root arrangements" $\mathcal{F}_h$, where $\mathcal{F} = \mathcal{F}_{A_n}$, $\mathcal{F}_{B_n}$, $\mathcal{F}_{D_n}$ is as in the preceding sections. Using Proposition 3.3.1 we will manage to extend to the induced case the formulas for Poincarè polynomials.
Case A_n.
Let us call by $\mathcal{F}_{A_n,h}$ the arrangement induced by $\mathcal{F}_{A_n}$ and let the series we want to compute be

$$\Phi_{A,h} = \Phi_{A,h}(q,t,h) = t + \sum_{n=2}^{\infty} P_{A_{n-1},h}(q) \frac{t^n}{n!}$$

where $P_{A_{n-1},h}(q)$ is the Poincarè polynomial of the model $Y_{\mathcal{F}_{A_{n-1},h}}$.

As before, we set $\lambda_{A,h}$ to be the contribution to $\Phi_{A,h}$ provided by trees, so $e^{\lambda_{A,h}} - 1 = \Phi_{A,h}$. The same considerations as in Theorem 3.2.1 lead us to the formula

$$\Phi^{(1)}_{A,h} = (1 + \Phi_{A,h}) + \Phi^{(1)}_{A,h} \left(\sum_{r \geq 1} \frac{q^{hr} - q}{q-1} \frac{\lambda^r_{A,h}}{r!} \right)$$

that gives

$$\lambda^{(1)}_{A,h} = 1 + \frac{\lambda^{(1)}_{A,h}}{q-1} [e^{q^h \lambda_{A,h}} - q e^{\lambda_{A,h}} + q - 1]$$

which is a recurrence relation.

Case B_n.

Let us call by $\mathcal{F}_{B_n,h}$ the arrangement induced by $\mathcal{F}_{B_n}$and let us consider the series

$$\Phi_{B,h} = \Phi_{B,h}(q,t,h) = t + \sum_{n=2}^{\infty} P_{B_n,h}(q) \frac{t^n}{2^n\, n!}$$

where $P_{B_n,h}(q)$ is the Poincarè polynomial of the model $Y_{\mathcal{F}_{B_n,h}}$. Let $\lambda_{B,h}$ be the contribution to $\Phi_{B,h}$ provided by trees, and let $\mu_{B,h}$, $\gamma_{B,h}$ be respectively the contributions provided by the trees with strong vertices and by the trees with only one strong vertex.

As in Section 2, we have $\Phi_{B,h} = e^{\lambda_{B,h}}(\mu_{B,h} + 1) - 1$, $\mu_{B,h} = \dfrac{1}{1 - \gamma_{B,h}} - 1$ and $2\lambda_{B,h} = \lambda_{A,h}$ since all these facts depend on the shape and not on the marking of graphs.

So, we only need to give a formula for $\gamma_{B,h}$; the same reasoning as in Section 2 shows that

$$\gamma_{B,h} = \sum_{r \geq 2} \frac{q^{hr} - q}{q-1} \frac{\lambda^r_{B,h}}{r!} = \frac{e^{q^h \lambda_{B,h}} - q e^{\lambda_{B,h}} - (q^h - q)\lambda_{B,h}}{q-1}$$

Case D_n.

Here the relation among the graphs associated to the basis in the cases B_n and D_n gives the following formula for the Poincarè polynomial $P_{D_n,h}$:

$$P_{D_n,h} = P_{B_n,h} - \binom{n}{2} \frac{q^{2h} - q}{q-1} P_{B_{n-2},h}$$

3.4 Geometric bases for hyperplane arrangements

Let $\mathcal{G}$ be a building set, S a $\mathcal{G}$-nested set and let us consider the variety D_S. As we have seen, the elements of the basis of $H^*(D_S, \mathbb{Z})$ can be represented as monomials in a quotient of $\mathbb{Z}[c_A]$ $(A \in \mathcal{G})$, where the variables c_A correspond to the basic cohomology classes $[D_A]$ restricted to D_S (see Theorem 1.3.1).

This implies that there is a natural correspondence between the monomials of the basis of $H^*(D_S, \mathbb{Z})$ and the varieties which are intersection of the divisors in the boundary of $Y_{\mathcal{G}}$: this correspondence associates to the monomial m_f the irreducible variety $D_{supp\, f}$.

Since the monomials of the cohomology bases we studied are not necessarily squarefree, the above correspondence is not one-to-one.

In this section, in the case when $\mathcal{G}$ refines a hyperplane arrangement, we will exhibit squarefree bases for $H^*(D_S, \mathbb{Z})$, that is to say, bases the elements of which correspond to irreducible subvarieties of D_S obtained by intersecting divisors without multiplicities. We note that if $\mathcal{G}$ does not refine a hyperplane arrangement, squarefree bases may not exist.
Let then $\mathcal{G}$ be a refinement of a hyperplane arrangement, and let $x_1, \ldots, x_N \in (\mathbb{C}^n)^*$ be representatives for the lines orthogonal to the hyperplanes. Then $x_1, \ldots, x_N$ belong to $\mathcal{G}$ and generate $\mathcal{C}_{\mathcal{G}}$. We put $X = \{x_1, \ldots, x_N\}$, $< X >= \sum_{j=1}^{N} \mathbb{C} x_j$ and $m = dim < X >$.

We want to give a suitable total order in $\mathcal{G}$. Actually there are many possible ways to choose this order, therefore we fix our ideas by choosing the following one, which we will refer to in the example at the end of this section. As a first step, we associate to each element $G \in \mathcal{G}$ the monomial $x_{j_1} \cdots x_{j_k}$ $(j_1 < \ldots < j_k)$ obtained by selecting in X the elements belonging to G. Then we order $\mathcal{G}$ according to the following rules. Given any two (non constant) monomials A and B, we put $A < B$ if either of the following cases occurs:

1. B divides A, or

2. neither A divides B nor B divides A but, setting

$$A' = \frac{A}{gcd\,(A, B)} \qquad B' = \frac{B}{gcd\,(A, B)}$$

   and writing $A = x_{i_1} \cdots x_{i_k}$ $(i_1 < \ldots < i_k)$, $B = x_{r_1} \cdots x_{r_s}$ $(r_1 < \ldots < r_s)$, we have $i_1 < r_1$.

Now we need the following lemma:

Lemma 3.4.1 *Let $\Gamma \subset \mathcal{G}$ be a maximal $\mathcal{G}$-nested set.*
Then $|\Gamma| = m = dim < X >$.

Proof.
The proof is by induction on $|\mathcal{G} - \mathcal{F}_{\mathcal{G}}|$.
If $\mathcal{G} = \mathcal{F}_{\mathcal{G}}$, our claim coincides with Proposition 1.1 (2) in [5]. Let now $|\mathcal{G} - \mathcal{F}_{\mathcal{G}}| > 0$ and let G be minimal in $\mathcal{G} - \mathcal{F}_{\mathcal{G}}$. Then $\mathcal{G}' = \mathcal{G} - G$ is building and is still associated to the hyperplane arrangement X.

According to Proposition 2.5 in [4], we have that a maximal $\mathcal{G}$-nested set is either a maximal $\mathcal{G}'$-nested set or is obtained from a maximal $\mathcal{G}'$-nested set replacing one of its elements with G.

In both cases, the cardinality of the maximal $\mathcal{G}$-nested set is the same as the cardinality of some maximal $\mathcal{G}'$-nested set, which by induction is m.

■

In the case of $H^*(D_S, \mathbb{Z})$, because of the characterization of the supports of $\mathcal{G}, S$-admissible functions, we are interested in subsets $\Gamma \subset \mathcal{G}$ such that $\Gamma \cup S$ is $\mathcal{G}$-nested.

Lemma 3.4.2 *Let $\Omega \subset \mathcal{G}$ be a maximal element in $\mathcal{M} = \{\Gamma \subset \mathcal{G} \;:\; \Gamma \cup S \;\; is \;\; \mathcal{G} - nested\}$. Then $|\Omega| = m$.*

Proof.
This immediately follows from the above lemma since Ω must be a maximal $\mathcal{G}$ nested set containing S. ■

Let us keep for the monomials in $H^*(D_S, \mathbb{Z})$ the same notation as in Section 1. Since a squarefree monomial m_f is completely determined by $supp\, f$, we can describe a squarefree basis for $H^*(D_S, \mathbb{Z})$ by means of subsets of $\mathcal{G}$. Let us then focus on the following algorithm which produces suitable subsets of $\mathcal{G}$. Take the support of a $\mathcal{G}, S$-admissible function g; it already defines a monomial in our squarefree basis. Now suppose that there is an $A \in supp\, g$ such that $h = d^S_{(supp\, g)_A, A} > 2$ and let $A_1, \ldots, A_k$ be all the maximal elements (if any) in $(supp\, g)_A$.

There is at least one maximal $\mathcal{G}$-nested set which includes $S \cup supp\, g$. Since the cardinality of such a set is m, we deduce that there exists at least one element $B \in \mathcal{G}$ satisfying the following conditions:

1. $\{B\} \cup S \cup supp\, g$ is $\mathcal{G}$-nested;

2. $B \subset A$ and, for every A_j, either $A_j \cap B = \{0\}$ or $A_j \subset B$;

3. $\dim B = \dim (\sum_{j \; s.t. \; A_j \subset B} A_j) + 1$.

Remark. Note that the above conditions make sense also in the case when $(supp\, g)_A = \emptyset$.

We choose, among all such elements, the minimal one with respect to the total order in $\mathcal{G}$. We call this element “the exceptional element of g at A” and denote it by A^e. Then we define the function $\tilde{g}$, taking values 0,1, such that $supp\, \tilde{g} = supp\, g \cup A^e$: $\tilde{g}$ defines a monomial in our squarefree basis.

Now, if there are elements $C \in supp\, \tilde{g}$ such that $d^S_{(supp\, \tilde{g})_C, C} > 2$, we can apply again the above algorithm and so further. We call by $\mathcal{SB}_{\mathcal{G},S}$ the set of monomials which can be constructed by means of the above algorithm.

Theorem 3.4.3 *The set $\mathcal{SB}_{\mathcal{G},S}$ gives a squarefree basis of $H^*(D_S, \mathbb{Z})$.*

Proof.

We observe that $|\mathcal{B}_{\mathcal{G},S}| = |\mathcal{SB}_{\mathcal{G},S}|$ by construction, so it is sufficient to prove that every element in $\mathcal{B}_{\mathcal{G},S}$ can be expressed as a $\mathbb{Z}$-linear combination of monomials in $\mathcal{SB}_{\mathcal{G},S}$.

Given a monomial $m_f \in \mathcal{B}_{\mathcal{G},S}$, we can consider the oriented labeled graph G_f associated to it (see Definition 3.3.2) and its subgraph G_f^1 obtained as follows: consider the leaves of G_f which are labeled with 1 and take all the paths in G_f that end with one of these leaves and that are made by vertices labeled with 1.

Since G_f^1 is a forest (maybe empty), we can consider the roots $C_1, \dots C_r$ of its connected components and give the following definitions.

Definition 3.4.1
The "squarefree rank" of m_f is the number $sr(f) = dim\,(\sum_{i=1}^r C_i)$.

We note that the sum of subspaces in the definition is direct, since $supp\, f \cup S$ has to be $\mathcal{G}$-nested.

Definition 3.4.2 *Given an oriented marked forest G and a vertex B, we will call B-subtree of G the tree obtained by considering the subgraph of G that stems out of B, with the same marking except that the vertex B (i.e. the root) is considered unmarked.*

Now, given a monomial $m_f \in \mathcal{B}_{\mathcal{G},S}$ that is not squarefree, there necessarily exists at least an element $A \in supp\, f$ such that $1 < f(A) < d^S_{(supp\, f)_A, A}$.

Let us take all the minimal (with respect to inclusion) elements $M_1, \dots M_t$ with this property. We will prove the theorem by proving, by reverse induction on the squarefree rank of monomials in $\mathcal{B}_{\mathcal{G},S}$, the following proposition:
a) Each element m_f of $\mathcal{B}_{\mathcal{G},S}$ can be expressed as a $\mathbb{Z}$-linear combination of monomials m_θ in $\mathcal{SB}_{\mathcal{G},S}$ s.t. G_f^1 is a subgraph of $supp\,\theta$.
The first step consists in observing that if, for a certain $m_f \in \mathcal{B}_{\mathcal{G},S}$, $sr(f)$ is maximal, then m_f is squarefree, hence it already belongs to $\mathcal{SB}_{\mathcal{G},S}$.

Let us then take a monomial $m_g \in \mathcal{B}_{\mathcal{G},S}$ that is not squarefree and with the squarefree rank equal to p. By induction, we assume that proposition *a)* is true for each monomial in $\mathcal{B}_{\mathcal{G},S}$ with the squarefree rank greater than p.

There must exist an element $M_1 = A \in supp\, g$ such that $1 < g(A) < d^S_{(supp\, g)_A, A}$ and A is minimal with respect to inclusion.

Keeping the same notation as in the construction of $\mathcal{SB}_{\mathcal{G},S}$, we call by $A_1, \dots, A_k$ the maximal elements (if any) in $(supp\, g)_A$ and consider A^e and $\tilde{g}$.

Let $Q = \prod_{L \in (supp\, \tilde{g})_{A^e}} c_L$. We can now apply the following "squarefree algorithm at A" : we use the polynomial

$$P^S_{(supp\, \tilde{g})_{A^e}, A^e} = Q(\sum_{A^e \subset E} c_E)$$

which belongs to I_S, to substitute in m_g the factor $c_A Q$.

We thus get a $\mathbb{Z}$-linear combination of monomials that are either zero or can be of the following three types, which we write in a more general form:

1. monomials obtained by elements m_f of $\mathcal{B}_{\mathcal{G},S}$ such that $A \in supp\, f$, G^1_f includes G^1_g as a subgraph and the A-subtree of G_f coincides with the A-subtree of G_g, by substituting $c_A Q$ with $-c_N Q$, where $A \subsetneq N$.

2. monomials obtained by elements m_f of $\mathcal{B}_{\mathcal{G},S}$ such that $A \in supp\, f$, G^1_f includes G^1_g as a subgraph and the A-subtree of G_f coincides with the A-subtree of G_g, by substituting $c_A Q$ with $-c_T Q$, where $A^e \subsetneq T \subsetneq A$.

3. monomials obtained by elements m_f of $\mathcal{B}_{\mathcal{G},S}$ such that $A \in supp\, f$, G^1_f includes G^1_g as a subgraph and the A-subtree of G_f coincides with the A-subtree of G_g, by substituting $c_A Q$ with $-c_{A^e} Q$.

Let us take a monomial m_h of type 1. In the proof of the Theorem 3.1.1, we described an algorithm which allowed us to show that every monomial in $H^*(D_S, \mathbb{Z})$ can be expressed as an integer linear combination of monomials in $\mathcal{B}_{\mathcal{G},S}$. The same algorithm now allows us to express m_h as an integer linear combination of monomials $m_\tau \in \mathcal{B}_{\mathcal{G},S}$ such that $A \in supp\, \tau$, G^1_τ includes G^1_g as a subgraph and the A-subtree of G_τ coincides with the A-subtree of G_g.

Furthermore, $\tau(A) = g(A) - 1$. If $\tau(A) = 1$, we have that $sr(\tau) > sr(g)$, therefore we treat m_τ by induction and we manage to write it as an integer linear combination of monomials in $\mathcal{SB}_{\mathcal{G},S}$ which satisfy the conditions of proposition *a)*; otherwise we can apply again to m_τ the squarefree algorithm at A, getting elements of type 2,3 or elements m_θ of type 1 with $\theta(A) = \tau(A) - 1$. Therefore, after a finite number of steps, we are reduced to study elements of types 2 and 3.

Given an element m_ν of type 2 we note that, using the algorithm of Theorem 3.1.1, it can be written as an integer linear combination of monomials $m_\varrho \in \mathcal{B}_{\mathcal{G},S}$ s.t. G^1_ϱ includes $G^1_g \cup T$. We note that the monomials m_ϱ belong to $\mathcal{B}_{\mathcal{G},S}$ since it must be $dim\, T > dim(\sum_{j\, s.t.\, A_j \subset T} A_j) + 1$ because of the choice of A^e.

Thus we have $sr(\varrho) > sr(g)$ and we can conclude by induction.

Let now m_γ be a monomial of type 3 and let us suppose that m_γ does not belong to $\mathcal{SB}_{\mathcal{G},S}$.

If $\gamma(A)=1$, then let us consider the monomial $m_{\hat{\gamma}}=\dfrac{m_\gamma}{c_{A^e}}$. We note that $m_{\hat{\gamma}}$ is in $\mathcal{B}_{\mathcal{G},S}$ and that $sr(\hat{\gamma})>sr(g)$. Applying induction, we can write $m_{\hat{\gamma}}$ as an integer linear combination of monomials $m_{\hat{\zeta}}\in\mathcal{SB}_{\mathcal{G},S}$ such that their supports contain $G_g^1\cup A$ as a subgraph. If we multiply this linear combination by c_{A^e}, we find that m_γ is equal to an integer linear combination of monomials $m_{\hat{\zeta}}c_{A^e}$ which still belong to $\mathcal{SB}_{\mathcal{G},S}$.

If instead $\gamma(A)>1$, then we can apply again to $m_{\hat{\zeta}}$ the "squarefree algorithm at A". This means that, calling by A^{ee} the exceptional element of γ at A and $\widetilde{\gamma}$ the 0,1-valued function such that $supp\,\widetilde{\gamma}=supp\,\gamma\cup A^{ee}$, we use the polynomial

$$P^S_{(supp\,\widetilde{\gamma})_{A^{ee}},A^{ee}}$$

to substitute in m_γ the factor $c_A \prod_{L\in(supp\,\widetilde{\gamma})_{A^{ee}}} c_L$.

