Compactifications of configuration spaces

1 Introduction

Let us consider the moduli spaces $M_{0,n+1}$ of $n+1$-pointed curves of genus 0.

Definition 1.1

$$M_{0,n+1} = \left\{ (p_0, \ldots, p_n) \in \underbrace{\mathbf{P}^1 \times \ldots \times \mathbf{P}^1}_{n+1 \ times} \mid p_i \neq p_j \ \forall\, i \neq j \right\} / SL(2, \mathbb{C})$$

where $SL(2,\mathbb{C})$ acts componentwise.

Using $SL(2,\mathbb{C})$ we can put $p_0 = \infty$. Therefore we can represent the elements of $M_{0,n+1}$ in a canonical way as follows:

$$M_{0,n+1} = \{ (p_1, \ldots, p_n) \in \mathbb{C}^n \mid p_i \neq p_j \ \forall\, i \neq j \} / G_0$$

where G_0 is the group of projective transformations which fix ∞, i.e.,

$$G_0 = \{ az + b \mid a \in \mathbb{C}^*,\ b \in \mathbb{C} \}.$$

In these notes we will recall the construction of the Mumford-Deligne compactification $\overline{M}_{0,n+1}$ of $M_{0,n+1}$ using the theory of wonderful models of hyperplane arrangements which De Concini and Procesi developed in [1]. We will focus on the relation between this construction and the compactifications of certain configuration spaces which we find in Section 5 of [5]. The first compactification is constructed assuming the point of view of algebraic geometry, the second concerns differential geometry. In fact, in [5] the following differential manifolds are compactified:

$$C_n = \{ (p_1, \ldots, p_n) \in \mathbb{C}^n \mid p_i \neq p_j \ \forall\, i \neq j \} / G_2$$

$$C_{n,m} = \{ (p_1, \ldots, p_n; q_1, \ldots, q_m) \mid p_i \in \mathcal{H},\ q_j \in \mathbb{R},\ p_i \neq p_j \ \forall\, i \neq j,\ q_s \neq q_t \ \forall\, s \neq t \} / G_1$$

where G_2 is the real Lie group of dimension 3

$$G_2 = \{ az + b \mid a \in \mathbb{R},\ a > 0,\ b \in \mathbb{C} \}$$

G_1 is the real Lie group of dimension 2

$$G_1 = \{ az + b \mid a \in \mathbb{R},\ a > 0,\ b \in \mathbb{R} \}$$

and $\mathcal{H}$ is the Lobacevsky plane.

Sections from 2 to 5 are devoted to the description of the wonderful model which is isomorphic to $\overline{M}_{0,n+1}$ and of a closed embedding of $\overline{M}_{0,n+1}$ into a product of complex projective spaces of dimension 1. If we look at the explicit coordinates of this embedding we can understand in a deeper way the definition of the differential configuration spaces we deal with.

Nevertheless the reader who is only interested in Kontsevich‘s construction of differential configuration spaces can skip these sections and start reading from Section 6.

2 A wonderful model of the braid arrangement

Let us consider $\mathbb{C}^n$ and the braid arrangement, that is to say, the hyperplane arrangement given by the hyperplanes $z_{ij} : x_j - x_i = 0$, where $x_i \in (\mathbb{C}^n)^*$ $(i = 1, \ldots, n)$ are the coordinate functions. We note that the intersection of all the hyperplanes is the subspace $N = \mathbb{C}(\underbrace{1, \ldots, 1}_{n\ times})$. We can thus consider the quotient $V = \mathbb{C}^n/N$ equipped with the arrangement $\mathcal{A}^*_{n-1}$ provided by the images of the hyperplanes z_{ij} via the quotient map $\pi : \mathbb{C}^n \mapsto V$. We can immediately see that $\mathcal{A}^*_{n-1}$ is a root arrangement of type A_{n-1}. We will call by t_{hk} the functionals in V^* the zeroes of which form the hyperplane $\pi(z_{hk})$ in V and such that $(t_{hk}, t_{hk}) = 2$ (where (,) is the scalar product in V^*). Then $\{t_{hk} \mid h, k = 1, \ldots, n\} \cup \{-t_{hk} \mid h, k = 1, \ldots, n\}$ is a root system (which we will denote by $\Phi_{\mathcal{A}_{n-1}}$) of type A_{n-1} and we observe that the set $\{t_{12}, t_{23}, \ldots, t_{(n-1)n}\}$ can be taken as a basis.

As a matter of notation, the set of the subspaces $A \subset V^*$ such that the "perpendicular" subspace $A^\perp$ belongs to $\mathcal{A}^*_{n-1}$ will be called by $\mathcal{A}_{n-1}$. Furthermore, we will denote by by $\mathcal{F}_{\mathcal{A}_{n-1}}$ the set of all the subspaces generated by the irreducible root subsystems of $\Phi_{\mathcal{A}_{n-1}}$. These subspaces can be described by means of a collection of subsets of $\{1, 2, \ldots, n\}$; in fact, given a subset $\overline{\Delta} = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ with $|\overline{\Delta}| \geq 2$, the subspace $\Delta \subset V^*$, generated by all the functionals t_{ij} such that $\{i, j\} \subset \overline{\Delta}$, belongs to $\mathcal{F}_{\mathcal{A}_{n-1}}$ (a basis of $\Delta \cap \Phi_{\mathcal{A}_{n-1}}$ is given by $t_{i_1 i_2}, t_{i_2 i_3}, \ldots, t_{i_{p-1} i_p}$). Furthermore, it can be shown that all the elements of $\mathcal{F}_{\mathcal{A}_{n-1}}$ can be obtained in this way.

Let us now call by ψ the projection map $\psi : V \mapsto \mathbb{P}(V)$ and consider the projectivization of $\mathcal{A}^*_{n-1}$. We call by $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ the complement in $\mathbb{P}(V)$ of the union of the images $\psi(D)$ $(D \in \mathcal{A}^*_{n-1})$.

By construction, given $A \in \mathcal{F}_{\mathcal{A}_{n-1}}$, the rational map

$$\pi_A : V \mapsto V/A^\perp$$

is defined outside $A^{\perp}$ and thus there is a morphism

$$\phi_{\mathcal{F}_{\mathcal{A}_{n-1}}} \ : \ \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \mapsto \prod_{A \in \mathcal{F}_{\mathcal{A}_{n-1}}} \mathbb{P}\left(V/A^{\perp}\right)$$

The graph of $\phi_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is a closed subset of

$$\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \times \prod_{A \in \mathcal{F}_{\mathcal{A}_{n-1}}} \mathbb{P}\left(V/A^{\perp}\right)$$

which embeds as open set into

$$\mathbb{P}(V) \times \prod_{A \in \mathcal{F}_{\mathcal{A}_{n-1}}} \mathbb{P}\left(V/A^{\perp}\right)$$

Finally we have an embedding

$$\widehat{\phi}_{\mathcal{F}_{\mathcal{A}_{n-1}}} \ : \ \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \mapsto \mathbb{P}(V) \times \prod_{A \in \mathcal{F}_{\mathcal{A}_{n-1}}} \mathbb{P}\left(V/A^{\perp}\right)$$

as a locally closed subset. This construction allows us to give the definition:

Definition 2.1 *We denote by* $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ *the closure of* $\widehat{\phi}_{\mathcal{F}_{\mathcal{A}_{n-1}}}(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}})$ *in*

$$\mathbb{P}(V) \times \prod_{A \in \mathcal{F}_{\mathcal{A}_{n-1}}} \mathbb{P}\left(V/A^{\perp}\right)$$

Remark. A similar definition can be given starting by any subspace arrangement $\mathcal{G}^*$ in V. The corresponding variety will be called by $\widehat{Y}_{\mathcal{G}}$.