In the same way as before we divide the resulting monomials in three classes and we call monomials "of type j" the monomials in the j-th class ($j=1,2,3$). As before, the monomials of type 1 can be expressed as integer linear combinations of monomials of types 2 and 3 and of monomials in $\mathcal{SB}_{\mathcal{G},S}$, while the monomials of type 2 are easily treated by induction.

It there remains to study the monomials of type 3. Let m_ε be such a monomial: we have then $\varepsilon(A)<\gamma(A)$ and, if $\varepsilon(A)=1$, we can apply induction to $\dfrac{m_\varepsilon}{c_{A^{ee}}\,c_{A^e}}$. Otherwise we can use again the squarefree algorithm at A and continue. This process of course ends after $g(A)-1$ steps, and at last we will have written m_g as an integer linear combination of monomials in $\mathcal{SB}_{\mathcal{G},S}$ which satisfy condition *a)*. This concludes our proof.

■

Example

Let us consider $\mathcal{G}=\mathcal{F}_{\mathcal{A}_{n-1}}$, the building set of irreducibles which refines the reflection arrangement of type A_{n-1}. We keep the notation of Chapter 2, Section 2, so we denote by t_{hk} $(h,k=1,\ldots,n)$ the representatives of the functionals orthogonal to the reflecting hyperplanes.

We can give a total order on the set $\{t_{hk}\}$ by the following rule: $t_{ij}<t_{hk}$ if $i<h$ or if $i=h$ and $j<k$.

This provides us, according to the rules mentioned at the beginning of this section, a total order on $H^*(Y_{\mathcal{F}_{\mathcal{A}_{n-1}}},\mathbb{Z})$.

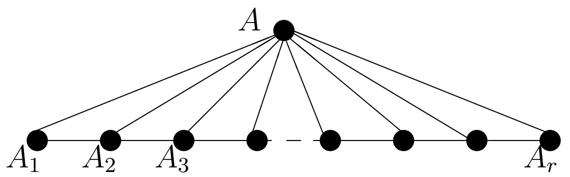
Now let us take a $\mathcal{G},\emptyset$-admissible monomial m_g and let A, A_1, $\ldots$, A_k be the same as in the definition of the squarefree basis. We will show in this example how to obtain the exceptional element A_1^e.

First, let us call by $A_{k+1},\ldots,A_r$ the leaves which are connected by an edge with A in the Yuzvinsky graph (see Chapter 2, Section 2) of the nested set $(supp\,g)_A$. Furthermore, let Γ, $\Gamma_1,\ldots,\Gamma_k$ be the subsets of $\{1,\ldots,n\}$

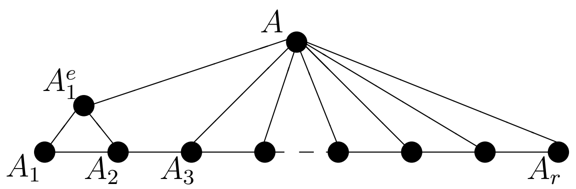
associated to A, $A_1 \ldots A_k$ respectively and, if $t > k$, let $\Gamma_t = \{a_t\}$, where a_t is the label of the leaf A_t.

It is easily seen that the order we have given in $H^*(Y_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ implies that $A_i < A_j$ if and only if $min\,\Gamma_i < min\,\Gamma_j$ $(i, j \leq k)$. We use the above relation to extend this order also to the leaves $A_{k+1}, \ldots, A_r$.

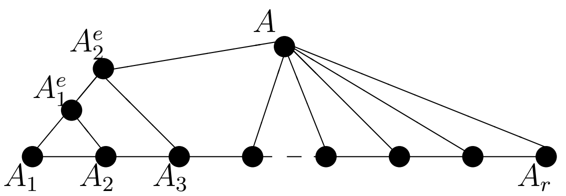
We can now reorder (if necessary) the A_j's in such a way that $A_1 < A_2 < \ldots < A_r$. The corresponding graph is



Then the construction of the exceptional element immediately reveals that A_1^e is the subspace generated by functionals $x_i - x_j$ for $i, j \in \Gamma_1 \cup \Gamma_2$. This gives rise to the following graph:



The next step is

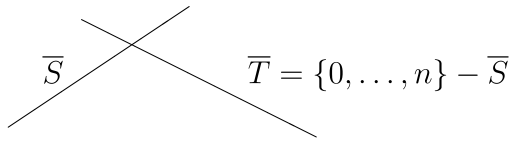


Then we can add a vertex A_3^e connected by two edges with A_2^e and A_4 and we can go on until we draw the vertex A_{r-3}^e connected by two edges to A_{r-4}^e and A_{r-2}.

3.5 A squarefree basis made by Keel generators of $H^*(\overline{M}_{0,n+1}, \mathbb{Z})$.

In this section we want to compare explicitly the description of the cohomology ring $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ (arising from the theory of models of arrangements) with the presentation of $H^*(\overline{M}_{0,n+1}, \mathbb{Z})$ due to Keel (see [16]). Let us first recall Keel's results and notation. Given a divisor

$D =$



in $\overline{M}_{0,n+1}$, its fundamental class in $H^2(\overline{M}_{0,n+1}, \mathbb{Z})$ is denoted by $\left[D^S\right]$. Then we have:

Theorem 3.5.1 *(see [16])*

$$H^*(\overline{M}_{0,n+1}, \mathbb{Z}) = \mathbb{Z}\left[\left[D^S\right] \mid \overline{S} \subset \{0, \ldots, n\}, \quad 2 \leq |\overline{S}| \leq n-1\right] /K$$

where K is the ideal generated by the following relations:

1. $\left[D^S\right] = \left[D^T\right]$, *where* $\overline{T} = \{0, \ldots, n\} - \overline{S}$.
2. *For any four distinct elements* $i, j, k, l \in \{0, \ldots, n\}$*:*

$$\sum_{\substack{i,j \in \overline{S} \\ k,l \notin \overline{S}}} \left[D^S\right] = \sum_{\substack{i,k \in \overline{S} \\ j,l \notin \overline{S}}} \left[D^S\right] = \sum_{\substack{i,l \in \overline{S} \\ j,k \notin \overline{S}}} \left[D^S\right]$$

3. $\left[D^S\right]\left[D^T\right]$ *unless* $\overline{S} \cap \overline{T} = \emptyset$ *or* $\overline{S} \cap \overline{T} = \overline{S}$ *or* $\overline{S} \cap \overline{T} = \overline{T}$.

Since Keel's presentation is in terms of fundamental classes of divisors, we can immediately write the explicit isomorphism between the rings $\mathbb{Z}[c_A]/I$ ($\cong H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$) and $\mathbb{Z}\left[\left[D^S\right]\right]/K$ ($\cong H^*(\overline{M}_{0,n+1}, \mathbb{Z})$). In fact this isomorphism is provided by the map $\Gamma^* : \mathbb{Z}[c_A]/I \mapsto \mathbb{Z}\left[\left[D^S\right]\right]/K$, where $\Gamma : \overline{M}_{0,n+1} \mapsto \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is the isomorphism described in Chapter 2, Section 2. Therefore, as a consequence of Proposition 2.3.1, given $\overline{A} \subsetneq \{1, \ldots, n\}$ and considering the class c_A associated in $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ to the divisor D_A, we have that $\Gamma^*(c_A) = \left[D^A\right]$. In order to describe $\Gamma^*(c_U)$ (where we put $\overline{U} = \{1, \ldots, n\}$) in terms of Keel generators, we can use the relations in $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$

$$r_{ij} : \sum_{\{i,j\} \subset \overline{A} \subset \{1,\ldots,n\}} c_A = 0$$

(in the notations of Chapter 1, calling by T_{ij} the irreducible associated to $\{i, j\}$, this is equivalent to saying that the polynomial $P^{\emptyset}_{\emptyset, T_{ij}}$ belongs to $I_\emptyset$).

Hence we can write

$$\Gamma^*(c_U) = -\sum_{\{i,j\}\subset \overline{A}\subsetneq\{1,\dots,n\}} \left[D^A\right]$$

for any choice of the subset $\{i,j\} \subset \{1,\dots,n\}$.

The isomorphism Γ^* allows us to provide a $\mathbb{Z}$-basis of $H^*(\overline{M}_{0,n+1},\mathbb{Z})$ in terms of Keel generators simply by considering the basis $\mathcal{B}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ provided by the Theorem 3.1.1 and taking $\Gamma^*(\mathcal{B}_{\mathcal{F}_{\mathcal{A}_{n-1}},\emptyset})$. We note that this basis is not made by monomials because of $\Gamma^*(c_U)$. Anyway monomial bases in terms of Keel generators can easily be deduced. For example one (which will be denoted by $\mathcal{SB}_{\overline{M}_{0,n+1}}$) is provided by the following construction.

Let us start from the squarefree basis $\mathcal{SB}_{\mathcal{F}_{\mathcal{A}_{n-1}},\emptyset}$ (which is referred to the order chosen in the example at the end of Section 4) and consider $m \in \mathcal{SB}_{\mathcal{F}_{\mathcal{A}_{n-1}},\emptyset}$. If c_U does not divide m, then we take $\Gamma^*(m)$ as an element of the basis $\mathcal{SB}_{\overline{M}_{0,n+1}}$. If instead c_U divides m then we find the exceptional element U^e of m at U (we are using the notations of Section 4) and we take $\Gamma^*(\frac{mc_{U^e}}{c_U})$ as an element of $\mathcal{SB}_{\overline{M}_{0,n+1}}$.

Proposition 3.5.2 *The set $\mathcal{SB}_{\overline{M}_{0,n+1}}$ is a squarefree monomial $\mathbb{Z}$-basis (expressed in terms of Keel's generators) of $H^*(\overline{M}_{0,n+1},\mathbb{Z})$.*

Proof.

Let us prove the equivalent claim that $(\Gamma^*)^{-1}(\mathcal{SB}_{\overline{M}_{0,n+1}})$ is a basis for $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}},\mathbb{Z})$. Since the cardinality of $\mathcal{SB}_{\overline{M}_{0,n+1}}$ is equal to the one of $\mathcal{SB}_{\mathcal{F}_{\mathcal{A}_{n-1}},\emptyset}$ by construction, it suffices to prove that the elements of $\mathcal{SB}_{\mathcal{F}_{\mathcal{A}_{n-1}},\emptyset}$ are integer linear combinations of the elements in $(\Gamma^*)^{-1}(\mathcal{SB}_{\overline{M}_{0,n+1}})$. The only non trivial case we have to check is provided by the elements $\mu \in \mathcal{SB}_{\mathcal{F}_{\mathcal{A}_{n-1}},\emptyset}$ that are divided by c_U. Now in $(\Gamma^*)^{-1}(\mathcal{SB}_{\overline{M}_{0,n+1}})$ there is the monomial $m = \frac{\mu c_{U^e}}{c_U}$. Substituting in m the variable c_{U^e} by means of the polynomial $P^{\emptyset}_{(supp\,\mu)_{U^e},U^e}$, we can write $-m$ as a sum of $d^{\emptyset}_{(supp\,\mu)_U,U^e}-1$ terms all of which belong to $\mathcal{SB}_{\mathcal{F}_{\mathcal{A}_{n-1}},\emptyset}$. Furthermore we note that one of these terms is μ itself while all the other terms are not divisible by c_U and belong to $(\Gamma^*)^{-1}(\mathcal{SB}_{\overline{M}_{0,n+1}})$. Then we have found an expression for μ as $\mathbb{Z}$-linear combination (the coefficients are equal to -1) of monomials in $(\Gamma^*)^{-1}(\mathcal{SB}_{\overline{M}_{0,n+1}})$. ■

Remark.
At the same way we can find "squarefree monomial $\mathbb{Z}$-bases " (in terms of Keel's generators) for the integer cohomology of all the subvarieties of $\overline{M}_{0,n+1}$ which are intersection of irreducible boundary divisors. In these cases we consider the nested set $S \subset \mathcal{F}_{\mathcal{A}_{n-1}} - V^*$ which identify such a subvariety and start the construction from the squarefree basis $\mathcal{SB}_{\mathcal{F}_{\mathcal{A}_{n-1}},S}$ of $H^*(D_S,\mathbb{Z})$.

Chapter 4

Symmetric group representations

In this chapter we will deal with the root hyperplane arrangement $\mathcal{A}_n^*$ of type A_n and with two symmetric group actions on the cohomology ring $H^*(\mathcal{M}_{\mathcal{A}_n}, \mathbb{C})$: the action of the Weyl group $\mathcal{S}_{n+1}$ and an extended $\mathcal{S}_{n+2}$ action coming from the isomorphism between $Y_{\mathcal{F}_{\mathcal{A}_n}}$ and $\overline{M}_{0,n+2}$. Our main result consists in proving (see Theorem 4.3.1) a recursive relation between the characters of these two representations. Furthermore, an explicit description of the $\mathcal{S}_{n+2}$ module $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ is provided.

4.1 Projection maps and cohomology

First we recall some notation: in Chapter 2, Section 2, we introduced the base $\{t_{12}, \dots, t_{1n}\}$ of V^*, and we called by $\{v_2, \dots, v_n\}$ its dual base in V. In what follows we will always consider V equipped with the basis $\{v_2, \dots, v_n\}$.

Let now π be the Hopf bundle projection

$$\pi : \quad V - \{0\} \mapsto \mathbf{P}_{\mathbb{C}}^{n-2}$$

with fiber $\mathbb{C}^*$, which identifies $z \in V$ with λz for $\lambda \in \mathbb{C}^*$; the restriction π' of π to $\mathcal{M}_{\mathcal{A}_{n-1}}$ maps $\mathcal{M}_{\mathcal{A}_{n-1}}$ onto $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \cong \mathcal{M}_{\mathcal{A}_{n-1}}/\mathbb{C}^*$ and $\pi' : \mathcal{M}_{\mathcal{A}_{n-1}} \mapsto \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ is a trivial bundle. We are interested in finding some relations between the $\mathcal{S}_n$ actions on the cohomology rings of $\mathcal{M}_{\mathcal{A}_{n-1}}$ and $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$: since in general there are no $\mathcal{S}_n$-equivariant sections of the given bundle, we are going to study an $\mathcal{S}_n$-equivariant covering map

$$\gamma : \quad \mathcal{M}_{\mathcal{A}_{n-1}} \mapsto \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \times \mathbb{C}^*$$

that will provide us an $\mathcal{S}_n$-isomorphism involving cohomology rings.

Let's define γ in the following way: given $p \in \mathcal{M}_{\mathcal{A}_{n-1}}$, we put

$$\gamma(p) = (\overline{\pi}(p)\,, Q^2(p))$$

where, if $p_\beta = 0$ is the equation of the hyperplane H_β, Q is the polynomial given by $Q = \prod p_\beta$, ($\beta \in \Phi^+$, the set of positive roots). Clearly $deg\, Q^2 = |\Phi| = (n-1)n$.

We note that if $\gamma(p) = \gamma(q)$, for $p, q \in \mathcal{M}_{\mathcal{A}_{n-1}}$, then $\overline{\pi}(p) = \overline{\pi}(q)$, i.e. $p = zq$ with $z \in \mathbb{C}^*$.

Furthermore, since $Q^2(p) = z^{|\Phi|}Q^2(q)$, we have that $z \in \Gamma$, the cyclic group of $|\Phi|$-th roots of 1.

It follows that γ provides a $|\Phi|$-sheets covering; this implies that

$$\Gamma \setminus \mathcal{M}_{\mathcal{A}_{n-1}} \cong \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \times \mathbb{C}^* \quad \text{via } \gamma.$$

The role that γ plays in cohomology depends on the following well known fact:

Lemma 4.1.1 *Let X be a variety and G a finite group which acts on X. Then*

$$H^*(G \setminus X, \mathbb{C}) \cong (H^*(X, \mathbb{C}))^G$$

In our case the action of Γ in cohomology is trivial since $\Gamma \subset \mathbf{S}^1$ and the action of a connected continuous group in cohomology is trivial. So we deduce:

Proposition 4.1.2

$$H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C}) \cong H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C}) \otimes (\frac{\mathbb{C}[\varepsilon]}{\varepsilon^2})$$

as $\mathcal{S}_n$-modules, where the action of $\mathcal{S}_n$ on $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$ is the natural one, the action of $\mathcal{S}_n$ on $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ is the one obtained from the $\mathcal{S}_n$ action induced on $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ from $\mathcal{M}_{\mathcal{A}_{n-1}}$, $deg\, \varepsilon = 1$ and $w\varepsilon = \varepsilon$ for every $w \in \mathcal{S}_n$.

Proof
This follows from Lemma 4.1.1, the observation above and the fact that γ is a $\mathcal{S}_n$-spaces map , when we consider $\mathcal{S}_n$ acting on the base space $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ with the action induced from $\mathcal{M}_{\mathcal{A}_{n-1}}$ via π'.
In fact, if $w \in \mathcal{S}_n$ and $p \in \mathcal{M}_{\mathcal{A}_{n-1}}$, $w\overline{\pi}(p) = \overline{\pi}(wp)$ by definition of the involved actions and $Q^2(wp) = Q^2(p)$ since w permutes the hyperplanes of $\mathcal{A}_{n-1}$. ■

We are now ready to extend the $\mathcal{S}_n$ action on $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$ to an $\mathcal{S}_{n+1}$ one, starting from an $\mathcal{S}_{n+1}$ action on $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ that we will introduce in the next section.

4.2 The $\mathcal{S}_{n+1}$ action on $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$

Let us consider, for $n \geq 2$, the moduli space $M_{0,n+1}$ of $n+1$-pointed curves of genus 0. We recall that

$$M_{0,n+1} = SL(2) \setminus \left\{ (p_0, \dots, p_n) \in \underbrace{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}_{n+1 \ times} \mid p_i \neq p_j \ \forall\, i \neq j \right\}$$

and that, via the $SL(2)$ action, we can write every element $p \in M_{0,n+1}$ as

$$p = [(0,1),\ (1,0),\ (1,1),\ (x_1, y_1), \dots,\ (x_{n-2}, y_{n-2})]$$

In Chapter 2, Section 2, we called by $\widehat{M}_{0,n+1}$ the set of the elements of $M_{0,n+1}$ written in canonical way:

$$\widehat{M}_{0,n+1} = \left\{ (q_1, \dots, q_{n-2}) \in \underbrace{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}_{n-2 \ times} \mid q_i \neq q_j,\ q_i \neq 1, 0, \infty \right\}$$

We defined, in terms of the basis $\{v_2, \dots, v_n\}$ of V, the map $\phi : \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \mapsto \widehat{M}_{0,n+1}$ as:

$$\phi(\gamma_1, \dots, \gamma_{n-1}) = ((\gamma_1, \gamma_2), (\gamma_1, \gamma_3), \dots, (\gamma_1, \gamma_{n-1}))$$

and we observed (see Theorem 2.2.1) that ϕ is an isomorphism between $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $M_{0,n+1}$.

Now the main remark is that on $M_{0,n+1}$ there is a natural $\mathcal{S}_{n+1}$ action, given by the permutation of the coordinates, and that, since this action commutes with the $SL(2)$ action, we can view it as an action on $\widehat{M}_{0,n+1}$.

As a consequence, we can lift this action, via the map ϕ, to the complement $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$.

We want to show a compatibility relation between this lifted action and the $\mathcal{S}_n$ action induced on $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ via the fiber projection π'.

Theorem 4.2.1 *If one identifies $\mathcal{S}_n \subset \mathcal{S}_{n+1} =< s_0, s_1, \dots, s_{n-1} >$ with the subgroup generated by $< s_1, \dots, s_{n-1} >$, then the action of $\mathcal{S}_{n+1}$ on $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ via ϕ^{-1} and the $\mathcal{S}_n$ action induced on $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ via the fiber projection π' are compatible.*

Proof
It's sufficient to check the statement for simple reflections.
Let us take the element $(1, \gamma_2, \dots, \gamma_{n-1}) \in \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and start from the reflection s_1.
On one hand, considering s_1 as an element of the Weyl group $\mathcal{S}_n$ acting via π', we have:

$$s_1\,(1, \gamma_2, \dots, \gamma_{n-1}) = (1, 1-\gamma_2, \dots, 1-\gamma_{n-1})$$

On the other hand, thinking of s_1 as an element of $\mathcal{S}_{n+1}$ acting on $M_{0,n+1}$:

$$s_1\,[(0,1),\ (1,0),\ (1,1),\ (1,\gamma_2),\ldots,\ (1,\gamma_{n-1})] =$$

$$= [(0,1),\ (1,1),\ (1,0),\ (1,\gamma_2),\ldots,\ (1,\gamma_{n-1})]$$

which, modulo $SL(2)$ acting by $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ can be written in canonical form:

$$= [(0,1),\ (1,0),\ (1,1),\ (1,1-\gamma_2),\ldots,\ (1,1-\gamma_{n-1})]$$

After translating in $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ via ϕ^{-1} we get the desired result. As for s_2, on one hand we have

$$s_2\,(1,\gamma_2,\ldots,\gamma_{n-1}) = (\gamma_2,1,\gamma_3,\ldots,\gamma_{n-1})$$

On the other hand, thinking of s_2 as an element of $\mathcal{S}_{n+1}$:

$$s_2\,[(0,1),\ (1,0),\ (1,1),\ (1,\gamma_2),\ldots,\ (1,\gamma_{n-1})] =$$

$$= [(0,1),\ (1,0),\ (1,\gamma_2),\ (1,1),(1,\gamma_3),\ldots,\ (1,\gamma_{n-1})]$$

Via the action of $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma_2} \end{pmatrix}$ we can write:

$$= \left[(0,1),\ (1,0),\ (1,1),\ (1,\frac{1}{\gamma_2}),\ldots,\ (1,\frac{\gamma_{n-1}}{\gamma_2})\right]$$

At the end we get:

$$s_2\,(1,\gamma_2,\ldots,\gamma_{n-1}) = (\gamma_2,1,\gamma_3,\ldots,\gamma_{n-1})$$

Finally, for s_j $(j \geq 3)$ we have

$$s_j\,(1,\gamma_2,\ldots,\gamma_{n-1}) = (1,\ldots,\gamma_j,\gamma_{j-1},\ldots,\gamma_{n-1})$$

therefore the claim follows noticing that

$$s_j\,[(0,1),\ (1,0),\ (1,1),\ (1,\gamma_2),\ldots,\ (1,\gamma_{n-1})] =$$

$$= [(0,1),\ (1,0),\ (1,1),\ (1,\gamma_2),\ldots,\ (1,\gamma_j),\ (1,\gamma_{j-1}),\ldots,\ (1,\gamma_{n-1})]$$

$\blacksquare$

Let us now focus on cohomology rings. Let $\mathbf{S}$ be the symmetric algebra of V^* and let $\mathbf{F}$ be the quotient field of $\mathbf{S}$. Let $\Omega(V)$ be the exterior algebra of the $\mathbf{F}$-vector space $\mathbf{F} \otimes_{\mathbb{C}} V^*$.
We recall the following well known results (see [1], [2], [22], [23]):

Theorem 4.2.2 *The cohomology ring $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$ is isomorphic to the $\mathbb{C}$ algebra $\mathcal{R}(\mathcal{A}_{n-1}) \subset \Omega(V)$ generated by 1 and by the forms $\omega_\beta = \frac{dp_\beta}{p_\beta}$, where $p_\beta = 0$ is the equation of the hyperplane H_β ($\beta \in \Phi^+$).*

The relations involving the ω_β's are the following ones:

$$\omega_\beta \omega_\delta = -\omega_\delta \omega_\beta \quad \beta, \delta \in \Phi^+ \tag{4.1}$$

$$\omega_\beta \omega_\varrho = \omega_\beta \omega_{\varrho-\beta} - \omega_\varrho \omega_{\varrho-\beta} \quad \textit{if } \varrho - \beta \textit{ is a positive root} \tag{4.2}$$

Theorem 4.2.3 *The cohomology ring $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ is isomorphic to the subalgebra $\widehat{\mathcal{R}}(\mathcal{A}_{n-1})$ of $\mathcal{R}(\mathcal{A}_{n-1})$ generated by 1 and elements $\theta_\beta = \omega_\beta - \omega_{\alpha_1}$ for $\beta \neq \alpha_1$.*

From now on we will identify the rings $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ and $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$ with the algebras $\widehat{\mathcal{R}}(\mathcal{A}_{n-1})$ and $\mathcal{R}(\mathcal{A}_{n-1})$ respectively. Let us put, for $\beta \in \Phi^+$, $\mathcal{L}(\beta) = j$ if j is the greatest index for which $a_j \neq 0$ in the expression $\beta = \sum_{i=1}^{n-1} a_i \alpha_i$. Then the preceding theorem and relations (4.1) and (4.2) can easily provide us a base of $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$.

Corollary 4.2.4 *A basis of $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ is given by the elements of type*

$$\theta_{\beta_1} \cdots \theta_{\beta_r} \qquad r = 1, \ldots, n-1$$

with the property that $\mathcal{L}(\beta_1) > \mathcal{L}(\beta_2) \ldots > \mathcal{L}(\beta_r)$.

The action of $\mathcal{S}_{n+1}$ we have constructed gives rise to a linear representation of $\mathcal{S}_{n+1}$ on $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$; this allows us to state the following theorem:

Theorem 4.2.5 *The action of $\mathcal{S}_{n+1}$ on $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ can be extended, via the isomorphism of Proposition 4.1.2, to an $\mathcal{S}_{n+1}$ action on $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$. This action is compatible with the natural $\mathcal{S}_n$ action on $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$ if we identify $\mathcal{S}_n \subset \mathcal{S}_{n+1} = < s_0, s_1, \ldots, s_{n-1} >$ with the subgroup $< s_1, \ldots, s_{n-1} >$.*

Proof.
Let

$$\lambda : H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C}) \otimes (\frac{\mathbb{C}[\varepsilon]}{\varepsilon^2}) \mapsto H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$$

be the $\mathcal{S}_n$-modules isomorphism of Proposition 4.1.2. Since we already have an s_0 action that makes $H^*(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ an $\mathcal{S}_{n+1}$-module, it's sufficient to let s_0 act trivially on $\frac{\mathbb{C}[\varepsilon]}{\varepsilon^2}$ (i.e. to put $s_0 \varepsilon = \varepsilon$) and then to extend to $H^*(\mathcal{M}_{\mathcal{A}_{n-1}}, \mathbb{C})$ via λ. ∎

4.3 A recursive formula for representations

In the preceding sections we have shown how to construct an action of $\mathcal{S}_{n+2}$ on $H^*(\mathcal{M}_{\mathcal{A}_n}, \mathbb{C})$ for every $n \geq 1$.
Now we can prove a remarkable recursive relation which connects the $\mathcal{S}_{n+1}$ and $\mathcal{S}_{n+2}$ actions on the cohomology ring. We express it in terms of characters, introducing the following notation: let $\chi_{n+1}(i,n)$ be the character of the natural $\mathcal{S}_{n+1}$ action on $H^i(\mathcal{M}_{\mathcal{A}_n}, \mathbb{C})$, $\chi_{n+2}(i,n)$ be the character of the extended $\mathcal{S}_{n+2}$ action on $H^i(\mathcal{M}_{\mathcal{A}_n}, \mathbb{C})$ and let p_{n+1} be the character of the standard representation of $\mathcal{S}_{n+1}$.

Theorem 4.3.1 *For $n \geq 2$ we have:*

$$\chi_{n+1}(i,n) = \chi_{n+1}(i,n-1) + p_{n+1}\chi_{n+1}(i-1,n-1)$$

Proof.
Thanks to Theorem 4.2.5, it suffices to prove the analogue of the statement for the cohomology rings $H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$: we will denote by the superscript "$\widehat{\ }$ "the corresponding characters .

Let us now focus on the map $\eta : \widehat{\mathcal{M}}_{\mathcal{A}_n} \mapsto \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$, given by omitting the last coordinate, and the correspondent injective map in cohomology

$$\eta^* : H^i(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C}) \mapsto H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$$

It turns out by construction that if we consider the extended actions of $\mathcal{S}_{n+1} =< s_0, s_1, \ldots, s_{n-1} >$ on $H^i(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C})$ and of $\mathcal{S}_{n+2} =< s_0, s_1, \ldots, s_n >$ on $H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$, the map η^* is $< s_0, s_1, \ldots, s_{n-1} >$-equivariant.

Let us call by $\Omega_n^i \subset H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$ the image of η^*. Keeping the notation introduced in Corollary (4.2.4), we note that Ω_n^i is the $\mathbb{C}$-subalgebra generated by elements θ_β with $\beta \in \Phi^+$ ($\beta \neq \alpha_1$) and $\mathcal{L}(\beta) < n$.
Now we have, by Corollary 4.2.4, that

$$H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C}) = \Omega_n^i \oplus (N \cdot \Omega_n^{i-1}) \tag{4.3}$$

as $\mathbb{C}$-vector spaces, where $N = \bigoplus_{i=0}^{n-1} \mathbb{C}\,\theta_{\alpha_n+\cdots+\alpha_{n-i}}$ and" $\cdot$ " is the product.
Here $\Omega_n^{-1} = \{0\}$ and $\Omega_n^0 = \mathbb{C}$. For $i = 1$, this means that

$$H^1(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C}) = \Omega_n^1 \oplus N.$$

But we can also write:

$$H^1(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C}) = \Omega_n^1 \oplus T \tag{4.4}$$

where T is a $< s_0, \ldots, s_{n-1} >$-invariant complement of Ω_n^1.
Now, if we look at $H^1(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$ as $< s_1, \ldots, s_{n-1} >$-module, we see that $N \cong P_n \oplus I_n$ where we used the following notation: for every $n \geq 2$, I_n is a one dimensional space affording the trivial representation of $\mathcal{S}_n$ and P_n is a $n-1$-dimensional space affording the standard representation of $\mathcal{S}_n$.
It follows that $T \cong P_n \oplus I_n$ as $< s_1, \ldots, s_{n-1} >$-module, so, by the branching rule, we can conclude that $T \cong P_{n+1}$ as $< s_0, s_1, \ldots, s_{n-1} >$-module. We need now the following:

Lemma 4.3.2

$$H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C}) = \Omega_n^i \oplus (T \cdot \Omega_n^{i-1})$$

as $< s_0, s_1, \ldots, s_{n-1} >$*-modules.*

Proof of the Lemma.
Let us show that

$$H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C}) = \Omega_n^i + (T \cdot \Omega_n^{i-1}) \tag{4.5}$$

Let $z = \mu_0^i + \sum_{j=0}^{n-1} \theta_{\alpha_n + \cdots + \alpha_{n-j}} \mu_j^{i-1}$ be an element of $H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$, with $\mu_0^i \in \Omega_n^i$ and $\mu_j^{i-1} \in \Omega_n^{i-1}$ (here we are using equation (4.3)). Now we can write $\theta_{\alpha_n + \cdots + \alpha_{n-j}} = \gamma_{1j} + \gamma_j$ where $\gamma_{1j} \in \Omega_n^1$, $\gamma_j \in T$, because of (4.4). Then

$$z = \mu_0^i + \left(\sum_{j=0}^{n-1} \gamma_{1j} \mu_j^{i-1} \right) + \left(\sum_{j=0}^{n-1} \gamma_j \mu_j^{i-1} \right)$$

Since the second term on the right belongs to Ω_n^i and the third term is in $T \cdot \Omega_n^{i-1}$, this proves (4.5). Then a simple dimension argument shows that the sum in (4.5) is direct. But the spaces involved are $< s_0, \ldots, s_{n-1} >$-invariant, so this direct sum is a direct sum of $< s_0, \ldots, s_{n-1} >$-modules. ■

We note that, since $\dim(T \cdot \Omega_n^{i-1}) = \dim(T) \dim(\Omega_n^{i-1})$ by Corollary (4.2.4), there is an obvious $< s_0, \ldots, s_{n-1} >$-isomorphism between $T \cdot \Omega_n^{i-1}$ and $T \otimes \Omega_n^{i-1}$. Hence it follows, recalling that $\Omega_n^i = \eta^*(H^i(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}, \mathbb{C}))$ and denoting by $\widehat{\chi}_{n+1,0}(i,n)$ the character of the $< s_0, \ldots, s_{n-1} >$ action on $H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$, that:

$$\widehat{\chi}_{n+1,0}(i,n) = \widehat{\chi}_{n+1}(i, n-1) + p_{n+1} \widehat{\chi}_{n+1}(i-1, n-1)$$

Since $< s_0, \ldots, s_{n-1} >$ is conjugate to $< s_1, \ldots, s_n >$ in $\mathcal{S}_{n+2}$, then $H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$ viewed as $< s_0, \ldots, s_{n-1} >$-module is isomorphic to $H^i(\widehat{\mathcal{M}}_{\mathcal{A}_n}, \mathbb{C})$ viewed as $< s_1, \ldots, s_n >$-module, i.e. as natural $\mathcal{S}_{n+1}$-module. So we get the desired:

$$\widehat{\chi}_{n+1}(i,n) = \widehat{\chi}_{n+1}(i, n-1) + p_{n+1} \widehat{\chi}_{n+1}(i-1, n-1)$$

■

As a first corollary we can show a nice relation involving characters.

Corollary 4.3.3 *For $n \geq 2$ we have:*

$$\chi_{n+1}(n,n) - p_{n+1}\chi_{n+1}(n-1,n) + p_{n+1}^2\chi_{n+1}(n-2,n)+$$

$$+\cdots(-1)^n p_{n+1}^n \chi_{n+1}(0,n) = 0$$

Proof.
The recursive formula of Theorem 4.3.1 gives, for the top degree n:

$$\chi_{n+1}(n,n) = \chi_{n+1}(n,n-1) + p_{n+1}\chi_{n+1}(n-1,n-1)$$

But $\chi_{n+1}(n,n-1) = 0$ being $n-1$ the top degree of the graded ring $H^i(\mathcal{M}_{\mathcal{A}_{n-1}},\mathbb{C})$. Thus we get:

$$\chi_{n+1}(n,n) = p_{n+1}\chi_{n+1}(n-1,n-1) \tag{4.6}$$

But we also have, again applying Theorem 4.3.1:

$$\chi_{n+1}(n-1,n) = \chi_{n+1}(n-1,n-1) + p_{n+1}\chi_{n+1}(n-2,n-1) \tag{4.7}$$

Substituting in (4.6) we obtain:

$$\chi_{n+1}(n,n) = p_{n+1}\chi_{n+1}(n-1,n) - p_{n+1}^2\chi_{n+1}(n-2,n-1)$$

and, inductively, we prove our claim since both $\chi_{n+1}(0,n)$ and $\chi_{n+1}(0,n-1)$ are the trivial character. ■

As another consequence of Theorem 4.3.1 we can give a quick proof of the following theorem, due to Lehrer and Solomon (see [19]):

Theorem 4.3.4 *Let χ_n be the character of the $\mathcal{S}_n$ action on $H^*(\mathcal{M}_{\mathcal{A}_{n-1}},\mathbb{C})$. Then, for $n \geq 2$ we have:*

$$\chi_n = 2\, Ind_{\mathcal{S}_2}^{\mathcal{S}_n}(1)$$

Proof.
Let us prove the claim by induction on n, the case $n = 2$ being obvious.
First of all we note, applying Theorem 4.3.1, that for $n \geq 3$

$$\chi_n = (1+p_n)\widetilde{\chi}_n \tag{4.8}$$

$\widetilde{\chi}_n$ being the character of the extended $\mathcal{S}_n$ action on $H^*(\mathcal{M}_{\mathcal{A}_{n-2}},\mathbb{C})$.
Next we need to recall the following (well known) fact:

Lemma 4.3.5 *Let G be a group acting on the $\mathbb{C}$-vector space M and let $H \subset G$ be a subgroup. We have the G-modules isomorphism:*

$$Ind_H^G(M') \cong Ind_H^G(\mathbb{C}) \otimes_{\mathbb{C}} M$$

where M' is M considered as H-module and the action on $\mathbb{C}$ is the trivial one.