The variety $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is a "wonderful model" for $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$, in the sense specified by point (1) of the following

Theorem 2.1 *(see [1])*
(1) $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ *is a smooth irreducible variety equipped with a proper map* $\pi : \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} \mapsto \mathbb{P}(V)$ *which is an isomorphism on the preimage of* $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ *and such that the complement of this preimage is a divisor with normal crossings.*
(2) *The complement* $\mathcal{D}$ *of* $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ *in* $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ *is the union of smooth irreducible divisors* D_G *indexed by the elements* $G \in \mathcal{F}_{\mathcal{A}_{n-1}} - \{V^*\}$ *(*D_G *is the*

only irreducible divisor such that $\pi(D_G) = G^{\perp}$*).*
(3) *Let* G *be a minimal (with respect to inclusion) element in* $\mathcal{F}_{\mathcal{A}_{n-1}}$*. If we put* $\mathcal{F}'_{\mathcal{A}_{n-1}} = \mathcal{F}_{\mathcal{A}_{n-1}} - \{G\}$*, and denote by* $\overline{\mathcal{F}}_{\mathcal{A}_{n-1}}$ *the family in* $(V)^*/G$ *given by the elements* $\{A+G/G \,:\, A \in \mathcal{F}'_{\mathcal{A}_{n-1}}\}$*, we have that the varieties* $\widehat{Y}_{\mathcal{F}'_{\mathcal{A}_{n-1}}}$ *and* $\widehat{Y}_{\overline{\mathcal{F}}_{\mathcal{A}_{n-1}}}$ *are wonderful models and that* $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ *can be obtained from* $\widehat{Y}_{\mathcal{F}'_{\mathcal{A}_{n-1}}}$ *by blowing up a subvariety isomorphic to* $\widehat{Y}_{\overline{\mathcal{F}}_{\mathcal{A}_{n-1}}}$*.*

De Concini and Procesi proved this in [1] by using the explicit description of an open affine covering of $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$. We will recall the construction of the above mentioned open charts. Using this construction we will be able to prove in the next section that the model $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is isomorphic to the moduli space of stable $n+1$ pointed curves of genus 0. For this purpose we need to introduce the notion of "nested set" (see [1]) by means of the following definitions. This notion is close to the one introduced by Fulton and MacPherson in their paper [3] on models of configuration spaces.

Definition 2.2 *A set* S *of subspaces in* $\mathcal{F}_{\mathcal{A}_{n-1}}$ *is called nested if it contains* V^* *and, given any its subset* $\{U_1, \ldots, U_k\}$ *of pairwise non comparable elements, one has* $U = U_1 \oplus \cdots \oplus U_k$ *and* $U \notin S$*. If we represent* $\{U_1, \ldots, U_k\}$ *by means of subsets of* $\{1, \ldots, n\}$ *this is equivalent to say that, for every* $i \neq j$*, the subsets representing* U_i *and* U_j *are either disjoint or one included into the other.*

Let now S be a nested set of subspaces in $(V)^*$. For every set $A \subset (V)^*$, $A \neq \{0\}$, the set

$$S^A = \{(V)^*\} \cup \{B \in S \mid A \subset B\}$$

is linearly ordered (with respect to inclusion) and non empty. We let $p_S(A)$ to be the minimum of S^A. We will write $p_S(v)$ instead of $p_S(\mathbb{C}v)$ if v is a vector in $(V)^*$.

Definition 2.3 *A basis* b *of* $(V)^*$ *is called "adapted" to* S *if, for all* $A \in S$*, the set*

$$b_A := b \cap A = \{v \in b | p_S(v) \subset A\}$$

is a basis of A*. A "marking" of a basis* b *adapted to* S *is a choice, for all* $A \in S$*, of an element* $x_A \in b$ *with* $p_S(x_A) = A$*.*

One can easily observe that, given a nested set S, one can always find a basis b adapted to S and a marking for b.

Consider now a space of functions $\mathbb{C}^b$ with coordinates u_x indexed by the elements of b and, given $A \in S$, set $u_A := u_{x_A}$ where $x_A \in b$ is the marked element associated to A. Calling by H^b the affine subspace of $\mathbb{P}(\mathbb{C}^b)$ described by $u_{x_{V^*}} = 1$, we can define a map:

$$\rho_S \,:\, H^b \mapsto \mathbb{P}(\mathbb{C}^b)$$

by means of the following relation:

$$v = u_v \prod_{B \supset A} u_B \quad \text{if } \; A = p(v) \text{ and } \; v \; \text{ is not marked} \tag{1}$$

$$v = \prod_{B \supset A} u_B \qquad \text{if } v = x_A \tag{2}$$

where the elements of b have been chosen as projective coordinates on the target space. Note that the image of ρ_S lies in the affine subspace defined by $x_{V^*} = 1$. This map is easily seen to be birational and, since b is a basis of $(V)^*$, we can consider it as a map

$$\rho_S \,:\, H^b \mapsto \mathbb{P}(V)$$

Proposition 2.2 *The map ρ_S restricts to an isomorphism between the open set where all the coordinates u_A ($A \in S - \{V^*\}$) are different from 0 and the open set where the coordinates $x_A \in b$ are different from 0, and maps the hyperplane defined by $u_A = 0$ in the projectivized subspace corresponding to $A^\perp$.*

If we now consider the variety

$$\widehat{Y}_S \subset \mathbb{P}(V) \times \prod_{A \in S} \mathbb{P}\left(V/A^\perp\right)$$

constructed according to Definition 2.1, we have that

Proposition 2.3 *(see [1]) The map ρ_S lifts to an open embedding of H^b into $\widehat{Y}_S$.*

Proof.
This essentially follows from the fact that the composition of ρ_S with the rational map

$$\widehat{\pi}_A \,:\, \mathbb{P}(V) \mapsto \mathbb{P}\left(V/A^\perp\right) \quad (A \in S)$$

is given by the formulas (1), (2), if we choose on $\mathbb{P}\left(V/A^\perp\right)$ the projective coordinates coming from the basis b_A of A. Thus as monomials in the u_x, these coordinates are all divisible by the monomial expressing x_A; we deduce that the map $\pi_A\,\rho_S$ is a morphism to the affine part $\mathbb{P}^0\left(V/A^\perp\right) \subset \mathbb{P}\left(V/A^\perp\right)$ where $x_A = 1$. We can then form a morphism (again denoted by ρ_S)

$$\rho_S \;:\; H^b \mapsto \mathbb{P}(V) \times \prod_{A\in S} \mathbb{P}\left(V/A^\perp\right)$$

the image of which is easily seen to be equal to the intersection between $\widehat{Y}_S$ and $\mathbb{P}(V) \times \prod_{A\in S} \mathbb{P}^0\left(V/A^\perp\right)$. ■

We will denote by $\mathcal{U}_S^b$ the open set in $\widehat{Y}_S$ provided by the previous proposition and identify with ρ_S the restriction to $\mathcal{U}_S^b$ of the projection from $\widehat{Y}_S$ to $\mathbb{P}(V)$. Moreover we observe (see [1], page 465), that ρ_S depends only on the marked elements of the basis b.

Now, let us describe one possible way to select adapted marked bases for S. Choose for every $B \in S$ a basis $b(B)$ of B made by vectors not contained in any $C \subset B$, $C \in S$. Choose a vector $x_B \in b(B)$ for every $B \in S$. Then these vectors are linearly independent and thus can be completed to a basis b which is adapted to S and in which they are the marked vectors.

If we fix the bases $b(B)$ $(B \in S)$ and perform the above algorithm in all the possible ways (that is to say, if we choose the marked vectors in all the possible ways), we get a family Θ of adapted marked bases. Since the open sets $\mathcal{U}_S^b$ depend only on the marking of the basis b, this gives rise to a finite family $\mathcal{V} = \{\mathcal{U}_S^{b'} \mid b' \in \Theta\}$ of open sets.

Proposition 2.4 *(see [1])*

1. *The variety $\widehat{Y}_S$ is covered by the open sets $\mathcal{U}_S^b$ in the family $\mathcal{V}$.*

2. *Given a minimal element $A \in S$ and put $S' = S - \{A\}$, $\widehat{Y}_S$ is the blow-up of $\widehat{Y}_{S'}$ along the proper transform Z_A of the subspace $A^\perp$ which is a smooth subvariety. Furthermore Z_A is canonically isomorphic to $\widehat{Y}_\Lambda$ where $\Lambda := \{(B + A)/A \in (V)^*/A \mid B \in S'\}$.*

Proof.
We observe that S' is still nested and we want to study the two varieties $\widehat{Y}_S$, $\widehat{Y}_{S'}$. By their very construction, there is a birational morphism $p : \widehat{Y}_S \mapsto \widehat{Y}_{S'}$.

Let us consider a basis b adapted to S and marked. Then b is also adapted to S' and, up to forget the marking on the element $x_A \in b$, is marked for S'. It follows that the map ρ_S equals the composition of p restricted to $\mathcal{U}_S^b$ with $\rho_{S'}$.

We want to explicit the relations between the coordinate charts $\mathcal{U}_S^b$ and $\mathcal{U}_{S'}^b$; in order to do this we will denote by u_v the coordinates in $\mathcal{U}_S^b$ and by u'_v the coordinates in $\mathcal{U}_{S'}^b$. We observe that we have $u_v = u'_v$ if $p_S(v) \neq A$ or $v = x_A$ and $u'_v = u_v u_A$ if $p_S(v) = A$ and $v \neq x_A$.

These are exactly the explicit maps of the blow up of $\mathcal{U}_{S'}^b$ along the subvariety $u'_A = 0, u'_v = 0$ ($p_S(v) = A$ and $v \neq x_A$) in the charts

$$p \,:\, \mathcal{U}_S^b \mapsto \mathcal{U}_{S'}^b$$

In particular, in the case of the claim 2), since A is a minimal element in S, if we start from an adapted basis b of S and we mark it for S' in such a way that no marked vector belongs to A, we can complete the marking to S in $m = dim\, A$ different ways. Let us call b_i ($i = 1, \ldots, m$) the marked bases we get; the associated charts $\mathcal{U}_S^{b_i}$ cover the blow up of $\mathcal{U}_{S'}^b$ along the subspace defined by the equations $u'_v = 0$ ($v \in A$), hence the induced map $\cup_i \mathcal{U}_S^{b_i} \mapsto \mathcal{U}_{S'}^b$ is a proper map.

Now, using the formulas (1) and (2), we can conclude that the variety we blow up in $\mathcal{U}_{S'}^b$ is exactly the proper transform of the subspace $A^\perp$. In fact we have

$$v = u'_v \prod_{B \supsetneq A,\, B \in S'} u'_B$$

for $v \in b_A$ and thus the claim follows dividing by

$$\prod_{B \supsetneq A,\, B \in S'} u'_B$$

These observations allow us to prove the claims 1) and 2) by induction on the cardinality of S. ∎

Let us now focus on the variety $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$. Let us take a nested set S and a marked basis b adapted to it.

Lemma 2.5 *Given any $x \in (V)^* - \{0\}$, suppose $A = p_S(x) \in S$. Then $x = x_A P_x(u)$, where $P_x(u)$ is a polynomial depending only on the variables u_v with v such that $p_S(v) \subseteq A$ and $v \neq x_A$.*

Proof.

Since b_A is a basis of A, we have an expression

$$x = \sum_{v \in b_A} a_v v = x_A (a_{x_A} + \sum_{\substack{v \in b_A \\ v \neq x_A}} a_v \frac{v}{x_A})$$

If we substitute for the v's their expression in terms of the u's provided by the relations (1), (2), we get the requested polynomial $P_x(u)$. ■

Now, given $G \in \mathcal{F}_{\mathcal{A}_{n-1}}$, the previous lemma allows us to define polynomials $P_x^G(u)$, $x \in G$, by the formula $x = x_A P_x^G(u)$.

Let us denote by Z_G the subvariety in H^b defined by the vanishing of these polynomials. Then we observe that we have a regular morphism

$$H^b - Z_G \mapsto \mathbb{P}(V/G^{\perp})$$

Definition 2.4 *Given a nested set S, we define the open set $\mathcal{U}_S^b$ or $\mathcal{U}_S^b(\mathcal{F}_{\mathcal{A}_{n-1}})$ as the complement in H^b of the union of all the varieties Z_G, $G \in \mathcal{F}_{\mathcal{A}_{n-1}}$.*

The open set $\mathcal{U}_S^b$ has been defined in such a way that all the rational morphisms

$$\mathcal{U}_S^b \mapsto \mathbb{P}(V/G^{\perp})$$

are well-defined; therefore we obtain an embedding j_S^b of $\mathcal{U}_S^b$ in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$. By construction, and by the formula $x_A = \prod_{A \subset B} u_B$, we have that the complement in $\mathcal{U}_S^b$ of the divisors $u_A = 0$ ($A \in S$), maps to the open set $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ injectively, while the divisor $u_A = 0$ maps to the projectivization of $A^{\perp}$.

The fact that the maps j_S^b are open embeddings (as S, b vary) easily follows from the diagram

$$\begin{array}{ccc} \mathcal{U}_S^b & \xrightarrow{j_S^b} & \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} \\ {\scriptstyle i}\downarrow & & {\scriptstyle \pi'}\downarrow \\ H^b & \xrightarrow{i_S^b} & \widehat{Y}_S \end{array}$$

since i_S^b is the open embedding of Proposition 2.3 and π' is a birational map. From now on we will identify $\mathcal{U}_S^b$ with its image $j_S^b(\mathcal{U}_S^b)$.

Theorem 2.6 *(for the proof see [1], Theorem 3.1.1) We have $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} = \cup_S \mathcal{U}_S^b$, where S ranges over all the maximal nested sets in $\mathcal{F}_{\mathcal{A}_{n-1}}$. In particular $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is smooth.*

3 A reduced construction

The definition of the De Concini-Procesi model $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ which has been given in Section 1 can be extended to any subspace arrangement represented by a collection of subspaces in V^* which satisfy some combinatorial properties (the "building" properties, see [1]). In some special cases one can see that it is not necessary to embed the complement of the arrangement in the product of the various $\mathbb{P}(V/A^{\perp})$, but it suffices to consider only the $\mathbb{P}(V/B^{\perp})$ with $dim\, B = 2$. In particular this happens when the arrangement we deal with is a root hyperplane arrangement. This includes our case of the braid arrangement.