Proof of the Lemma.

Let $\mathbb{C}[G]$ and $\mathbb{C}[H]$ be the group algebras of G and H; we can write:

$$Ind_H^G(M') = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} M' \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (\mathbb{C} \otimes_{\mathbb{C}} M) \cong$$

$$\cong (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}) \otimes_{\mathbb{C}} M = Ind_H^G(\mathbb{C}) \otimes_{\mathbb{C}} M$$

■

The above lemma, applied to the $\mathcal{S}_n$-module $H^*(\mathcal{M}_{\mathcal{A}_{n-2}}, \mathbb{C})$ shows that, in terms of characters:

$$Ind_{\mathcal{S}_{n-1}}^{\mathcal{S}_n}(\chi_{n-1}) = Ind_{\mathcal{S}_{n-1}}^{\mathcal{S}_n}(1)\, \widetilde{\chi}_n \tag{4.9}$$

But $Ind_{\mathcal{S}_{n-1}}^{\mathcal{S}_n}(1) = (1+p_n)$, so, comparing (4.8) and (4.9), we get for $n \geq 3$

$$Ind_{\mathcal{S}_{n-1}}^{\mathcal{S}_n}(\chi_{n-1}) = \chi_n$$

that, by induction, gives:

$$\chi_n = Ind_{\mathcal{S}_{n-1}}^{\mathcal{S}_n}(2\, Ind_{\mathcal{S}_2}^{\mathcal{S}_{n-1}}(1)) = 2\, Ind_{\mathcal{S}_2}^{\mathcal{S}_n}(1)$$

■

Remark.

These results are also true, with the same proof, when the characteristic of the coefficient field we are dealing with and $|\Phi|$, the order of the cyclic group Γ, are coprime.

The key point is Lemma 4.1.1, who fails to be true if the order of the group G and the characteristic of the coefficient field are not coprime.

We notice that the extended symmetric group action and the recursive relation of Theorem 4.3.1 have been independently studied by Mathieu in [21], by different methods.

4.4 The Euler characteristic of $\widehat{\mathcal{M}}_{\mathcal{A}_n}/\mathcal{S}_j$

In this section, as an application of the Theorems 4.3.1 and 4.3.4, we will compute the Euler characteristic $\chi(\widehat{\mathcal{M}}_{\mathcal{A}_n}/\mathcal{S}_j)$ of the quotient spaces $\widehat{\mathcal{M}}_{\mathcal{A}_n}/\mathcal{S}_j$ ($1 \leq j \leq n+2$, that is to say, $\mathcal{S}_j$ is identified with a subgroup of $\mathcal{S}_{n+2}$ and we denote by $\mathcal{S}_1$ the trivial group made by the identity). The interest of this computation lies in the fact that $\chi(\widehat{\mathcal{M}}_{\mathcal{A}_n}/\mathcal{S}_j)$ (which, by the isomorphism described in Chapter 2, is equal to $\chi(M_{0,n+2}/\mathcal{S}_j)$) plays a crucial role in the computation of the Euler characteristic of the moduli spaces $M_{1,n}$, $M_{2,n}$ of n-pointed curves of genus 1 and 2 and of their compactifications $\overline{M}_{1,n}$, $\overline{M}_{2,n}$, as it is shown in [3].

Obviously, we start by recalling the Euler characteristic of $M_{0,n}$. This is provided by the following

Theorem 4.4.1 *For $n \geq 3$*

$$\chi(M_{0,n}) = (-1)^{n-3}(n-3)! \tag{4.10}$$

Proof.

Consider the fibration $\pi : M_{0,n+1} \to M_{0,n}$ with fiber $\mathbf{P}^1 - \{n \text{ points}\}$. This gives the recursive formula

$$\chi(M_{0,n+1}) = (2-n)\chi(M_{0,n}) \tag{4.11}$$

with initial data $\chi(M_{0,3}) = 1$. ■

Let us now notice that when $n-j \geq 3$ the quotient map $q : M_{0,n} \to M_{0,n}/S_j$ is unramified, since any automorphism of $\mathbf{P}^1$ fixing three or more points is the identity. This implies that

$$\chi(M_{0,n}/S_j) = \frac{\chi(M_{0,n})}{j!} = \frac{(-1)^{n-3}(n-3)!}{j!} \tag{4.12}$$

In the case when $n-j < 3$ it is convenient to proceed in a different way, using the results of the preceding section. Generalizing the notation introduced there, let us call by $\chi_j(i,n)$ (resp. $\chi_j(n)$) the character of the action of the symmetric group $\mathcal{S}_j$ on $H^i(\mathcal{M}_{\mathcal{A}_n}, \mathbb{Q})$ (resp. $H^*(\mathcal{M}_{\mathcal{A}_n}, \mathbb{Q})$). Furthermore, we denote by p_n and I_n respectively the characters of the standard and trivial representations of $\mathcal{S}_n$, and by $(\ ,\ )_{\mathcal{S}_j}$ the inner product in the space of class functions on S_j (in the sequel we may omit the subscript $\mathcal{S}_j$ if it is clear to which symmetric group we are referring to).

Lemma 4.4.2 *For $n \geq 3$,*

$$(\chi_n(n-2), I_n)_{\mathcal{S}_n} = 1 \tag{4.13}$$

$$(\chi_{n-1}(n-2), I_{n-1})_{\mathcal{S}_{n-1}} = 1 \tag{4.14}$$

$$(\chi_{n-2}(n-2), I_{n-2})_{\mathcal{S}_{n-2}} = n-2. \tag{4.15}$$

Proof.

This is a consequence of Theorem 4.3.4. In fact we can write

$$(\chi_{n-1}(n-2), I_{n-1})_{\mathcal{S}_{n-1}} = (Ind_{\mathcal{S}_2}^{\mathcal{S}_{n-1}}(I_2), I_{n-1})_{\mathcal{S}_{n-1}}$$

which, by Frobenius reciprocity law, is equal to

$$(I_2, Res_{\mathcal{S}_2}^{\mathcal{S}_{n-1}}(I_{n-1}))_{\mathcal{S}_2} = (I_2, I_2)_{\mathcal{S}_2} = 1$$

This gives relation (4.14). As for relation (4.13) we note that $Res_{\mathcal{S}_{n-1}}^{\mathcal{S}_n}(\chi_n(n-2))$ is equal to $\chi_{n-1}(n-2)$ and therefore

$$1 = (\chi_{n-1}(n-2), I_{n-1})_{\mathcal{S}_{n-1}} =$$

$$= (\chi_n(n-2), Ind_{\mathcal{S}_{n-1}}^{\mathcal{S}_n}(I_{n-1}))_{\mathcal{S}_n} = (\chi_n(n-2), I_n + p_n)_{\mathcal{S}_n}$$

Since dim $H^0(M_{0,n}, \mathbb{Q}) = 1$ we know that $(\chi_n(n-2), I_n)_{\mathcal{S}_n} \geq 1$. This implies that $(\chi_n(n-2), I_n)_{\mathcal{S}_n} = 1$ and $(\chi_n(n-2), p_n)_{\mathcal{S}_n} = 0$.

It remains to prove the last assertion, which can be formulated as

$$(Res_{\mathcal{S}_{n-2}}^{\mathcal{S}_{n-1}}(\chi_{n-1}(n-2)), I_{n-2})_{\mathcal{S}_{n-2}} = n-2.$$

Then we can write

$$(Res_{\mathcal{S}_{n-2}}^{\mathcal{S}_{n-1}}(\chi_{n-1}(n-2)), I_{n-2})_{\mathcal{S}_{n-2}} = (\chi_{n-1}(n-2), I_{n-1} + p_{n-1})_{\mathcal{S}_{n-1}} =$$

$$= 1 + (\chi_{n-1}(n-2), p_{n-1})_{\mathcal{S}_{n-1}}$$

which, applying Theorem 4.3.4, is equal to

$$1 + (I_2, Res_{\mathcal{S}_2}^{\mathcal{S}_{n-1}}(p_{n-1}))_{\mathcal{S}_2}.$$

The second addendum can be easily computed using the branching rule and is equal to $n-3$. This completes the proof. ∎

We are now ready to compute Euler characteristics.

Theorem 4.4.3 *The Euler characteristic of $M_{0,n}/\mathcal{S}_n$ and of $M_{0,n}/\mathcal{S}_{n-1}$ is equal to 1 for every $n \geq 3$.*

The Euler characteristic of $M_{0,n}/\mathcal{S}_{n-2}$ is equal to 0 if n is even and equal to 1 is n is odd.

Proof.

The assertions concerning $M_{0,n}/\mathcal{S}_n$ and $M_{0,n}/\mathcal{S}_{n-1}$ are immediate consequences of Theorem 4.1.1 and Lemma 4.4.2, since $H^0(M_{0,n}/\mathcal{S}_n, \mathbb{Q})$ and $H^0(M_{0,n}/\mathcal{S}_{n-1}, \mathbb{Q})$ are one dimensional spaces which afford the trivial representations I_n and I_{n-1} respectively.

Turning to $M_{0,n}/\mathcal{S}_{n-2}$, we want to prove by induction on n the following stronger proposition which implies our claim:

Proposition 4.4.4 *For every $n \geq 3$ and $0 \leq i \leq n-3$, in the irreducible decomposition of the $\mathcal{S}_{n-2}$ module $H^i(M_{0,n}, \mathbb{Q})$ the trivial representation I_{n-2} occurs exactly with multiplicity 1.*

Proof.

The base of induction ($n = 3$) is obvious. Now, given $n > 3$, it suffices to prove that, for every i, the multiplicity of I_{n-2} in the decomposition of $H^i(M_{0,n}, \mathbb{Q})$ is at least 1. In fact we then observe that the top cohomology of $M_{0,n}$ has degree $n-3$ and we can apply Lemma 4.4.2.

For $i = 0$ our assertion is trivial. Let us then suppose $i \geq 1$. From the relation of Theorem 4.3.1 we deduce

$$\chi_{n-2}(i, n-2) = \chi_{n-2}(i, n-3) + Res^{\mathcal{S}_{n-1}}_{\mathcal{S}_{n-2}}(p_{n-1})\chi_{n-2}(i-1, n-3),$$

that is to say,

$$\chi_{n-2}(i, n-2) = \chi_{n-2}(i, n-3) + (p_{n-2} + I_{n-2})\chi_{n-2}(i-1, n-3) \qquad (4.16)$$

If $i = 1$, we have $\chi_{n-2}(i-1, n-3) = I_{n-2}$, therefore I_{n-2} appears in the irreducible decomposition of $\chi_{n-2}(i, n-2)$. If $i \geq 2$, we observe that, by the inductive hypothesis, $(\chi_{n-3}(i-1, n-3), I_{n-3}) = 1$. But by Frobenius reciprocity law

$$1 = (\chi_{n-3}(i-1, n-3), I_{n-3}) = (\chi_{n-2}(i-1, n-3), I_{n-2} + p_{n-2}).$$

Now $(\chi_{n-2}(i-1, n-3), I_{n-2}) = 0$ since we have already proven that the only subspace of $H^*(M_{0,n-1}, \mathbb{Q})$ which affords the trivial representation I_{n-2} is $H^0(M_{0,n-1}, \mathbb{Q})$. Then we have

$$1 = (\chi_{n-2}(i-1, n-3), p_{n-2}).$$

Therefore, in the equation (4.16) we find the product $p_{n-2}p_{n-2}$ as an addendum: its decomposition into irreducibles (see [8] Chap. 4) is equal to $I_{n-2} + p_{n-2}$ plus two other irreducible characters. ∎

4.5 The $\mathcal{S}_{n+1}$ action on $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and its integer cohomology ring

In the preceding sections we used the isomorphism between $M_{0,n+1}$ and $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ to obtain an extended action of $\mathcal{S}_{n+1}$ on $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and on its complex cohomology ring. Now, if we consider the isomorphism $\Gamma : \overline{M}_{0,n+1} \cong \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ we can extend to the boundary that reasoning since $\overline{M}_{0,n+1}$ has a natural $\mathcal{S}_{n+1}$ action; as a consequence we find an $\mathcal{S}_{n+1}$ action on $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and on

$H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$, compatible with the natural $\mathcal{S}_n$ action which derives from the linear action of $\mathcal{S}_n$ on V as reflection group

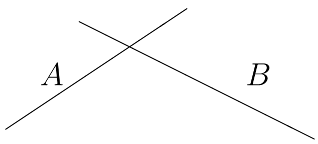
The importance of these extended symmetric group representations was first pointed out in the context of moduli spaces (see [13]): using the theory of models of arrangements they can be studied in an elementary way since the elements of the Yuzvinsky basis of $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ turn out to be permuted by $\mathcal{S}_n$. In fact, if we represent (as in Chapter 2, Section 2) these elements by means of forests on n leaves, we see that the $\mathcal{S}_n$ action is the one which permutes the n numbered leaves.

Let us now start by focusing on the $\mathcal{S}_{n+1}$ action on $\overline{M}_{0,n+1}$. We think of $\overline{M}_{0,n+1}$ as the set of tree-like stable pointed curves, according to the description in Chapter 2, Section 2: the symmetric group action is then the one which permutes the $n+1$ marked points.

It is interesting to study what is the effect of this action on the divisors in the boundary: since the cohomology ring is generated by the cohomology classes of these divisors, this will allow us to recover the $\mathcal{S}_{n+1}$ action on $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$.

Recall that an irreducible divisor in $\overline{M}_{0,n+1}$ can be represented by the picture

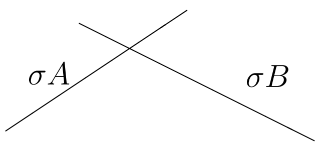
$D =$



where $A \subset \{0, 1, \dots, n\}$ and $B = \{0, 1, \dots, n\} - A$ satisfy $|A| \geq 2$, $|B| \geq 2$.

Let us now identify $\mathcal{S}_{n+1}$ with the group of the permutations on the numbers $\{0, \dots, n\}$; using the conventions of Section 3, we can write $\mathcal{S}_{n+1} =< s_0, \dots, s_{n-1} >$ where s_i represents the transposition $(i, i+1)$. Then, if we consider $\sigma \in \mathcal{S}_{n+1}$ and put $\sigma A = \{\sigma(y) \mid y \in A\}$, $\sigma B = \{\sigma(z) \mid z \in B\}$, we have

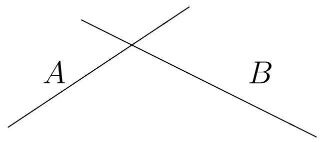
$\sigma D =$



But we can also represent D (via the isomorphism Γ of Theorem 2.2.4) as a divisor in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, using the notation of forests on n numbered leaves, introduced in Chapter 2. With a slight abuse of notation, in what follows we will indicate for simplicity by the same symbol the subsets of $\{0, \dots, n\}$

and the associated subspaces in V^* (i.e. we will omit the superscript "—"). In particular we can identify the irreducible divisors with the subsets of $\{1, \ldots, n\}$ of cardinality greater than or equal to 2 (see Proposition 2.2.2). The connection between these two representations of the divisors (and thus of the objects in the boundary) was described in the Proposition 2.3.1 and it is given by the following rule. Let us consider $A \subset \{0, 1, \ldots, n\}$ and $B = \{0, 1, \ldots, n\} - A$, $|A| \geq 2$, $|B| \geq 2$, and suppose that $0 \in B$. Then we associate to the divisor

$D =$



the subset $A \subset \{1, \ldots, n\}$.

We can then transfer the $\mathcal{S}_{n+1}$ action on the divisors of $\overline{M}_{0,n+1}$ to an $\mathcal{S}_{n+1}$ action on the set

$$\mathcal{L} = \{\, A \in \mathcal{P}(\{1, \ldots, n\}) \mid |A| \geq 2,\ A \neq \{1, \ldots, n\} \,\}$$

which parametrizes these divisors. We will denote this action by the symbol "$\star$", and observe that we have, for $\sigma \in \mathcal{S}_{n+1}$ and $T \in \mathcal{L}$,

$$\sigma \star T = \begin{cases} \sigma T & \text{if } 0 \notin \sigma T \\ \{0, 1, \ldots, n\} - \sigma T & \text{if } 0 \in \sigma T \end{cases} \tag{4.17}$$

The cohomological interpretation of this action is immediate, since the cohomology ring $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ is generated by the classes c_A ($A \in \mathcal{L}$) of the divisors in the boundary: if $\sigma \in \mathcal{S}_{n+1}$, we have that $\sigma(c_A) = c_{\sigma \star A}$. We then note that this action is compatible with the $\mathcal{S}_n$ one, once we identify $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ with the subgroup $\mathcal{S}_n =< s_1, \ldots, s_{n-1} >$.