Theorem 3.1 *The restriction to $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ of the projection*

$$\prod_{A\in\mathcal{F}_{\mathcal{A}_{n-1}}} \mathbb{P}(V/A^{\perp}) \mapsto \prod_{\substack{A \in \mathcal{F}_{\mathcal{A}_{n-1}} \\ dim\, A = 2}} \mathbb{P}(V/A^{\perp})$$

induces a closed embedding

$$\zeta \,:\, \widehat{Y}_{\mathcal{F}} \mapsto \prod_{\substack{A \in \mathcal{F}_{\mathcal{A}_{n-1}} \\ dim\, A = 2}} \mathbb{P}(V/A^{\perp})$$

Proof.

We can prove the theorem using local coordinates. The following lemma allows us to choose a suitable collection of open charts.

Lemma 3.2 *We can choose an open covering $\mathcal{U}$ of $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ made by open charts U_S^b of the following kind: S is a maximal nested set and b is a marked basis, consisting of roots, adapted to S.*

Let us consider $U_S^b \in \mathcal{U}$: we will prove that ζ restricted to U_S^b is an open embedding by showing that there is a local inverse

$$\eta_S^b \,:\, \zeta(U_S^b) \mapsto \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$$

For every $E \in S$, let us call by γ_E the element of b which belongs to $E - \bigcup_{C\in S_E} C$. Moreover, for every $D \in \mathcal{F}_{\mathcal{A}_{n-1}}$, let us call by π_D the projection

$$\pi_D \,:\, \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} \mapsto \mathbb{P}\left(V/D^{\perp}\right)$$

Therefore we can define η^b_S by defining, for every $D \in \mathcal{F}_{\mathcal{A}_{n-1}}$, the morphism $\pi_D \, o \, \eta^b_S$ (note that the case dim $D = 2$ is obvious).

Let us first consider the case $D \in S$. We note that we can find a basis $\gamma_D, \gamma_D - \mu_1, \ldots, \gamma_D - \mu_{\text{dim}D-1}$ of D where the μ_i and the $\gamma_D - \mu_i$ are roots and $(\gamma_D, \mu_i) \neq 0$ for every $i = 1, \ldots, \dim D - 1$. This means that the two dimensional subspaces $< \gamma_D, \mu_i >$ spanned by γ_D and μ_i ($i = 1, \ldots, \text{dim}D - 1$) belong to $\mathcal{F}_{\mathcal{A}_{n-1}}$.

Furthermore, if we take a point $p \in \zeta(U^b_S)$ and denote by $[p_{\gamma_D}, p_{\mu_i}]$ its homogeneous coordinates in $\mathbb{P}\left(V/ < \gamma_D, \mu_i >^{\perp}\right)$ with respect to the basis dual to γ_D, μ_i, we can consider $p_{\gamma_D} = 1$ by the definition of U^b_S.

Now, given $C \in S_D$, we can write

$$\gamma_C = \sum_{r=1}^{\text{dim}D-1} a_r(C)\mu_r$$

for certain scalars $a_r(C)$. Therefore we can define $\pi_D \, o \, \eta^b_S(p)$ giving its projective coordinates in terms of the basis dual to γ_D, γ_C ($C \in S_D$): we put

$$\gamma_D\left(\pi_D \, o \, \eta^b_S(p)\right) = 1$$

and

$$\gamma_C\left(\pi_D \, o \, \eta^b_S(p)\right) = \sum_{r=1}^{\text{dim}D-1} a_r(C)p_{\mu_r}$$

Now it remains the case when $D \notin S$. Keeping the notation of Section 1, we call by $p_S(D)$ the minimal (with respect to inclusion) subspace in S which includes D. As before, we find a basis $\gamma_{p_S(D)}, \mu_1, \ldots, \mu_{\text{dim}p_S(D)-1}$, consisting of roots, of $p_S(D)$ such that $(\gamma_{p_S(D)}, \mu_l) \neq 0$ for every $l = 1, \ldots, \dim p_S(D) - 1$. After choosing a basis $e_1, \ldots, e_{\text{dim}D}$ of D, we can write, for every $j = 1, \ldots, \text{dim}D$

$$e_j = a_0(j)\gamma_{p_S(D)} + \sum_{r=1}^{\text{dim}p_S(D)-1} a_r(j)\mu_r$$

Therefore we can define the projective coordinates of $\pi_D \, o \, \eta^b_S(p)$ (in terms of the basis dual to $e_1, \ldots, e_{\text{dim}D}$) in the following way:

$$e_j\left(\pi_D \, o \, \eta^b_S(p)\right) = a_0(j) + \sum_{r=1}^{\text{dim}p_S(D)-1} a_r(j)p_{\mu_r}$$

(here $[p_{\gamma_{p_S(D)}}, p_{\mu_r}]$ are the homogeneous coordinates in $\mathbb{P}\left(V/ < \gamma_{p_S(D)}, \mu_r >^{\perp}\right)$ with respect to the basis dual to $\gamma_{p_S(D)}, \mu_r$, and we take $p_{\gamma_{p_S(D)}} = 1$). By construction, the above defined map η_S^b is a morphism and it is the inverse of ζ restricted to U_S^b. It follows that ζ restricted to any U_S^b is an isomorphism with its image; therefore $\zeta(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$ is smooth and ζ is an open embedding unless it has an exceptional subvariety, i.e. a subvariety $Z \subset \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ such that codim $Z = 1$ but codim $\zeta(Z) \geq 2$. But since $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is covered by a finite number of coordinate charts such a subvariety cannot exist. ∎

In the next section we will focus on the consequences of Theorem 3.1, showing that, for every integer $n \geq 3$, $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and the moduli space $\overline{M}_{0,n+1}$ of $n+1$-pointed stable curves of genus 0 are isomorphic.

4 The braid arrangement and the moduli space of pointed curves of genus 0

Let us start from a realization of the moduli space $M_{0,n+1}$ of $n+1$-pointed curves of genus 0.

Definition 4.1

$$M_{0,n+1} = \left\{ (p_0, \ldots, p_n) \in \underbrace{\mathbf{P}^1 \times \ldots \times \mathbf{P}^1}_{n+1 \; times} \mid p_i \neq p_j \; \forall \, i \neq j \right\} / SL(2, \mathbb{C})$$

where $SL(2,\mathbb{C})$ acts componentwise.

Given an element $p \in M_{0,n+1}$, after making $SL(2)$ to act, we can canonically write

$$p = [(0,1),\ (1,0),\ (1,1),\ (x_1, y_1), \ldots,\ (x_{n-2}, y_{n-2})]$$

As a matter of notation, here, and everywhere we deal with orbits, the brackets mean: "equivalence class of".

It follows that $M_{0,n+1}$ is in bijective correspondence with the set

$$\widehat{M}_{0,n+1} = \left\{ (q_1, \ldots, q_{n-2}) \in \underbrace{\mathbf{P}^1 \times \ldots \times \mathbf{P}^1}_{n-2 \; times} \mid q_i \neq q_j,\ q_i \neq 1, 0, \infty \right\}$$

Theorem 4.1 *There is a bijective map between $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $\widehat{M}_{0,n+1}$ that gives rise to an isomorphism between $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $M_{0,n+1}$.*

Proof.

Let us choose in V the basis $\{v_2, \dots, v_n\}$ dual to the basis $\{t_{12}, t_{13}, \dots, t_{1n}\}$ of V^*. We note that a set of representatives for the v_j's can be chosen as follows: $v_j = \pi((0, \dots, 0, 1, 0, \dots, 0))$ where the only non zero entry is the j-th.

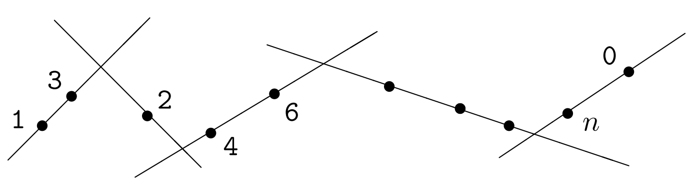
Then we can define a map $\phi : \widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \mapsto \widehat{M}_{0,n+1}$:

$$\phi(\gamma_1, \dots, \gamma_{n-1}) = ((\gamma_1, \gamma_2), (\gamma_1, \gamma_3), \dots, (\gamma_1, \gamma_{n-1}))$$

Note that if $\gamma_j = 0$ for a certain j, then $(\gamma_1, \dots, \gamma_{n-1}) \in H_{1(j+1)}$ and if $\gamma_i = \gamma_j$ $(i < j)$ then $(\gamma_1, \dots, \gamma_{n-1}) \in H_{(i+1)(j+1)}$. This implies that ϕ is well defined. The injectivity is trivial, while the surjectivity is a consequence of the above remarks, a right inverse being given by the map θ such that $\theta((1, r_1), \dots, (1, r_{n-2})) = (1, r_1, \dots, r_{n-2})$. ∎

The above theorem is the reason of the connections between the theory of hyperplane arrangements and the theory of moduli spaces of pointed curves of genus 0. In fact, consider the compactification $\overline{M}_{0,n+1}$ of $M_{0,n+1}$, that is to say, the moduli spaces of stable $n+1$-pointed curves of genus 0. What we want to point out is that the isomorphism of Theorem 4.1 between the open subvarieties $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}} \subset \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and $M_{0,n+1} \subset \overline{M}_{0,n+1}$ can be extended to the boundary, i.e., we have an isomorphism between $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and $\overline{M}_{0,n+1}$.

To prove this, we start by giving a description of the elements of $\overline{M}_{0,n+1}$ as connected tree-like stable $n+1$-pointed curves. This means that we are considering elements of this kind



Here each line represents an irreducible curve of genus 0 (i.e, $\mathbb{P}^1$), every double point represents a point of transversal intersection between the irreducible curves, the other special points, i.e. the punctures, are numbered from 0 to n and the stability is given by the request that the special points (punctures or double points) on each irreducible component are at least 3.

It is well known that there is a morphism

$$\mu_{n+1} : \overline{M}_{0,n+1} \mapsto \overline{M}_{0,n}$$

obtained by forgetting the point labeled with n and, if it is the case, collapsing some irreducible components. At the same way we can construct the maps μ_i which "forget" the point labeled with i ($i = 1, \dots, n-1$).

Let us then call $\overline{M}_{0,ijk}$ ($1 \leq i < j < k \leq n$) the moduli space $\overline{M}_{0,4}$ in which the points are labeled using the numbers i, j, k. A composition of some of the maps μ_i gives a morphism

$$\overline{M}_{0,n+1} \mapsto \overline{M}_{0,ijk}$$

Now the morphism we are interested in is

$$\nu \,:\, \overline{M}_{0,n+1} \mapsto \prod_{\substack{i,j,k \in \{1,\dots,n\} \\ i<j<k}} \overline{M}_{0,ijk}$$

which is given by the above described projections to each component.

Proposition 4.2 *The morphism ν is injective.*

Proof.

First we note that we can reduce ourselves to prove that, for every n, the map

$$\prod_{i=1}^{n} \mu_i \,:\, \overline{M}_{0,n+1} \mapsto \prod_{1 \leq i \leq n} \overline{M}_{0,n}$$

is injective. Therefore we have to check that an element $p \in \overline{M}_{0,n+1}$ is uniquely determined by its image $\prod_i \mu_i(p)$. For this we notice that if there is an irreducible component of p which has at least two marked points (say "i" and "j") different from "0" and at least four special points, p can be determined by knowing $\mu_i(p)$ and $\mu_j(p)$.

Let us now assume that $n \geq 6$ and that p has not irreducible components with the above mentioned properties. Then in every irreducible component of p there are at most two marked points and thus, being $n+1 \geq 7$, there are at least four irreducible components. In particular there are two irreducible components c_1, c_2 of p, which some marked points different from "0" belong to, and such that $c_1 \cap c_2 = \emptyset$. Now, if the point "i" belongs to c_1 and the point "j" belongs to c_2 ($i, j \neq 0$), p is determined by $\mu_i(p)$ and $\mu_j(p)$. Then our claim is proved after a case-by-case check for $3 \leq n \leq 5$. ■

Now we note that in the theory of De Concini - Procesi models we came across a map similar to $\prod_i \mu_i$, namely the map ζ of Theorem 3.1. In fact, given

$$\zeta \,:\, \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} \mapsto \prod_{\substack{A \in \mathcal{F}_{\mathcal{A}_{n-1}} \\ dim\, A = 2}} \mathbb{P}(V/A^{\perp})$$

we observe that the irreducible two dimensional subspaces $A \in \mathcal{F}_{\mathcal{A}_{n-1}}$ can be parametrized, according to the conventions established above, by the triples of integers i, j, k with $1 \leq i < j < k \leq n$. As a matter of notation, we will call $\mathbb{P}_{ijk}$ the projective space $\mathbb{P}(V/A^{\perp})$ when $\overline{A} = \{i, j, k\} \subset \{1, \ldots, n\}$.

Then we want to define in a suitable way an isomorphism

$$\gamma \,:\, \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \overline{M}_{0,ijk} \mapsto \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \mathbb{P}_{ijk}$$

Our request is that γ should be compatible with the isomorphism between the subvarieties $\zeta(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}})$ and $M_{0,n+1}$. Such a γ can be obtained by identifying $\mathbb{P}_{ijk}$ with $\overline{M}_{0,ijk}$ in the following way. Let $p = [(0,1),\,(1,0),\,(1,1),\,(x_3,y_3),\ldots,\,(x_n,y_n)]$ be a point of $M_{0,n+1}$ and let us put, for convenience of notation, $(x_2, y_2) = (1,1)$ and $(x_1, y_1) = (1,0)$.

Then the projection of p to $\overline{M}_{0,ijk}$ is given by $[(0,1),\,(x_i,y_i),\,(x_j,y_j),\,(x_k,y_k)]$ and it can be written in canonical way if we use $SL(2)$ to send (x_i, y_i) to $(1,0)$ and (x_j, y_j) to $(1,1)$ (keeping fixed $(0,1)$). The matrix of $SL(2)$ we use is (up to scalar)

$$\begin{pmatrix} \frac{y_j}{x_j} - \frac{y_i}{x_i} & 0 \\ -\frac{y_i}{x_i} & 1 \end{pmatrix}$$

(note that, for every $i = 1, \ldots, n$, $x_i \neq 0$).