4.6 The representation on H^2

Let us now focus on $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$. We want to give explicit formulas for the $\mathcal{S}_{n+1}$ action in terms of the elements of the Yuzvinsky basis and to determine the associated representation. Recall that the Yuzvinsky basis for $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ is given by the elements c_T with $T \subset \{1, \ldots, n\}$ and $|T| \geq 3$.

Since $\mathcal{S}_{n+1}$ is generated by transpositions and we already know the action of the subgroup $\mathcal{S}_n \cong< s_1, \ldots, s_{n-1} >$, it is sufficient to give formulas for the transpositions $\tau_j = (0, j)$ ($j = 1, \ldots, n$).

Let us first take $T \subset \{1, \ldots, n\}$ with $3 \leq |T| \leq n-2$ and let us write $T^C = \{0, \ldots, n\} - T$. Then we have, as a consequence of (4.17),

$$\tau_j(c_T) = c_{\tau_j \star T} = \begin{cases} c_T & \text{if } j \in T^C \\ c_{(T^C \cup \{j\}) - \{0\}} & \text{if } j \in T \end{cases} \tag{4.18}$$

We note that $|(T^C \cup \{j\}) - \{0\}| \geq 3$ so $\tau_j(c_T)$ is still in any case an element of the Yuzvinsky basis. If instead $|T| = n-1$, then the same computations hold but $|(T^C \cup \{j\}) - \{0\}| = 2$ and therefore we need to use the relations

$$r_{ij} \ : \ \sum_{\{i,j\} \subset A \subset \{1,\ldots,n\}} c_A = 0 \tag{4.19}$$

to rewrite $\tau_j(c_T)$ as a $\mathbb{Z}$-linear combination of the elements of the basis.

For example, if $n = 4$,

$$\tau_4(c_{\{1,2,3,4\}}) = c_{\{3,4\}} = -c_{\{1,3,4\}} - c_{\{2,3,4\}} - c_{\{1,2,3,4\}}$$

It then remains to compute $\tau_j(c_{\{1,\ldots,n\}})$. This can be done by means of the relations (4.19); therefore, using for instance r_{12}, we have to give a formula for $\tau_j(- \sum_{\{1,2\} \subset A \subsetneq \{1,\ldots,n\}} c_A)$.

Proposition 4.6.1 *We have*

$$\tau_j(c_{\{1,\ldots,n\}}) = \sum_{\{j\} \subsetneq A \subset \{1,\ldots,n\}} (|A| - 2) c_A$$

Proof. Let us consider B with $\{j\} \subsetneq B \subset \{1, \ldots, n\}$ and $|B| \leq n-1$. We can suppose that $1 \notin B$. Writing $\tau_j(c_{\{1,\ldots,n\}}) = -\sum_{\{1,j\} \subset A \subsetneq \{1,\ldots,n\}} \tau_j(c_A)$ we have that c_B does not appear in the decomposition of the terms $-\tau_j(c_A)$ when $|A| < n-1$, unless $A = (\{1, \ldots, n\} - B) \cup \{j\}$: in this case c_B appears with coefficient -1. Furthermore, if $|A| = n-1$, $-\tau_j(c_A)$ is equal to $-c_{\{j,(\{1,\ldots,n\}-A)\}}$ which can be written as $\sum_{\{h\} \subsetneq D \subset \{1,\ldots,n\}} c_D$, where $\{h\} = \{1, \ldots, n\} - A$. In this expression, c_B appears (with multiplicity 1) if and only if $h \in B$. Since there are $|B| - 1$ possible choices for the element h (in fact h can be any element of B except for j), the coefficient of c_B in $\tau_j(c_{\{1,\ldots,n\}})$ is $|B| - 2$. A simplified version of this reasoning proves our claim in the remaining cases (that is to say, $B = \{1, \ldots, n\}$ and $j \notin B$). ■

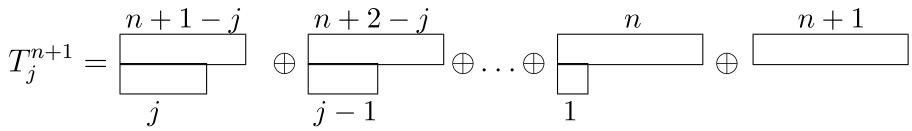
Let us now study the representations of $\mathcal{S}_n$ and $\mathcal{S}_{n+1}$ on $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$. Let us call by $D_j \subset H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ the $\mathbb{Z}$-linear subspace spanned by the Yuzvinsky basis elements c_T with $|T| = j$ ($j = 3, \ldots, n$).

If $3 \le j \le n-2$, the relation (4.18) implies that $D_j \oplus D_{n+1-j}$ is a $\mathcal{S}_{n+1}$ invariant subspace.

This in particular means that, when $j = n+1-j$ (that is to say, when n is odd and $j = \frac{n+1}{2}$), $D_{\frac{n+1}{2}}$ is an invariant subspace. Let us instead suppose that n is even and $n \ge 6$. Given j such that $3 \le j < \frac{n+1}{2}$, we want to determine the representations afforded by the subspaces $D_j \oplus D_{n+1-j}$.

We notice that the basis $\{c_T\}$ ($T \subset \{1, \dots, n\}$, $|T| = j$ or $n+1-j$) of $D_j \oplus D_{n+1-j}$ can as well be determined by using as indices only the subsets of cardinality j. In fact we can write c_{T^C} instead of c_T when $|T| = n+1-j$ (we observe that this will not generate confusion given that $0 \in T^C$).

Then the $\mathcal{S}_{n+1}$ action on the basis $\{c_T\}$ ($T \subset \{0, 1, \dots, n\}$, $|T| = j < \frac{n+1}{2}$) is easily seen to be the permutation action of the symmetric group on the subsets of $\{0, 1, \dots, n\}$ of cardinality j. This, by the "Young rule", corresponds to the following representation which we will denote by T_j^{n+1}:



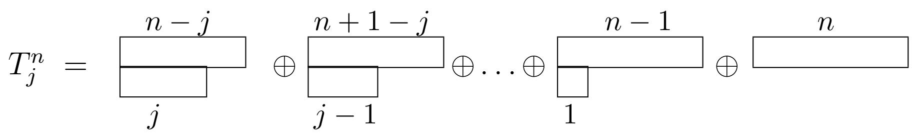
Let us now focus on D_{n-1} and D_n; we can write

$$H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z}) = \left(\bigoplus_{3 \le j \le n-2} D_j \right) \oplus D_{n-1} \oplus D_n$$

and also

$$H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z}) = \left(\bigoplus_{3 \le j \le n-2} D_j \right) \oplus T$$

where T is a $\mathcal{S}_{n+1}$ invariant complement of the $\mathcal{S}_{n+1}$ invariant subspace $\bigoplus_{3 \le j \le n-2} D_j$. If we consider D_{n-1} and $D_n = \mathbb{Z} c_{\{1,\dots,n\}}$ as $\mathcal{S}_n =< s_1, \dots, s_{n-1} >$ (invariant) submodules and denote, for every $0 \le j \le \frac{n}{2}$, by T_j^n the $\mathcal{S}_n$ representation



We can observe that, by the Young rule, we have the following $\mathcal{S}_n$ modules isomorphisms:

$$D_{n-1} \cong T_1^n \quad D_n \cong T_0^n \cong I_n$$

where we denote by I_n the trivial $\mathcal{S}_n$ representation. This means that we can write

$$T \cong T_1^n \oplus I_n$$

as $\mathcal{S}_n$ modules. Then one can immediately see that, by the branching rule, this forces T to satisfy

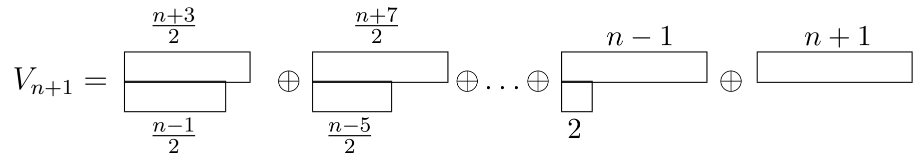
$$T \cong T_1^{n+1}$$

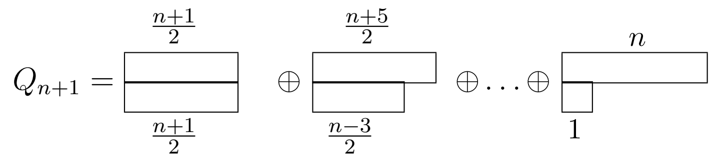
as $\mathcal{S}_{n+1}$ modules. Therefore we have

$$H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z}) \cong \left(\bigoplus_{3 \le j \le \frac{n}{2}} T_j^{n+1} \right) \oplus T_1^{n+1}$$

as $\mathcal{S}_{n+1}$ module.

If instead n is odd and $n \geq 7$, we have to take into account that the submodule $D_{\frac{n+1}{2}}$ is $\mathcal{S}_{n+1}$ invariant. Viewed as $\mathcal{S}_n$ module, it is isomorphic to $T^n_{\frac{n-1}{2}}$. Then, by the branching rule, the only possible associated representations could be





But one easily sees that the one dimensional subspace of $D_{\frac{n+1}{2}}$ spanned by the element

$$\sum_{\substack{T \subset \{1, \dots, n\} \\ |T| = \frac{n+1}{2}}} c_T$$

is $\mathcal{S}_{n+1}$ invariant, so in the decomposition of $D_{\frac{n+1}{2}}$ the trivial representation appears. This implies that the irreducible decomposition V_{n+1} is the right one.

Summarizing, we have that, when n is odd, $n \geq 7$,

$$H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z}) \cong \left(\bigoplus_{3 \le j \le \frac{n-1}{2}} T_j^{n+1} \right) \oplus V_{n+1} \oplus T_1^{n+1}$$

as $\mathcal{S}_{n+1}$ modules. In order to complete our analysis, it remains to notice that

$$H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_4}}, \mathbb{Z}) = V_6 \oplus T_1^6$$

$$H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_3}}, \mathbb{Z}) = T_1^5$$

and $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_2}}, \mathbb{Z})$ is the trivial $\mathcal{S}_4$ module.

Remarks.
1) In the observations above we used the fact that, as $\mathcal{S}_n$ modules,

$$D_j \cong T_j^n \quad \text{if } 3 \leq j \leq \frac{n}{2}$$

$$D_j \cong T_{n-j}^n \quad \text{if } \frac{n}{2} \leq j \leq n$$

This is an immediate consequence of the Young rule.

2) One should compare these results with Getzler's formula in [13].

4.7 The $\mathcal{S}_{n+1}$-equivariant immersion

In this section we will describe an $\mathcal{S}_{n+1}$-equivariant injective ring homomorphism

$$\nu \; : H^i(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z}) \mapsto H^i(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$$

Here we think of $H^i(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ equipped with the extended $\mathcal{S}_{n+1}$ action, while on $H^i(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$ we consider the $\mathcal{S}_{n+1} =< s_0, \ldots, s_{n-1} >$ action which comes from the $\mathcal{S}_{n+2} =< s_0, \ldots, s_n >$ extended action.

We describe first this immersion for $i = 2$. Let us consider the elements c_T ($T \subset \{1, \ldots, n\}$, $3 \leq |T| \leq n$) of the Yuzvinsky basis of $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$: we define the map ν by means of the relation

$$\nu(c_T) = c_T + c_{T \cup \{n+1\}}$$

where of course the addenda on the right are Yuzvinsky basis elements in $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$.

Theorem 4.7.1 *Let $\sigma \in \mathcal{S}_{n+1} =< s_0, \ldots, s_{n-1} >$ and $c_T \in H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ be an element of the Yuzvinsky basis. Then*

$$\nu(\sigma(c_T)) = \sigma(\nu(c_T))$$

where the symmetric group actions on the left and on the right are the ones described above.

Proof.

Let us suppose that $3 \leq |T| \leq n-2$. Then if $0 \notin \sigma T$ we have, (writing σc_T instead of $\sigma(c_T)$ for shortness)

$$\sigma c_T = c_{\sigma T}$$

and therefore

$$\nu(c_{\sigma T}) = c_{\sigma T} + c_{\sigma T \cup \{n+1\}} = \sigma(c_T + c_{T \cup \{n+1\}})$$

since $\sigma(n+1) = n+1$. If instead $0 \in \sigma T$, we have

$$\sigma c_T = c_{\{0,\ldots,n\}-\sigma T}$$

and therefore

$$\nu(\sigma c_T) = \nu(c_{\{0,\ldots,n\}-\sigma T}) = c_{\{0,\ldots,n\}-\sigma T} + c_{(\{0,\ldots,n\}-\sigma T)\cup\{n+1\}}$$

But in $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$ we have

$$\sigma(c_T + c_{T\cup\{n+1\}}) = c_{\{0,\ldots,n,n+1\}-\sigma T} + c_{\{0,\ldots,n,n+1\}-(\sigma T \cup \{n+1\})}$$

Now, since $c_{\{0,\ldots,n,n+1\}-\sigma T} = c_{(\{0,\ldots,n\}-\sigma T)\cup\{n+1\}}$ and $c_{\{0,\ldots,n,n+1\}-(\sigma T\cup\{n+1\})} = c_{\{0,\ldots,n\}-\sigma T}$, the $\mathcal{S}_{n+1}$ equivariance is proved.

Let us now suppose that $|T| = n-1$. Then if $0 \notin \sigma T$ we proceed as before. If instead $0 \in \sigma T$, we observe that $\sigma c_T = c_{\{0,1,\ldots,n\}-\sigma T}$ is not an element of the Yuzvinsky basis, since $|\{0,1,\ldots,n\} - \sigma T| = 2$. Using the relations (4.19) we can write

$$c_{\{0,1,\ldots,n\}-\sigma T} = -\sum_{(\{0,1,\ldots,n\}-\sigma T)\subsetneq A \subset \{1,\ldots,n\}} c_A$$

Then

$$\nu(c_{\{0,1,\ldots,n\}-\sigma T}) = -\sum_{(\{0,1,\ldots,n\}-\sigma T)\subsetneq A \subset \{1,\ldots,n\}} (c_A + c_{A\cup\{n+1\}}) \tag{4.20}$$

On the other hand, as we observed before, in $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$ we have

$$\sigma c_T + \sigma c_{T\cup\{n+1\}} = c_{\{0,\ldots,n,n+1\}-\sigma T} + c_{\{0,\ldots,n,n+1\}-(\sigma T\cup\{n+1\})}$$

But since $|\{0,\ldots,n,n+1\} - (\sigma T \cup \{n+1\})| = 2$ we can use the relations (4.19) adapted to the $\mathbb{Z}$-module $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$:

$$\sigma c_T + \sigma c_{T\cup\{n+1\}} = c_{\{0,\ldots,n,n+1\}-\sigma T} - \sum_{\{0,1,\ldots,n,n+1\}-(\sigma T\cup\{n+1\})\subsetneq A \subset \{1,\ldots,n,n+1\}} c_A$$

which is easily seen to be equal to (4.20).

It remains to study the case when $|T| = n$, that is to say, the case of the element $c_{\{1,\ldots,n\}}$. If $\sigma\{1,\ldots,n\} = \{1,\ldots,n\}$ the invariance follows immediately. Otherwise, we see that it is sufficient to check the statement for a transposition $\tau_j = (0,j) \in \mathcal{S}_{n+1}$. Then, by the Proposition 4.6.1,

$$\tau_j c_{\{1,\ldots,n\}} = \sum_{\{j\}\subsetneq A\subset\{1,\ldots,n\}} (|A|-2)c_A$$

and consequently

$$\nu(\tau_j c_{\{1,\ldots,n\}}) = \sum_{\{j\}\subsetneq A\subset\{1,\ldots,n\}} (|A|-2)(c_A + c_{A\cup\{n+1\}})$$

On the other hand, in $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$,

$$\tau_j c_{\{1,\ldots,n\}} + \tau_j c_{\{1,\ldots,n,n+1\}} = c_{\{j,n+1\}} + \sum_{\{j\}\subsetneq B\subset\{1,\ldots,n,n+1\}} (|B|-2)c_B$$

which, using the relations (4.19), can be rewritten as

$$= - \sum_{\{j,n+1\}\subsetneq D\subset\{1,\ldots,n,n+1\}} c_D + \sum_{\{j\}\subsetneq B\subset\{1,\ldots,n,n+1\}} (|B|-2)c_B =$$

$$= \sum_{\{j\}\subsetneq B\subset\{1,\ldots,n\}} (|B|-2)(c_B + c_{B\cup\{n+1\}})$$

This concludes the proof.

■

The result of the preceding theorem can be extended to the entire cohomology rings $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ $(n \geq 3)$ since they are generated by their components of degree 2.

Theorem 4.7.2 *The map ν extends to an injective ring homomorphism*

$$\nu \;:\; H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z}) \mapsto H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$$

which is $\mathcal{S}_{n+1}$-invariant, where on the left we consider the $\mathcal{S}_{n+1}$ extended action, while on the right we consider the $\mathcal{S}_{n+1} =< s_0,\ldots,s_{n-1} >$ action which comes from the $\mathcal{S}_{n+2} =< s_0,\ldots,s_n >$ extended action.

Proof.

We will give an explicit description of the extended map ν. For this we use the squarefree basis introduced in Chapter 3, Section 4, and studied in particular in the example at the end of that section.