Thus we obtain $\left[(0,1),\,(1,0),\,(1,1),\,(\dfrac{y_j}{x_j} - \dfrac{y_i}{x_i}, \dfrac{y_k}{x_k} - \dfrac{y_i}{x_i})\right]$. If we consider the isomorphism ϕ of Theorem 4.1 between $\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}}$ and $M_{0,n+1}$, we have that $\phi^{-1}(p) = (1, \frac{y_3}{x_3}, \ldots, \frac{y_n}{x_n})$.

Let us now study the projection of $\phi^{-1}(p)$ to $\mathbb{P}_{ijk}$. We recall that if $\overline{A} = \{i, j, k\} \subset \{1, \ldots, n\}$ then $A^{\perp}$ is the $n-3$-dimensional subspace the elements of which have the i-th, j-th and k-th components equal. This means that the projection of $\phi^{-1}(p)$ to $\mathbb{P}_{ijk}$ is $(\frac{y_j}{x_j} - \frac{y_i}{x_i}, \frac{y_k}{x_k} - \frac{y_i}{x_i})$ in the projective coordinates given by the vectors v_j and v_k (note that $k > j \geq 2$).

As a consequence, we can identify $\mathbb{P}_{ijk}$ with $\overline{M}_{0,ijk}$ via γ by choosing in $\mathbb{P}_{ijk}$ the projective coordinates given by v_j and v_k.

Now let us consider the diagram

$$\begin{array}{ccc} \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}} & & \overline{M}_{0,n+1} \\ \zeta\Big\downarrow & & \nu\Big\downarrow \\ \prod_{\substack{i,j,k \in \{1,\dots,n\} \\ i<j<k}} \mathbb{P}_{ijk} & \xleftarrow[\gamma]{\approx} & \prod_{\substack{i,j,k \in \{1,\dots,n\} \\ i<j<k}} \overline{M}_{0,ijk} \end{array}$$

Theorem 4.3 *The above diagram can be completed with an isomorphism* $\Gamma : \overline{M}_{0,n+1} \mapsto \widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$.

Proof

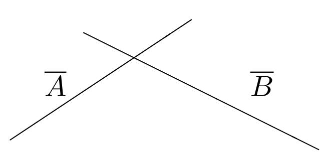
First we note that $\gamma(\nu(M_{0,n+1})) = \zeta(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}})$ and then, since $\gamma(\nu(\overline{M}_{0,n+1}))$ is closed, the closure of $\zeta(\widehat{\mathcal{M}}_{\mathcal{A}_{n-1}})$ is included in $\gamma(\nu(\overline{M}_{0,n+1}))$. But this closure is equal to $\zeta(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$. Since $\zeta(\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}})$ and $\gamma(\nu(\overline{M}_{0,n+1}))$ are closed and contain the same open dense subvariety, they must coincide. Then we observe that the map $\zeta^{-1} \circ \gamma \circ \nu$ is a well defined birational morphism between $\overline{M}_{0,n+1}$ and $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ which is also bijective, since we have proven that it is onto and furthermore ν is injective, γ is bijective and ζ is injective. Since the two varieties are smooth, this implies that $\zeta^{-1} \circ \gamma \circ \nu$ is an isomorphism.

■

5 Divisors in $\overline{M}_{0,n+1}$.

Let us now focus on the map Γ and in particular on the image of the subvarieties in the boundary of $\overline{M}_{0,n+1}$. Recall that an irreducible divisor D in the boundary of $\overline{M}_{0,n+1}$ can be represented (see [4]) by the picture

$D =$



where $\overline{A} \subset \{0,\dots,n\}$ and $\overline{B} = \{0,\dots,n\} - \overline{A}$ satisfy $|\overline{A}| \geq 2$, $|\overline{B}| \geq 2$. The divisor D is the one which contains as an open set the set of all the elements

δ of $\overline{M}_{0,n+1}$ which satisfy the following property: δ has two irreducible components such that the labels of the special points of each component are the elements of $\overline{A}$ and $\overline{B}$ respectively.

Now, given the model $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, let us call by $\hat{\pi}$ its projection to $\mathbb{P}(V)$.

Proposition 5.1 *Given D, $\overline{A}$ and $\overline{B}$ as before, let us suppose that $0 \in \overline{B}$. Furthermore, keeping the notation of Section 2, let us indicate by A the irreducible subspace in V^* associated to $\overline{A}$. Then we have that $\Gamma(D) = D_A$.*

Proof.

Let us consider a chart U_S^b in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, where S is a nested set not containing V^* and $A \in S$. The intersection between D_A and U_S^b is given by the equation $u_A = 0$ (recall that if $A \notin S$ the intersection is empty). Therefore, given an element p in $D_A \cap U_S^b$, it satisfies the following property:

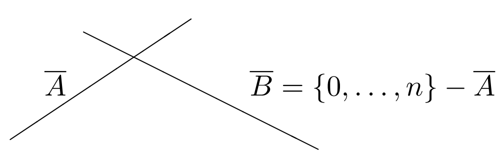
1. Given any triple (i, j, k) with $1 \leq i < j < k \leq n$ and $|\{i, j, k\} \cap \overline{A}| = 2$, the projection p_{ijk} of $\zeta(p)$ to $\mathbb{P}_{ijk}$ is 1,0,∞ when respectively $i \notin \overline{A}$, $j \notin \overline{A}$, $k \notin \overline{A}$.

This follows by construction of the chart U_S^b; let us consider for example the case $i \notin \overline{A}$. The projective coordinates on $\mathbb{P}_{ijk}$ are the ones provided by the basis v_j, v_k of $V/B^\perp$, where $\overline{B} = \{i, j, k\}$. Thus the projection to $V/B^\perp$ of an element $v = x_2 v_2 + \cdots + x_n v_n$ in V is given in coordinates by $(x_j - x_i, x_k - x_i)$ (here we put $x_1 = 0$) and the corresponding projective coordinates in $\mathbb{P}_{ijk}$ are $[x_j - x_i, x_k - x_i]$. But $t_{jk} \in B$ and its expression in terms of the coordinates of U_S^b is a multiple of u_A. Since $B \notin S$ and $D_A \cap U_S^b = \{u_A = 0\}$, given a point $p \in D_A \cap U_S^b$, the projective coordinates $[x_2, \ldots, x_n]$ of $\hat{\pi}(p)$ satisfy $t_{jk}((x_2, \ldots, x_n)) = 0$, that is to say, $x_j = x_k$.

Therefore, since $p_{ijk} = [x_j - x_i, x_k - x_i] \neq [0, 0]$ by construction of U_S^b, we have $p_{ijk} = [1, 1]$. At the same way we can treat the cases $j \notin \overline{A}$, $k \notin \overline{A}$.

Let us now consider the points of the divisor

$D =$



Let $q \in D$ and let us take a triple (i, j, k) with $1 \leq i < j < k \leq n$, $i \notin \overline{A}$ and $\{j, k\} \subset \overline{A}$. Then the projection $\gamma\, o\, \nu(q)_{ijk}$ of $\gamma\, o\, \nu(q)$ to $\mathbb{P}_{ijk}$ is provided by the cross-ratio (p_0, p_i, p_j, p_k) of the special points p_0, p_i, p_j, p_k where we have collapsed the points p_j, p_k to the double point of D. Then this cross ratio is equal to 1. Reasoning in the same way when $j \notin \overline{A}$ and $k \notin \overline{A}$ we can conclude that $\Gamma(q)$ satisfies the property 1.

Furthermore, looking at the tree like representation of an element $z \in \overline{M}_{0,n+1}$ and at the cross ratios (p_0, p_i, p_j, p_k) we immediately see that $\Gamma(z)$ satisfies the property 1 if and only if $z \in D$. Since Γ is an isomorphism this means that the set of elements in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ which satisfy property 1 is exactly $\Gamma(D)$. Therefore $D_A \cap U_S^b \subset \Gamma(D)$ and, since D_A and $\Gamma(D)$ are irreducible divisors in $\widehat{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$, it follows that $\Gamma(D) = D_A$. ■

6 A definition of differential compact configuration spaces

In this section we are going to describe a construction of the differential compactified configuration spaces which appear in [5]. Given two non negative integers n, m satisfying $2n + m \geq 2$ we can consider the quotient space

$$C_{n,m} = \{\, (p_1, \dots, p_n; q_1, \dots, q_m) \mid p_i \in \mathcal{H},\ \ q_j \in \mathbb{R},\ p_i \neq p_j\ \forall\, i \neq j,\ q_s \neq q_t\ \forall\, s \neq t \,\} / G_1$$

where G_1 is the real Lie group of holomorphic transformations which preserve the half-plane and the point ∞:

$$G_1 = \{az + b \mid a \in \mathbb{R},\ a > 0,\ b \in \mathbb{R}\}$$

and $\mathcal{H}$ is the Lobacevsky plane. Note that $C_{n,m}$ is a C^∞ manifold of dimension $2n + m - 2$. At the same way, given $n \geq 2$, we can introduce the C^∞ manifold

$$C_n = \{\, (p_1, \dots, p_n) \in \mathbb{C}^n \mid p_i \neq p_j\ \forall\, i \neq j \,\} / G_2$$

where G_2 is the real Lie group of dimension 3:

$$G_2 = \{az + b \mid a \in \mathbb{R},\ a > 0,\ b \in \mathbb{C}\}\,.$$

Let us now consider the map

$$\phi_{n,m}\ :\ C_{n,m} \mapsto \mathcal{L}_{n,m} = (S^1)^{n(n-1)+nm} \times (\mathbb{P}_{\mathbb{C}})^{\binom{n+m}{3}^3} \times (\mathbb{P}_{\mathbb{C}})^{n(n+m-1)}$$

defined by

$$\phi_{n,m}([(p_1, \dots, p_n; q_1, \dots, q_m)]) =$$

$$= \left(Arg(p_i - p_j),\ Arg(p_i - \overline{p}_j),\ Arg(p_r - q_k),\ \frac{\tau_s - \tau_l}{\tau_t - \tau_l},\ \frac{\gamma_s - p_l}{\gamma_s - \overline{p}_l} \right)$$

where in the formula $i > j$, $s > t$, τ_s, τ_l, τ_t are three distinct points among $\tau_1 = p_1, \dots, \tau_n = p_n, \tau_{n+1} = q_1, \dots, \tau_{n+m} = q_m$ and γ_s is a point among $p_1, \dots, \widehat{p_l}, \dots, p_n, q_1, \dots, q_m$. Of course, in the cases when n, m are small, in the above definiton we use only the coordinates which are well defined (for example, if $n \leq 2$ the quotients $\dfrac{p_s - p_l}{p_t - p_l}$ do not appear). In particular when $n = 1$, $m = 0$ the target space is a point. Analogously, we define the map

$$\phi_n \,:\, C_n \mapsto \mathcal{L}_n = (S^1)^{\binom{n}{2}} \times (\mathbb{P}_{\mathbb{C}})^{\binom{n}{3}3}$$

$$\phi_n([(p_1, \dots, p_n)]) = \left(Arg(p_r - p_k),\ \frac{p_k - p_i}{p_j - p_i} \right)$$

where $r > k$ and $k > j$.

Proposition 6.1 *The maps $\phi_{n,m}$ and ϕ_n are embeddings.*

Proof.

The proof in the two cases is similar: we will focus on $\phi = \phi_n$ for simplicity of notation. Let us call by $S^1(ij)$ the one dimensional torus in $\mathcal{L}_n$ such that the projection of $\phi_n([(p_1, \dots, p_n)])$ to $S^1(ij)$ is $Arg(p_i - p_j)$; at the same way, we denote by $\mathbb{P}_{\mathbb{C}}(ijk)$ the one dimensional complex projective space in $\mathcal{L}_n$ such that the projection of $\phi_n([(p_1, \dots, p_n)])$ to $\mathbb{P}_{\mathbb{C}}(ijk)$ is $\dfrac{p_k - p_i}{p_j - p_i}$. Let then x_{ij} denote the coordinate in $S^1(ij)$ and x_{ijk} the coordinate in $\mathbb{P}_{\mathbb{C}}(ijk)$. First we will show that ϕ is injective. Secondly we will prove that there exists an open set U containing $\phi(C_n)$ in $\mathcal{L}_n$ and a $\mathbb{C}^\infty$ map $\psi \,:\, U \mapsto C_n$ such that $\psi \circ \phi$ is the identity on C_n. This will imply that $d\phi$ is injective on C_n: ϕ is therefore an imbedding.

Furthermore, given an open set $V \subset C_n$, we have that $\phi(V) = \psi^{-1}(V) \cap \phi(C_n)$, that is to say, $\phi(V)$ is an open subset of $\phi(C_n)$ when $\phi(C_n)$ is given the topology induced by $\mathcal{L}_n$. This implies that ϕ is a homeomorphism between C_n and $\phi(C_n)$ with the above mentioned topology, hence an embedding.

Let us now prove the injectivity of ϕ. We use the following description of the moduli space $M_{0,n+1}$:

$$M_{0,n+1} = \{\, (p_1, \dots, p_n) \in \mathbb{C}^n \mid p_i \neq p_j \ \forall\, i \neq j \,\} / G_0$$

where G_0 is the group of projective transformations which fix ∞, i.e.,

$$G_0 = \{az + b \mid a \in \mathbb{C}^*,\ b \in \mathbb{C}\}\,.$$

Then we have shown (see Section 4) that the map

$$\gamma \circ \nu \,:\, M_{0,n+1} \mapsto \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \mathbb{P}_{ijk}$$

given by

$$[(p_1, p_2, \ldots, p_n)] \mapsto \left(\frac{p_k - p_i}{p_j - p_i} \right)$$

is an open embedding.

This means that the composite map

$$\pi_{\mathbb{P}} \circ \phi \,:\, C_n \mapsto \prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \mathbb{P}_{\mathbb{C}}(ijk)$$

where $\pi_{\mathbb{P}}$ is the projection onto $\prod_{\substack{i,j,k \in \{1,\ldots,n\} \\ i<j<k}} \mathbb{P}_{\mathbb{C}}(ijk)$, is not injective (since $G_2 \subset G_0$) and the fiber of a point $\pi_{\mathbb{P}} \circ \phi([(p_1, \ldots, p_n)])$ is given by $e^{i\theta}(p_1, \ldots, p_n)$. Now θ can be determined taking into account the coordinates $Arg(p_i - p_j)$ of the map ϕ, which therefore is proven to be injective.