Recall that, for every n, the order we chose on $\mathcal{F}_{\mathcal{A}_{n-1}}$ can be expressed by the following rule. Given $A, B \subset \{1,\ldots,n\}$, then $A < B$ if either of the following cases occurs:

1. $B \subset A$, or

2. neither $A \subset B$ nor $B \subset A$ but $min\,(A - A \cap B) < min\,(B - A \cap B)$

Since the map ν is defined on the elements c_A $(A \in \mathcal{G})$, which generate $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ as a ring, it can be extended to $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ in a natural way. Let us call by $I_\emptyset^{(n-1)}$ and $I_\emptyset^{(n)}$ the ideals such that

$$H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z}) \cong \mathbb{Z}[c_A]/I_\emptyset^{(n-1)} \quad (A \in \mathcal{F}_{\mathcal{A}_{n-1}})$$

$$H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z}) \cong \mathbb{Z}[c_A]/I_\emptyset^{(n)} \quad (A \in \mathcal{F}_{\mathcal{A}_n})$$

It is immediate to check that, for every generator $P^\emptyset_{\mathcal{H},B}$, we have $\nu(P^\emptyset_{\mathcal{H},B}) \in I_\emptyset^{(n)}$, that is to say, $\nu(I_\emptyset^{(n-1)}) \subset I_\emptyset^{(n)}$. Then ν is a ring homomorphism.

As for the injectivity, let us consider a monomial $m_f = c_{A_1} \cdots c_{A_r}$ of the squarefree basis of $H^i(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ $(i = 2r)$. We have

$$\nu(m_f) = \nu(c_{A_1}) \cdots \nu(c_{A_r}) = (c_{A_1} + c_{A_1 \cup \{n+1\}}) \cdots (c_{A_r} + c_{A_r \cup \{n+1\}})$$

which gives a sum of monomials which are either zero or belong to the squarefree basis of $H^i(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$.

In fact we have

$$\nu(m_f) = \sum_{J \subset \{1,\ldots,r\}} \left[\left(\prod_{i \in J} c_{A_i} \right) \left(\prod_{j \in \{1,\ldots r\} - J} c_{A_j \cup \{n+1\}} \right) \right]$$

Then the addenda on the right either have non nested support or are monomials m_g with the following property: if $A_j \cup \{n+1\}$ belongs to $supp\, g$ then, for every k such that $A_j \subset A_k$, we have $A_k \cup \{n+1\} \in supp\, g$.

Because of the choice of the order in $\mathcal{F}_{\mathcal{A}_n}$, the monomials m_g which satisfy the above mentioned condition are elements of the squarefree basis of $H^i(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$. This explicit description immediately proves the injectivity of ν, while the $\mathcal{S}_{n+1}$-equivariance follows by construction and by the Theorem 4.7.1.

■

The $\mathcal{S}_{n+1}$-equivariant immersion we constructed is interesting both in view of the study of the structure of the rings $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$ and in view of their representation theory. As for the representation theory, we note that in fact the map ν has properties similar to the ones of the map η^* we studied in Section 3 of the present chapter (see Theorem 4.3.1). Hopefully it could be used in a similar way as η^*, that is to say, it could be the key to study some recursive relations between the symmetric group representations of $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_n}}, \mathbb{Z})$ and $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$.

Chapter 5

Generalized Poincarè series

5.1 The aim of this chapter

In this last chapter we will focus on the natural (i.e., non extended) action of the symmetric group $\mathcal{S}_n$ on $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$. In Chapter 4, Section 5, we determined the irreducible decomposition of this representation (and of the "extended" $\mathcal{S}_{n+1}$ representation) restricted to $H^2(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$; our aim now is, given any element $w \in \mathcal{S}_n$, to provide formulas for the trace of the corresponding operator.

As we pointed out in the Introduction, in [13] Getzler found a formula for the Legendre transform of the cyclic characteristic series of the rings $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$. Differentiating this formula we can find the "non cyclic" characteristic series of the rings $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ which encodes all the information concerning the non extended symmetric group action. In this paper we will deal with a different combinatorial object (the series $\mathcal{H}$ which will be defined later) which encodes the same information of the non cyclic characteristic. We start by considering the generalized Poincarè polynomial of $\widehat{Y}_{\mathcal{A}_{n-1}}$ with respect to w:

Definition 5.1.1 *Given $w \in \mathcal{S}_n$, we call by $P_{w,\mathcal{A}_{n-1}}(q)$ the "generalized Poincarè polynomial"*

$$P_{w,\mathcal{A}_{n-1}}(q) := \sum_i (tr\, w_{|\, H^{2i}(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})}) q^i$$

Now we can view $w \in \mathcal{S}_n$ as an element of $\mathcal{S}_m$ for every $m > n$, by the obvious immersion $\mathcal{S}_n \mapsto \mathcal{S}_m$; this makes w to act on all the rings $H^*(\widehat{Y}_{\mathcal{A}_{m-1}}, \mathbb{Z})$ for $m > n$. It turns out that, in order to determine the generalized Poincarè polynomials $P_{w,\mathcal{A}_{m-1}}(q)$ as m varies, it is convenient to compute directly the two variables "generalized Poincarè series"

$$P_{w,A}(q,t) = \sum_{m=n}^{\infty} P_{w,\mathcal{A}_{m-1}}(q) \frac{t^m}{m!}$$

We give in Section 2 some formulas for $P_{w,A}(q,t)$ for any $w \in \mathcal{S}_n$. The method we apply is a generalization of the combinatorial blow-up method used in the proof of Theorem 3.2.1. In fact, in particular, when w is the identity, we recover the formula for the ordinary Poincarè series.

We then note that a substantial simplification to the equations provided in Section 2 can be obtained if we consider, instead of the series $P_{w,A}(q,t)$, a universal graded series $\mathcal{H}$ in some formal graded variables S_j and $P_k^{(d)}$ ($j \geq 1$, $k \geq 1$, $d \geq 0$) that is connected to the generalized Poincarè series in the way explained by the following steps:

1. Given $w \in \mathcal{S}_n$ with decomposition $w = c_1 \cdots c_l$, where $c_1, \ldots, c_l$ are non trivial disjoint cycles of length $\lambda_1 \geq \ldots \geq \lambda_l > 1$ respectively, consider the polynomial $\mathcal{H}_l$ which is the homogeneous component of degree l of $\mathcal{H}$.

2. Then substitute in $\mathcal{H}_l$ the formal variables S_j (and $P_k^{(d)}$) with some special functions from $\mathbb{Z}^j$ ($\mathbb{Z}^k$ respectively) to $\mathbb{Z}[[q,t]]$.

3. Finally put the numbers $\lambda_1, \ldots, \lambda_l$ as inputs of these special functions and sum over all the possible permutations of $\lambda_1, \ldots, \lambda_l$.

4. The universal series $\mathcal{H}$ is constructed in such a way that, after the steps 1,2 and 3, $P_{w,A}^{(\sum_{i=1}^{l} \lambda_i)}(q,t)$ is obtained (here the superscript (n) means "n-th derivative with respect to t").

The third section of this chapter is devoted to finding a nice formula for the formal series $\mathcal{H}$ (see Theorem 5.3.4), which is obtained by purely combinatorial methods. In particular, as a by-product of our analysis, we prove some theorems on certain remarkable sums of functions defined on rooted trees (see Theorems 5.3.2 and 5.3.3).

Finally, in Section 4, we deal with the complex root arrangements $\mathcal{B}_n^*$ of type B_n and on their De Concini - Procesi models of irreducibles $\widehat{Y}_{\mathcal{F}_{\mathcal{B}_n}}$: we provide a formal series $\mathcal{H}_B$ which, by the same methods as above, gives us the generalized Poincarè series with respect to the elements of some subgroups (isomorphic to $\mathcal{S}_n$) of the Weyl group of type B_n.

Remark. We recall that, in view of the isomorphism $H^*(\widehat{Y}_{\mathcal{G}}, \mathbb{Z}) \cong H^*(Y_{\mathcal{G}}, \mathbb{Z})$, the reasonings and results of the present chapter are the same in the compact and non compact case.

5.2 The generalized Poincarè series

5.2.1 The generalized Poincarè series with respect to a cycle

We will compute the generalized Poincarè series by applying a technique similar to the splitting technique used in the proof of Theorem 3.2.1.

Let us first study the Poincarè series $P_{c,A}(q,t)$, where $c \in \mathcal{S}_n$ is a cycle. We may assume, taking a conjugate, that $c = (1, 2, \ldots, r)$. Let us then focus on an element θ, of the Yuzvinsky basis for $H^*(\widehat{Y}_{\mathcal{A}_{r-1}}, \mathbb{Z})$, that is fixed by c. Thus the graph associated to θ is a forest $\mathcal{T}'$ whose connected components are s ($s \geq 1$) identical copies of a symmetric rooted tree $\mathcal{T}$.

In fact, if there is a vertex v_1 connected by an edge to some leaves (say $a_1, \ldots, a_t$), it is easily seen that it must be $t|r$ and $a_1, \ldots, a_t$ are all the representatives, among $\{1, \ldots, r\}$, of a certain congruence class (say the class of 1) modulo $\dfrac{r}{t}$.

Furthermore, there must be vertices $v_2, \ldots, v_{\frac{r}{t}}$ which play the same role as v_1 with respect to the other congruence classes.

Then we have $c(v_1) = v_2 \ldots c(v_2) = v_3 \ldots c(v_{\frac{r}{t}}) = v_1$, therefore we can apply again the same considerations and so on.

This allows us to describe the shape of $\mathcal{T}$ in the following way. We will say that a vertex of $\mathcal{T}$ belongs to the t-th class if the oriented path that connects it with the root is made by t edges. Then in $\mathcal{T}$ there are d_1 outgoing edges from the root, d_2 outgoing edges from each one of the d_1 vertices which belong to the first class, d_3 outgoing edges from each one of the $d_1 d_2$ vertices which belong to the second class, and so on, until we reach the last class of vertices, say the k-th.

Note that it must be $s \prod_{j=1}^{k} d_j = r$ and $d_j > 2 \; \forall \, j$.

We will indicate by $F(\mathcal{T})$ the collection of elements of the Yuzvinsky basis whose graphs have the shape described above. The following functions are strictly related with the contribution that the elements of $F(\mathcal{T})$ give to the Poincarè series:

Definition 5.2.1 *Given the positive natural numbers d and m, with $d > 1$, $d|m$, $d \neq m$, we define the function*

$$T(d,m) = \sum_{\substack{(d_1, \ldots, d_k) \\ s.t. \; \prod_{j=1}^{k} d_j = d \\ d_j > 1 \, \forall j}} \left(\prod_{j=1}^{k} \left(\frac{q_j^{d_j - 1} - q_j}{q_j - 1} \right) \right)$$

where $q_j = q^{\frac{m}{d_j \cdots d_k}}$. Furthermore, if $d = 1$ and $m > 1$ we put

$$T(1,m) = 1$$

Definition 5.2.2 *Given a positive natural number r, we put:*

$$\overline{S}(r) = \sum_{\substack{d|r \\ d \neq r}} T(d,r)$$

We now call by Y_c the set of the elements of the Yuzvinsky bases of $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ ($n \geq 3$, $n \geq r > 1$) that are fixed by c. Looking at the graphs of the elements in Y_c, we want to single out, if it exists, a "singular" vertex.

Definition 5.2.3 *Let us consider the graph Γ of an element $\gamma \in Y_c$. A vertex v of Γ is called "singular" if it satisfies the following conditions:*

1. *The subtree that stems from v has among its leaves the leaves $1, 2, \ldots, r$.*
2. *The condition 1 is not satisfied by the vertices which follow v in the orientation.*

Theorem 5.2.1 *We have the following formula for the generalized Poincarè series $P_{c,A}(q,t)$:*

$$P^{(r)}_{c,A}(q,t) = \overline{S}(r)(1+\Phi) + \Phi^{(1)} \sum_{\substack{j \geq 0 \\ m|r \\ m \neq r}} T(m,r) \frac{q^{j+\frac{r}{m}-1} - q}{q-1} \frac{\lambda^j}{j!}$$

Proof

The definition of singular vertex of a graph in Y_c makes the proof similar to that of Theorem 3.2.1.

For every $n \geq 2$ we will regard the Yuzvinsky basis for $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}, \mathbb{Z})$ as the set of marked forests described above, with the n leaves identified with the numbers from 1 to n.

We can now split Y_c in two parts: the subset made by I-elements and the subset made by II-elements, where I-elements are the elements the graphs of which have not a singular vertex, II-elements are the elements the graphs of which have a singular vertex.

Let us then compute the contribution of I-elements to $P_{c,A}(q,t)$. Let γ be a I-element: then the part of the graph of γ that covers the leaves 1,2,..., r is a forest of type $\mathcal{T}'$ (with $s > 1$ since γ is a I-element), while the other part is the graph of a Yuzvinsky-type element constructed on the leaves labeled by the numbers greater than or equal to $r+1$. This gives the relation:

$$P^{(r)}_{c,A}(q,t) = \sum_{\substack{d|r \\ d \neq r}} T(d,r)(1+\Phi) + \text{contribution of II-elements}$$

Let us now work on II-elements: given a II-element ϱ, we can construct two new Yuzvinsky-type elements: ϱ' and ϱ''. The graph of ϱ' is obtained from the one of ϱ by collapsing to the singular vertex v, which becomes a leaf, the subtree ρ_v that stems out of v. The graph of ϱ'' is ρ_v (we note that we are considering $v \in \rho_v$).

We observe that a II-element can be uniquely determined by giving its associated couple (ϱ' , ϱ''). Therefore, in order to obtain the contribution to $P_{c,A}$ of II-elements, we can multiply the series originated respectively by elements of type ϱ' and ϱ''. The second one is easily shown to be

$$\sum_{\substack{j \geq 0 \\ m|r \\ m \neq r}} T(m,r)\frac{q^{j+\frac{r}{m}-1} - q}{q-1}\frac{\lambda^j}{j!}$$

The contribution to the series due to the elements of type ϱ' is $\Phi^{(1)}$. In this case the first derivative is needed since the elements of type ϱ' have an artificial leaf (the singular vertex). Summing up we have:

$$\text{contribution of II-elements} = \Phi^{(1)} \sum_{\substack{j \geq 0 \\ m|r \\ m \neq r}} T(m,r)\frac{q^{j+\frac{r}{m}-1} - q}{q-1}\frac{\lambda^j}{j!}$$

and this proves the theorem. ■

5.2.2 Extracted graphs and the general case

Now our purpose is to study the general case of an element $w \in \mathcal{S}_n$ with disjoint cycle decomposition $w = c_1 c_2 \cdots c_l$, where c_j is a cycle of length $\lambda_j > 1$. We can assume, up to conjugation, that w permutes the first $\sum_{j=1}^{l} \lambda_j$ leaves and therefore we can take $n = \sum_{j=1}^{l} \lambda_j$. It is convenient to introduce the following notation:

Definition 5.2.4 *For every j, we indicate by L_j the set of the λ_j leaves that are permuted by c_j.*

Our next step is to state the new appropriate definition of singular vertex:

Definition 5.2.5 *Let the graph A be associated to an element α of the Yuzvinsky basis of $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_m}}, \mathbb{Z})$ $(m \geq n)$ and let α be fixed by w. A vertex v of A is called "singular" if the following conditions are satisfied:*

1. The subtree that stems from v has among its leaves some of the sets L_j.

2. Some of the sets, say $L_{i_1}, \ldots, L_{i_p}$ ($p \geq 1$), are not leaves of the subtrees stemming from any other vertices that follow v in the orientation.

The sets $L_{i_1}, \ldots, L_{i_p}$ are called then "adjacent to v".

Now we note that we can extract from A two new marked graphs in this way:

Definition 5.2.6 *The graph $\widetilde{A}$ is called "extracted from A" if it is constructed according to the following rules:*

1. Take as vertices all the singular vertices of A.

2. Connect two of them by an oriented edge if they are connected by an oriented path in A.

3. Associate to each vertex a label corresponding to the number of sets L_j adjacent to it.

Definition 5.2.7 *The graph $\widetilde{\widetilde{A}}$ is called "fully extracted from A" if it has the same vertices, edges and orientation as $\widetilde{A}$, but the following different marking: if $L_{i_1}, L_{i_2}, \ldots, L_{i_p}$ are the sets adjacent to a singular vertex v of A, the mark of v in $\widetilde{\widetilde{A}}$ is the set $\{ L_{i_1}, L_{i_2}, \ldots, L_{i_p} \}$.*

Extracted and fully extracted graphs will play an important role in the sequel. As a first example, we will use fully extracted graphs in the computation of the generalized Poincarè series $P_{w,A}(q,t)$, given that they allow us to generalize the splitting in I-elements and II-elements that we used in the case $P_{c,A}(q,t)$ of Theorem 5.2.1. In fact we will sum separately the contributions coming from the elements such that their graphs give rise to the same fully extracted graph. Before starting to compute $P_{w,A}(q,t)$ we need to define some special functions and to introduce the notion of "contraction" of a list of integers.

Let us consider a list of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ (maybe with repetitions) and let us construct some new lists $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ according to the following rules: choose a partition of $\{1, \ldots, p\}$ in k sets $J_1, \ldots, J_k$ such that

1. $\min\{x \,|x \in J_1\} < \min\{x \,|x \in J_2\} < \ldots < \min\{x \,|x \in J_k\}$;

2. for every j ($1 \leq j \leq k$), either $|J_j| = 1$ or, if $|J_j| > 1$ then we have $MCD_j = MCD\{\lambda_i \mid i \in J_j\} > 1$.

Then, for every j such that $|J_j| > 1$, choose a number $\gamma_j > 1$ which divides MCD_j; if $J_j = \{r\}$ $(1 \leq r \leq p)$, then take $\gamma_j = \lambda_r$; finally form the list $(\gamma_1, \gamma_2, \ldots, \gamma_k)$.

We say that $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ is "contracted from" $(\lambda_1, \lambda_2, \ldots, \lambda_p)$, and we denote by

$$(\lambda_1, \lambda_2, \ldots, \lambda_p) \stackrel{J_1,\ldots,J_k}{\longrightarrow} (\gamma_1, \gamma_2, \ldots, \gamma_k)$$

the process of "contraction" described above.

We note that if the chosen partition is $J_1 = \{1\}$, $J_2 = \{2\}$, $J_p = \{p\}$, the associated contraction is the trivial one

$$(\lambda_1, \lambda_2, \ldots, \lambda_p) \stackrel{J_1,\ldots,J_p}{\longrightarrow} (\lambda_1, \lambda_2, \ldots, \lambda_p)$$

Furthermore, two contractions

$$(\lambda_1, \lambda_2, \ldots, \lambda_p) \stackrel{D_1,\ldots,D_t}{\longrightarrow} (\delta_1, \delta_2, \ldots, \delta_t)$$

$$(\lambda_1, \lambda_2, \ldots, \lambda_p) \stackrel{J_1,\ldots,J_k}{\longrightarrow} (\gamma_1, \gamma_2, \ldots, \gamma_k)$$

will be considered equal if and only if $k = t$, $J_s = D_s$ $\forall s$ and $\gamma_r = \delta_r$ $\forall r$.

Definition 5.2.8 *Let us put, for every list $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ of natural numbers strictly greater than 1*

$$\overline{S}_p(\lambda_1, \lambda_2, \ldots, \lambda_p) = \prod_{s=1}^{p} \overline{S}(\lambda_s)$$

We observe that the functions $\overline{S}_p$ are symmetric. Then we can define, in a recursive way, the symmetric functions S_p, for all positive integers p.

Definition 5.2.9 *Let $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ be as before; we define recursively*

$$S_p(\lambda_1, \lambda_2, \ldots, \lambda_p) = \overline{S}_p(\lambda_1, \lambda_2, \ldots, \lambda_p) +$$

$$\sum_{(\lambda_1,\lambda_2,\ldots,\lambda_p) \stackrel{J_1,\ldots,J_k}{\longrightarrow} (\gamma_1,\gamma_2,\ldots,\gamma_k)} S_k(\gamma_1, \gamma_2, \ldots, \gamma_k) \prod_{t=1}^{k} \left(\frac{q^{-\gamma_t + \sum_{h \in J_t} \lambda_h} - q^{\gamma_t}}{q^{\gamma_t} - 1} \right)$$

where the sum ranges over all the possible different non trivial contractions.

Note that, if $p = 1$, $S_1(\lambda_1) = \overline{S}(\lambda_1)$.

We also need to introduce the following special functions:

Definition 5.2.10 *Let us put, for every list* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *of natural numbers strictly greater than 1*

$$\overline{P}_p(\lambda_1, \lambda_2, \ldots, \lambda_p) = \left(\sum_{\substack{j \geq 0 \\ d_1|\lambda_1,\ d_1 \neq \lambda_1 \\ \ldots \\ d_p|\lambda_p,\ d_p \neq \lambda_p}} T(d_1, \lambda_1) \cdots T(d_p, \lambda_p) \frac{q^{j-1+\sum_s \frac{\lambda_s}{d_s}} - q}{q-1} \frac{\lambda^j}{j!} \right)$$

and, in a recursive way,

Definition 5.2.11 *Let* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *be as before; we define recursively*

$$P_p(\lambda_1, \lambda_2, \ldots, \lambda_p) = \overline{P}_p(\lambda_1, \lambda_2, \ldots, \lambda_p) +$$

$$\sum_{(\lambda_1,\lambda_2,\ldots,\lambda_p) \overset{J_1,\ldots,J_k}{\longrightarrow} (\gamma_1,\gamma_2,\ldots,\gamma_k)} P_k(\gamma_1, \gamma_2, \ldots, \gamma_k) \prod_{t=1}^{k} \left(\frac{q^{-\gamma_t + \sum_{h \in J_t} \lambda_h} - q^{\gamma_t}}{q^{\gamma_t} - 1} \right)$$

where the sum ranges over all the possible different non trivial contractions.

Remark
In the above formulas, the recursion ends when the list $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ is composed by numbers that are pairwise coprime or when $p = 1$ (in both cases the only possible contraction is the trivial one).

The motivation for all the definitions we have given stems from the following construction. Let us call by Y_w the set of all the elements of the Yuzvinsky bases which are fixed by w. Then let us take a fully extracted graph $\widetilde{\widetilde{A}}$ and look at a certain vertex v, marked by $\{L_{i_1}, L_{i_2}, \ldots, L_{i_p}\}$. Furthermore, let ν be the number of outgoing edges in $\widetilde{\widetilde{A}}$ at v.

We want to compute the "contribution of v" to $P_{w,A}^{(\sum_{j=1}^{l} \lambda_j)}(q,t)$. This means that we have to consider all the elements in Y_w which have fully extracted graph $\widetilde{\widetilde{A}}$, and sum, over all these elements, the contributions of the subtrees that stem from v. In doing this, we collapse to a single "artificial" leaf the subtrees that stem from a singular vertex h if h follows v in the orientation of $\widetilde{\widetilde{A}}$. What we get at the end are exactly the functions $P_p(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p})^{(\nu)}$, where the ν-th derivative depends on the fact that we are counting ν artificial leaves in correspondence with the singular vertices that follow v in the orientation.
Furthermore, we note that if, in the graph of an element of Y_w, the sets $L_{i_1}, L_{i_2}, \ldots, L_{i_p}$ are not adjacent to any singular vertex, their contribution to $P_{w,A}^{(\sum_{j=1}^{l} \lambda_j)}(q,t)$ is $S_p(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p})$.

Remark
1) It is clear from the definitions that the complexity of the computation of the functions P_m and S_m sensibly grows if m grows and the numbers λ_j are not pairwise coprime.
2) From now on, if $\mathcal{L} = \{L_{i_1}, L_{i_2}, \ldots, L_{i_p}\}$, we will also write $P(\mathcal{L})$ and $S(\mathcal{L})$ for $P_p(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p})$ and $S_p(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p})$ (recall that P_m and S_m are symmetric functions).

We are almost ready to give the formula for $P_{w,A}^{(\sum_{j=1}^{l} \lambda_j)}(q,t)$: to express it in compact form, we need to introduce this further definition.

Definition 5.2.12 *Given a fully extracted graph* $\widetilde{\widetilde{A}}$*, we associate to it the series* $\mathcal{P}(\widetilde{\widetilde{A}})$ *which is a product of series that contains a factor for each vertex of* $\widetilde{\widetilde{A}}$*: if* v *is a vertex marked by* $\{ L_{i_1}, L_{i_2}, \ldots, L_{i_p} \}$ *and it has* ν *outgoing edges, the corresponding factor is* $P_p(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p})^{(\nu)}$.

Given a subset $J = \{ L_{i_1}, L_{i_2}, \ldots, L_{i_r} \}$ of the set $\mathcal{L} = \{ L_1, L_2, \ldots, L_l \}$, we call by J^C its complement. Let now Γ_{J^C} be the family of automorphism classes of fully extracted graphs characterized by the following property: an automorphism class E belongs to Γ_{J^C} if, taken a representative η of E, the marks of η (which are subsets of J^C) give a partition of J^C. Of course, if $J = \mathcal{L}$, then $\Gamma_{J^C} = \emptyset$.

Furthermore, given any graph A, we indicate by comp(A) the number of its connected components. We put comp$(\emptyset)= 0$.

We can finally state the theorem which provides a formula for the generalized Poincarè series; the proof of the theorem essentially follows the same ideas of the proof of Theorem 5.2.1, using the new definitions and observations introduced in the present subsection.

Theorem 5.2.2

$$P_{w,A}^{(\sum_{j=1}^{l} \lambda_j)}(q,t) = S(\mathcal{L})(1+\Phi) + \sum_{J\subset\mathcal{L},\, J\neq\mathcal{L}} S(J) \left[\sum_{[B]\in\Gamma_{J^C}} \Phi^{(comp(B))}\mathcal{P}(B) \right] \tag{5.1}$$

Unfortunately, formula (5.1) is not easy to be computed as far as the number l of distinct cycles of w grows, mostly because Γ_{J^C} becomes quite a complicated object to study. In the next paragraph we are going to show how to overcome this difficulty.

5.3 The formal series $\mathcal{H}$ and sums over trees

We note that in the formula (5.1) the numbers λ_j appear as inputs of the functions $P_j^{(d)}$ and S_m. Let us look at the right side of (5.1): given an addendum of this sum we can view it as a monomial μ in $P_j^{(d)}$ and S_m; we observe then that there are other monomials which differ from μ only for a permutation of the inputs λ_j.

It turns out that if we take into account the associated symmetric group action, we can simplify (and we can study in a deeper way) our formulas for the generalized Poincarè series. In fact we can shift the problem to that of searching for a "universal" series $\mathcal{H}$ in certain formal graded variables S_m, $P_j^{(d)}$ $(m, j \geq 1\,, d \geq 0)$, which represent respectively the functions S_m, P_j and the derivatives $P_j^{(d)}$ defined in the preceding section.

The formal series $\mathcal{H}$, as mentioned in the introduction, should be constructed so that it satisfies what follows:

1. The graduation is obtained by giving degree m to the variables S_m and degree j to the variables $P_j^{(d)}$.

2. Let $w \in \mathcal{S}_n$ be an element with disjoint cycle decomposition $w = c_1 c_2 \cdots c_l$, where c_j is a cycle of length $\lambda_j > 1$. We consider the number l of its cycles, and take $\mathcal{H}_l$, that is to say, the homogeneous component of degree l of $\mathcal{H}$.

3. Then we create the polynomial $\mathcal{H}_l(\lambda_1, \ldots, \lambda_l)$, by transforming, in each term of $\mathcal{H}_l$, the formal variables S_m, P_j and derivatives in their concrete representative, and by making the symmetric group $\mathcal{S}_l$ to act.

   For example, we transform the term $S_2 P_1 (P_1^{(2)})^2 P_2$ of degree 7 into

$$\sum_{\varsigma \in \mathcal{S}_7} S_2(\lambda_{\varsigma(1)}, \lambda_{\varsigma(2)}) P_1(\lambda_{\varsigma(3)}) P_1^{(2)}(\lambda_{\varsigma(4)}) P_1^{(2)}(\lambda_{\varsigma(5)}) P_2(\lambda_{\varsigma(6)}, \lambda_{\varsigma(7)})$$

4. Our request is that this must be equal to the Poincarè series

$$P_{w,A}^{(\sum_{j=1}^{l} \lambda_j)}(q,t)$$

Let us then construct the series $\mathcal{H}$ having in mind the formula (5.1).
First of all, we should change the sum on automorphism classes of fully extracted graphs $\widetilde{\widetilde{A}}$ in a sum on automorphism classes of extracted graphs $\widetilde{A}$ (this means that we are "forgetting" the difference among the various sets L_j), by associating to an extracted graph $\widetilde{A}$ the polynomial $\mathcal{Q}(\widetilde{A})$, in the variables $P_j^{(d)}$, constructed in the following way:

Definition 5.3.1 *Given any oriented labeled forest B, the polynomial $\mathcal{Q}(B)$ is a product of monomials that contains a factor for each vertex of B: if v is a vertex of B labeled by the number s and with ν outgoing edges, the corresponding factor is $\dfrac{P_s^{(\nu)}}{s!}$. Furthermore we put $\mathcal{Q}(\emptyset) = 1$.*

Given an oriented labeled forest $\widetilde{A}$, we can construct the rooted tree $\widehat{A}$ that equals $\widetilde{A}$ if $\widetilde{A}$ is connected, and otherwise is obtained by adding to $\widetilde{A}$ a new root vertex u which is connected by an edge to all the roots of the connected components of $\widetilde{A}$.

Then we note that the cardinality $Aut(\widetilde{A})$ of the automorphism group of the graph $\widetilde{A}$ is equal to the product $\prod_v \gamma_{sym,v}$ where v ranges over the vertices of $\widehat{A}$ and $\gamma_{sym,v}$ is determined in this way: delete v and consider the connected components of the subgraph of $\widehat{A}$ that stems from v. Suppose that they can be partitioned in k automorphism classes with the following cardinalities: $a_1, a_2, \ldots, a_k$. Then $\gamma_{sym,v} = a_1!a_2!\cdots a_k!$. We observe that $Aut(\emptyset) = 1$.

We are now ready, having in mind formula (5.1), to define $\mathcal{H}$:

Definition 5.3.2

$$\mathcal{H} = \left(\sum_{r\geq 0} \frac{S_r}{r!}\right)\left[\sum_{[\widetilde{A}]}(1+\Phi)^{(comp(\widetilde{A}))}\frac{\mathcal{Q}(\widetilde{A})}{Aut(\widetilde{A})}\right] \tag{5.2}$$

where $[\widetilde{A}]$ ranges over all automorphism classes of oriented labeled forests ($\emptyset$ included) and $S_0 = 1$.

We note that the motivation for the coefficients $\frac{1}{r!}$ and $\frac{1}{Aut(\widetilde{A})}$ is provided by our intention to make the symmetric group to act on $\mathcal{H}$.

From the definition we immediately deduce the following statement, which we aimed at:

Proposition 5.3.1 *With the same notation as above, we have:*

$$P_{w,A}^{(\sum_{j=1}^{l}\lambda_j)}(q,t) = \mathcal{H}_l(\lambda_1, \ldots, \lambda_l)$$

Our next goal is to show a remarkable simplification of formula (5.2), which we will obtain by studying in a deeper way the combinatorics of the sum

$$\sum_{[\widetilde{A}]}(1+\Phi)^{(\mathrm{comp}(\widetilde{A}))}\frac{\mathcal{Q}(\widetilde{A})}{Aut(\widetilde{A})} \tag{5.3}$$

We note that the expression (5.3) can be rearranged in the following way:

$$\sum_{[\widetilde{A}]}(1+\Phi)^{(\mathrm{comp}(\widetilde{A}))}\frac{\mathcal{Q}(\widetilde{A})}{Aut(\widetilde{A})} = \sum_{n\geq 0}(1+\Phi)^{(n)}\frac{\Gamma^n}{n!}$$

where

$$\Gamma = \sum_{\substack{[\widetilde{A}] \\ \widetilde{A} \text{ connected}}} \frac{\mathcal{Q}(\widetilde{A})}{Aut(\widetilde{A})}$$

and the sum ranges over all the automorphism classes of oriented, labeled, rooted trees (this time, $\emptyset$ excluded).
Thus the problem can be reduced to the one of finding a "nice" formula for Γ. This is provided by

Theorem 5.3.2

$$\Gamma = \sum_{n\geq 1} \frac{1}{n!} \left[\left(\sum_{j\geq 1} \frac{P_j}{j!} \right)^n \right]^{(n-1)} \tag{5.4}$$

where the derivation of a product of formal variables is performed according to the Leibniz rule.

Proof

We begin by stressing the case when there is only one variable, i.e. only P_1 is involved. This case leads to the following interesting relation which we state as an independent theorem:

Theorem 5.3.3

$$\sum_{\substack{[B] \\ B \textit{ tree} \\ \textit{all labels=1}}} \frac{\mathcal{Q}(B)}{Aut(B)} = \sum_{n\geq 1} \frac{1}{n!} \left[(P_1)^n \right]^{(n-1)} \tag{5.5}$$

where $[B]$ ranges over all the automorphism classes of oriented labeled rooted trees, with all the labels equal to 1.

Proof.

We think of P_1^n as $\underbrace{P_1 P_1 \cdots P_1}_{\text{n terms}}$, and we associate to each P_1 a label which determines its position. The process of differentiating $n-1$ times by applying the Leibniz rule will give, if we do not associate terms, n^{n-1} monomials in P_1 and its derivatives.

We observe that, if μ is such a monomial, it has been obtained in this way: starting from $\underbrace{P_1 P_1 \cdots P_1}_{\text{n terms}}$, first we have differentiated the P_1 which lies in the a_1-th position, then the one in the a_2-th position, and so on. Thus we can determine μ by renaming it $\mu(a_1, \ldots, a_{n-1})$. To $\mu(a_1, \ldots, a_{n-1})$ we can associate the oriented rooted tree $G(\mu(a_1, \ldots, a_{n-1}))$ according to the following rules. In correspondence with a_1 we take a vertex v_1 and an outgoing edge. Then, if $a_2 \neq a_1$, we draw, on the other end of the edge, another vertex v_2 and another outgoing edge and so on. If we come across $a_j = a_k$ with $j > k$, instead of drawing a vertex and an edge on the free end of the edge stemming from v_{j-1}, we draw another edge which starts from vertex v_k and we continue. At the end we add a vertex to all the remaining free ends of edges. Note that we are constructing $G(\mu(a_1, \dots, a_{n-1}))$ branch by branch (here we call branch a directed path from v_1 to a leaf), and that the tree $G(\mu(a_1, \dots, a_{n-1}))$ satisfies by construction the relation

$$\mathcal{Q}(G(\mu(a_1, \dots, a_{n-1}))) = \mu(a_1, \dots, a_{n-1})$$

We also observe that monomials that are determined by different sequences may give the same tree. In fact, what is really determinant to draw the graph, is not the sequence $(a_1, \dots, a_{n-1})$ but the "symbolic" sequence obtained from it by substituting the number a_1 and all the a_j equal to a_1 with the symbol x_1, a_2 and all the a_k equal to a_2 with the symbol x_2, and so on.

This provides us a method for counting the number of monomials $\mu(a_1, \dots, a_{n-1})$ that give (up to automorphism) the same tree with n vertices A. Let τ be the number of branches of A: then we note that there are $\frac{\tau!}{AutA}$ different symbolic sequences giving (up to automorphism) the tree A.

Furthermore, we see that each one of these symbolic sequences involves $n - \tau$ different symbols, given that the repetitions of symbols are in bijective correspondence with the branches. Therefore, each symbolic sequence corresponds to $\frac{n!}{\tau!}$ possible different sequences $(a_1, \dots, a_{n-1})$, since the a_j's belong to $\{1, 2, \dots, n\}$.

Now, considering formula (5.5), we see that, given an equivalence class of oriented trees $[A]$, the coefficient of $\mathcal{Q}(A)$ on the left side is $\frac{1}{AutA}$. But if we look on the other side at the number of monomials that give, according to the previous construction, a tree automorphic to A (therefore in particular these monomials are equal to $\mathcal{Q}(A)$) we find $\left(\dfrac{\tau!}{AutA}\dfrac{n!}{\tau!}\right)$. This, after dividing by $n!$, gives the thesis. ∎

Let us go back to the theorem 5.3.2: it is sufficient to prove the theorem when the number of variables is finite, and we will write here the proof for the two variable case. In fact the notation is simpler and the proof in the other cases is completely analogous. As before, looking at the right side of (5.4), we use an algorithm that associates a rooted tree to the monomials obtained by differentiating according to the Leibniz rule.

Let us consider the monomial $P_1^r P_2^s$ on the right side of (5.4). Observe that we can "forget" the coefficient $\frac{1}{2!}$ associated to the variable P_2: in fact it is associated to the variable P_2 also on the left side of (5.4), because of the definition of $\mathcal{Q}$. Let us fix a canonical way to put the variables in a list, for example $\underbrace{P_1 P_1 \cdots P_1}_{\text{r terms}} \cdot \underbrace{P_2 P_2 \cdots P_2}_{\text{s terms}}$ and let us take the $r + s - 1$-th derivative; we can determine each one of the resulting monomials μ by writing as before: $\mu = \mu(a_1, \ldots, a_{r+s-1})$.

We want now to associate a marked forest $G(\mu)$ to μ; this can be done as before, with the further conditions that the vertex we draw when we are differentiating a variable P_2 is labeled with 2 and that we put all the labels of the leaves equal to 1.

Let us fix a monomial $\delta = \delta(a_1, \ldots, a_{r+s-1})$ which is obtained by differentiating $\underbrace{P_1 P_1 \cdots P_1}_{\text{r terms}} \cdot \underbrace{P_2 P_2 \cdots P_2}_{\text{s terms}}$ and let A be the tree obtained by applying our algorithm to δ. Furthermore, let us suppose that, in δ, P_1 appears with exponent g_1 and P_2 with exponent g_2.

Our aim is to count the number of monomials μ, obtained from $\underbrace{P_1 P_1 \cdots P_1}_{\text{r terms}} \cdot \underbrace{P_2 P_2 \cdots P_2}_{\text{s terms}}$, that give rise to a graph automorphic to A.

Then, reasoning as in the proof of Theorem 5.3.3, we immediately see that this number is

$$\frac{(g_1+g_2)!}{AutA} \frac{r!}{g_1!} \frac{s!}{g_2!}$$

Taking into account that on the right side of (5.4) the coefficient of $P_1^r P_2^s$ (up to the coefficient $\frac{1}{2}$ associated to P_2) is $\begin{pmatrix} r+s \\ r \end{pmatrix}$, we find

$$\begin{pmatrix} g_1+g_2 \\ g_1 \end{pmatrix} \frac{1}{AutA} (r+s)!$$

monomials that, according to our algorithm, give rise to a graph equivalent to A.

We have to show that this number, up to multiply by $\frac{1}{(r+s)!}$, is equal to $\sum_{[A]\sim[B]} \frac{1}{Aut(B)}$, where $[A] \sim [B]$ means that $\mathcal{Q}(B) = \delta$ and that $[B]$ may differ from $[A]$ only in the marking of the leaves (so that $Aut(A)$ may be different from $Aut(B)$).

Now, given a certain $[B]$ such that $[A] \sim [B]$, we observe that the number a_B which satisfies $Aut(B) = \frac{Aut(A)}{a_B}$ coincides with the number of possible different ways to obtain from A a tree automorphic to B by substituting g_2 times in A a leaf labeled with 1 with a leaf labeled with 2. This leads us to the following equalities:

$$\sum_{[A]\sim[B]} \frac{1}{Aut(B)} = \sum_{[A]\sim[B]} \frac{a_B}{Aut(A)} = \frac{1}{AutA} \sum_{[A]\sim[B]} a_B$$

But we have

$$\sum_{[A]\sim[B]} a_B = \begin{pmatrix} g_1+g_2 \\ g_1 \end{pmatrix}$$

since the sum on the left counts all the possible ways to substitute g_2 leaves labeled with 1 with leaves labeled with 2.

In conclusion we can write

$$\sum_{[A]\sim[B]} \frac{1}{Aut(B)} = \binom{g_1+g_2}{g_1} \frac{1}{Aut(A)}$$

and this concludes the proof. ■

To sum up, we have obtained the following compact formula for $\mathcal{H}$:

Theorem 5.3.4

$$\mathcal{H} = \left(\sum_{r\geq 0} \frac{S_r}{r!}\right)\left(\sum_{n\geq 0} (1+\Phi)^{(n)} \frac{\Gamma^n}{n!}\right)$$

where Γ *is as in Theorem 5.3.2.*

Remark.

The above formula for $\mathcal{H}$ is direct and easy to be computed. Thus the complexity of the problem lies in the computability of the functions P_m and S_m. If for example we limit ourselves to study elements $w \in \mathcal{S}_n$ the cycles of which have length pairwise coprime, we observe that $S_p(\lambda_1,\ldots,\lambda_p) = \prod_{k=1}^{p} \overline{S}(\lambda_k)$; therefore we do not need to introduce the formal variables S_r for $r > 1$ and in this case we can write

$$\mathcal{H} = e^{S_1}\left(\sum_{n\geq 0} (1+\Phi)^{(n)} \frac{\Gamma^n}{n!}\right)$$

5.4 Some results for B_n arrangements

Let us now focus on a Coxeter arrangement $\mathcal{B}_n$, of type B_n, in $\mathbb{C}^n$. The associated Weyl group W_{B_n} may be viewed as the group of all the permutations and sign changes on the n coordinates in $\mathbb{C}^n$.

Let us consider the associated De Concini-Procesi compact model of irreducibles $\widehat{Y}_{\mathcal{F}_{\mathcal{B}_n}}$ and its cohomology ring $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{B}_n}}, \mathbb{Z})$. The standard Poincarè series for $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{B}_n}}, \mathbb{Z})$ has already been computed in Chapter 3, Section 2.

In this section we are going to find the generalized Poincarè series $P_{w,B}(q,t)$ with respect to any element w of the subgroups $\Gamma_n \subset W_{B_n}$ (for $n \geq 2$) made by permutations. We observe that $\Gamma_n \simeq \mathcal{S}_n$ and that we will automatically find the Poincarè series with respect to the elements of all the subgroups which are conjugate to Γ_n.

Let us then consider $w \in \Gamma_n$, and let us suppose in addition that the permutation w has no fixed points (i.e. n is the minimum integer such that Γ_n contains w). The definition of $P_{w,B}(q,t)$ is

$$P_{w,B}(q,t) = \sum_{m=n}^{\infty} P_{w,\mathcal{B}_m}(q)\frac{t^m}{m!}$$

where $P_{w,\mathcal{B}_m}(q)$ $(m \geq n)$ is the Poincarè polynomial of $\widehat{Y}_{\mathcal{F}_{\mathcal{B}_n}}$ with respect to w. As before, it turns out that it is particularly convenient to search for a "universal" formal graded series $\mathcal{H}_B$.

Therefore our steps will be the following ones: first we will modify the functions $S_m(\lambda_1, \ldots, \lambda_m)$ and $P_r(\lambda_1, \ldots, \lambda_r)$ defined in Section 2; then we will compute the series $\mathcal{H}_B$ which, in the same way as the graded series $\mathcal{H}$ studied in Section 3, after choosing an appropriate homogeneous component and making some symmetric group to act, will provide us the requested generalized Poincarè series.

5.4.1 The symmetric functions $S_{p,B}$, $P_{p,B}$

In what follows we will refer to the properties of the Yuzvinsky bases for $H^*(\widehat{Y}_{\mathcal{F}_{\mathcal{B}_n}}, \mathbb{Z})$ which have been described in Chapter 2, Section 3.

Let us consider an element $w \in \Gamma_n \subset W_{B_n}$ and let $w = c_1 \cdots c_l$ be its decomposition in disjoint cycles of lengths $\lambda_1, \ldots, \lambda_l$ respectively. We can assume, up to conjugation, that w permutes the first $\sum_{j=1}^{l} \lambda_j$ leaves and therefore we can take $n = \sum_{j=1}^{l} \lambda_j$. Let L_j be, as in Section 2 of the present chapter, the set of leaves which are permuted by c_j $(j = 1, \ldots, l)$.

Given a w-invariant element θ and considered its graph, we can extend without changes the definition of singular vertex introduced in Section 2, without distinguishing if such a vertex if weak of strong.

Now, a new situation to be studied is when we have a singular strong vertex v with adjacent sets $L_{i_1}, \ldots, L_{i_p}$. We want to compute, in the spirit of Section 2, the contribution of v to the generalized Poincarè series $P_{w,B}^{(\sum_{j=1}^{l}\lambda_j)}(q,t)$.

We will call by $\widehat{S}_{p,B}(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p})$ such a contribution, while if, in the graph of θ, the sets $L_{i_1}, L_{i_2}, \ldots, L_{i_p}$ are not adjacent to any singular vertex, their contribution to $P_{w,B}^{(\sum_{j=1}^{l}\lambda_j)}(q,t)$ will be denoted by $S_{p,B}(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p})$.

We observe that, given a contraction C

$$C = (\lambda_1, \lambda_2, \ldots, \lambda_p) \xrightarrow{J_1,\ldots,J_k} (\gamma_1, \gamma_2, \ldots, \gamma_k)$$

if some (more than 1) of the sets J_r have cardinality strictly greater than 1, we can find a sequence of consecutive non trivial contractions $C_1, \ldots, C_t$ (if

$t = 1$, then $C = C_1$) such that their composition gives C. We will denote this sequence by

$$C:\ C_1 \to \ldots \to C_t$$

Furthermore, given a contraction K equal to

$$(\beta_1, \beta_2, \ldots, \beta_p) \overset{D_1,\ldots,D_s}{\longrightarrow} (\delta_1, \delta_2, \ldots, \delta_s)$$

we call by $N(K)$ the polynomial defined by

$$N(K) = \prod_{t=1}^{s} \left(\frac{q^{-\delta_t + \sum_{h \in D_t} \beta_h} - q^{\delta_t}}{q^{\delta_t} - 1} \right)$$

This notation allows us to the give the following definition.

Definition 5.4.1

$$\widehat{S}_{p,B}(\lambda_1, \lambda_2, \ldots, \lambda_p) =$$

$$\sum_{C=(\lambda_1,\lambda_2,\ldots,\lambda_p) \overset{J_1,\ldots,J_k}{\longrightarrow} (\gamma_1,\gamma_2,\ldots,\gamma_k)} \sum_{C:\ C_1 \to \ldots \to C_t} \left[\left(\prod_{m=1}^{k} \left(\sum_{\substack{d|\gamma_m \\ d \neq \gamma_m}} q^{\frac{\gamma_m}{d}} T(d, \gamma_m) \prod_{u \in J_m} 2^{\pi(\frac{\lambda_u}{\gamma_m} d)} \right) \right) \prod_{j=1}^{t} N(C_j) \right]$$

$$S_{p,B}(\lambda_1, \lambda_2, \ldots, \lambda_p) =$$

$$\sum_{C=(\lambda_1,\lambda_2,\ldots,\lambda_p) \overset{J_1,\ldots,J_k}{\longrightarrow} (\gamma_1,\gamma_2,\ldots,\gamma_k)} \sum_{C:\ C_1 \to \ldots \to C_t} \left[\left(\prod_{m=1}^{k} \left(\sum_{\substack{d|\gamma_m \\ d \neq \gamma_m}} T(d, \gamma_m) \prod_{u \in J_m} 2^{\pi(\frac{\lambda_u}{\gamma_m} d)} \right) \right) \prod_{j=1}^{t} N(C_j) \right]$$

where the involved sums range over all the possible different contractions, $\pi(n) = 0$ if n is odd, $\pi(n) = 1$ if n is even, and the function $T(d, m)$ is the one defined in Section 2.

We also have to define the functions $P_{p,B}$

Definition 5.4.2

$$P_{p,B}(\lambda_1, \lambda_2, \ldots, \lambda_p) = 2^p \left(\prod_{j=1}^{p} 2^{\pi(\lambda_j)} \right) P_p(\lambda_1, \lambda_2, \ldots, \lambda_p)$$

where P_p is the function defined in Section 2 and $\pi(n) = 0$ if n is odd, $\pi(n) = 1$ if n is even.

We note that the function π allows us to take into account all the possible different w-invariant partitions of the leaves of a weak vertex.

5.4.2 The formal series $\mathcal{H}_B$

We can now define and compute the formal graded series $\mathcal{H}_B$ with respect to the group Γ_n: it is in the formal variables $S_{m,B}$, $\widehat{S}_{r,B}$ $P^{(d)}_{j,B}$ $(j, r, m \geq 1\ d \geq 0)$ and it will be constructed so that it satisfies what follows:

1. The graduation is obtained by giving degree m to the variables $S_{m,B}$ and $\widehat{S}_{m,B}$ and degree j to the variables $P^{(d)}_{j,B}$.

2. Given $w \in \Gamma_n$, and put $w = c_1 \cdots c_l$ its decomposition in cycles of lengths $\lambda_1, \ldots, \lambda_l$ respectively, we take $\mathcal{H}_{l,B}$, that is to say, the homogeneous component of degree l of $\mathcal{H}_B$.

3. Then we create the polynomial $\mathcal{H}_{l,B}(\lambda_1, \ldots, \lambda_l)$, by transforming, in each term of $\mathcal{H}_{l,B}$, the formal variables $S_{m,B}$, $\widehat{S}_{r,B}$ $P^{(d)}_{j,B}$ in their concrete representative, and by making the symmetric group $\mathcal{S}_l$ to act.

4. This must give the requested Poincarè series $P^{(\sum_{j=1}^{l} \lambda_j)}_{w,B}(q, t)$ multiplied by $2^{\sum_{j=1}^{l} \lambda_j}$.

Let us call by $\mathcal{H}^{weak}_B$ the contribution provided to $\mathcal{H}_B$ by the elements the graph of which contains only weak vertices. Reasoning in a similar way as in the preceding sections, we see that $\mathcal{H}^{weak}_B$ must be put equal to

$$\left(\sum_{r \geq 0} \frac{S_{r,B}}{r!} \right) \sum_{k \geq 0} \frac{1}{k!} \left[\sum_{[\widetilde{A}]} (\frac{\lambda}{2})^{(\mathrm{comp}(\widetilde{A}))} \frac{\mathcal{Q}(\widetilde{A})}{Aut(\widetilde{A})} \right]^k$$

where $[\widetilde{A}]$ ranges over all the automorphism classes of oriented labeled forests ($\emptyset$ included), $S_{0,B} = 1$ and the function λ is the one of Section 2.

Now we can substitute the expression in brackets with

$$\Lambda = \sum_{n \geq 0} \frac{\lambda^{(n)}}{2} \frac{\Gamma^n}{n!}$$

where Γ is defined as in Section 3, using $P_{p,B}$ instead of P_p. Then we can write in compact form

$$\mathcal{H}^{weak}_B = \left(\sum_{r \geq 0} \frac{S_{r,B}}{r!} \right) e^{\Lambda}$$

Let us now study how to construct $\mathcal{H}^{strong}_B$, which represents the contribution provided to $\mathcal{H}_B$ by trees with a strong root. First we note that, by elementary combinatorial arguments, if $\mathcal{H}_B^{1\,strong}$ represents the contribution provided by trees with a single strong vertex (i.e. the root) we have

$$\mathcal{H}_B^{strong} = \left(\frac{1}{1 - \mathcal{H}_B^{1\,strong}} - 1 \right)$$

Therefore we need to define a suitable $\mathcal{H}_B^{1\,strong}$: the same considerations as above suggest to us to put

$$\mathcal{H}_B^{1\,strong} = \left(\sum_{r \geq 0} \frac{\widehat{S}_{r,B}}{r!} \right) \sum_{k \geq 2} \frac{(q\Lambda)^k}{k!} + \left(\sum_{r \geq 1} \frac{\widehat{S}_{r,B}}{r!} \right) (q\Lambda + 1)$$

where we put $\widehat{S}_{0,B} = 1$.

At the end, in order to find the formal series $\mathcal{H}_B$, the various contributions that we have described above must be summed according to the following relation:

$$\mathcal{H}_B = \mathcal{H}_B^{weak} + \left(\frac{1}{1 - \mathcal{H}_B^{1\,strong}} - 1 \right)$$

Thus our considerations can be summarized by the following statement.

Theorem 5.4.1 *We have the following formula for the formal graded series $\mathcal{H}_B$:*

$$\mathcal{H}_B = \left(\sum_{r \geq 0} \frac{S_{r,B}}{r!} \right) e^{\Lambda} + \left(\frac{1}{1 - \left[\sum_{k \geq 2} \frac{(q\Lambda)^k}{k!} + e^{q\Lambda} \sum_{r \geq 1} \frac{\widehat{S}_{r,B}}{r!} \right]} - 1 \right)$$

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