Let us now construct the inverse to ϕ. We start by considering the open set $U \subset \mathcal{L}_n$,

$$U = \{(x_{ij}, x_{lkr}) \mid x_{12k} \neq 0,\ x_{12k} \neq 1,\ x_{12j} \neq x_{12s} \forall\, k, j, s\}$$

Note that $\phi(C_n) \subset U$; then we define the $\mathbb{C}^{\infty}$ function $\psi \,:\, U \mapsto C_n$ in the following way:

$$\psi((x_{ij}, x_{lkr})) = [(q_1, q_2, \ldots, q_n)]$$

where

- $q_1 = i$
- $q_2 = i + cosx_{12} + isinx_{12}$
- $q_3 = i + (cosx_{12} + isinx_{12})x_{123}$
- ....
- $q_n = i + (cosx_{12} + isinx_{12})x_{12n}$

It is immediate to check that ψ is well defined and that ψ, restricted to $\phi(C_n)$, is the inverse to ϕ. ■

Definition 6.1 *The space $\overline{C}_{n,m}$ (resp. $\overline{C}_n$) is the closure of the image of $\phi_{n,m}$ (resp. ϕ_n) in the target space.*

In the next section, following [5], we will show how to give to the compactifications $\overline{C}_n$ and $\overline{C}_{n,m}$ the structure of smooth manifolds with corners (a manifold with corners of dimension d is defined analogously to a manifold with boundary with the only difference that the manifold is covered locally by open parts of a simplicial cone $(\mathbb{R}_{\geq 0})^d$).

7 Trees and open charts

Let us first describe a continuous section s^{cont} of the natural projection map

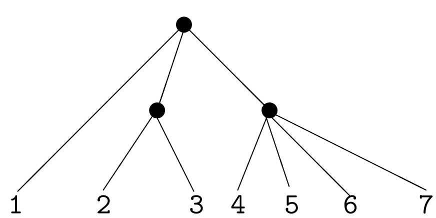
$$Conf_n = \{\, (p_1, \ldots, p_n) \in \mathbb{C}^n \mid p_i \neq p_j \;\forall\, i \neq j \,\} \mapsto C_n$$

Given a point $p = [(p_1, \ldots, p_n)] \in C_n$ we put $s^{cont}(p) = (q_1, \ldots, q_n)$, where $(q_1, \ldots, q_n)$ is in the fiber of p and

1. the diameter of the set $\{q_1, \ldots, q_n\}$ is equal to 1
2. the center of the minimal circle in $\mathbb{C}$ containing $\{q_1, \ldots, q_n\}$ is 0.

We will say that $\{q_1, \ldots, q_n\}$ is a configuration of points "in standard position". In every G_2-orbit in $Conf_n$ there is one and only one point which gives rise to a configuration in standard position.

Let us now introduce a family of open charts in C_n which are parametrized by a family of rooted oriented trees. The trees we are dealing with are all the rooted oriented trees with n leaves labeled with the numbers from 1 to n and such that the number of edges which stem from each vertex (which is not a leaf) is greater than or equal to two. For instance:



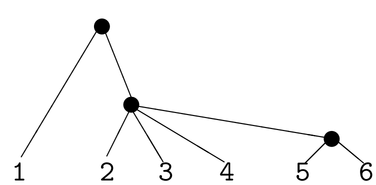
Let us call by T such a tree and denote by $Star(v)$ the set of edges which start from a given vertex v. We can then parametrize an open set U_T in C_n in the following way:

1. for every vertex v of T (except leaves) we provide a configuration c_v of points in standard position labeled by the set $Star(v)$;
2. for every vertex v except leaves and the root of the tree, we provide the scale $s_v > 0$ with which we should put in c_u a copy of c_v instead of the corresponding point p_{uv} (here u is a vertex of T which precedes v in the orientation and uv is the edge which stems from u and ends in v).

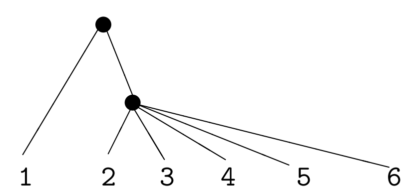
Then we have a continuous atlas $\mathcal{U} = \bigcup_T U_T$ which covers C_n. The compactification $\overline{C}_n$ is achieved by formally allowing some of the scales s_v to be equal to 0. Then $\overline{C}_n$ turns out to be a topological manifold with corners, with strata C_T labeled by the admissible trees T. According to the construction, C_T is isomorphic to the product $\prod_v C_{Star(v)}$, where v ranges over all the vertices of T except leaves.

In order to introduce a smooth structure on $\overline{C}_n$ it is now sufficient to choose, for every $m \leq n$, a smooth section s^{smooth} of the projection $Conf_m \mapsto C_m$ instead of s^{cont}. Then the coordinates near a point in a stratum C_T are given by the scales $s_v \in \mathbb{R}_{\geq 0}$ close to 0 and by the local coordinates in the manifolds $C_{Star(v)}$. Let us show with an example that what we obtain is a compatible family of C^∞ open charts which cover $\overline{C}_n$. Looking at the trees which parametrize the open charts, we note that it is sufficient to prove the compatibility between two charts U_T and U_S when the trees T and S differ only for an elementary ramification. Let us consider for instance:

T



S



Let us choose, for every m, the section s^{smooth} which associates to a point $[(p_1, \ldots, p_m)] \in C_m$ the point $(i, i + e^{i\theta}, p'_3, \ldots, p'_m)$ in its fiber, that is to say, one uses the group G_2 to put $p_1 = 1$ and $|p_2 - p_1| = 1$. Then we have the following open charts:

$$U_T \subset \; C_2 \; \times \mathbb{R}_{\geq 0} \times \qquad C_4 \qquad \times \mathbb{R}_{\geq 0} \times \; C_2 \; \subset \mathbb{R}^9$$

which can be written, using the sections s^{smooth}, in this way:

$$U_T \subset ([-\pi, \pi]) \times \mathbb{R}_{\geq 0} \times ([-\pi, \pi] \times \mathbb{C} \times \mathbb{C}) \times \mathbb{R}_{\geq 0} \times ([-\pi, \pi]) \subset \mathbb{R}^9$$

and

$$U_S \subset \ C_2 \ \times \mathbb{R}_{\geq 0} \times \qquad C_5 \qquad \subset \mathbb{R}^9$$

which can be written as

$$U_S \subset ([-\pi, \pi]) \times \mathbb{R}_{\geq 0} \times ([-\pi, \pi] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}) \subset \mathbb{R}^9$$

The coordinates in the case of U_T are given by

$$(\gamma_2, s, \gamma_4, q_5, q_6, \nu, \gamma_8) \in ([-\pi, \pi]) \times \mathbb{R}_{\geq 0} \times ([-\pi, \pi] \times \mathbb{C} \times \mathbb{C}) \times \mathbb{R}_{\geq 0} \times ([-\pi, \pi])$$

If the corresponding point does not belong to the boundary (i.e., if $s > 0$, $\nu > 0$), it is the following point in C_n:

$$[(i, i+e^{i\gamma_2}+si, i+e^{i\gamma_2}+si+se^{i\gamma_4}, i+e^{i\gamma_4}+sq_5, i+e^{i\gamma_4}+sq_6+s\nu i, i+e^{i\gamma_4}+sq_6+s\nu i+s\nu e^{i\gamma_8})]$$

Analogously, the coordinates in the case of U_S are

$$(\theta_2, t, \theta_4, u_5, u_6, u_7) \in ([-\pi, \pi]) \times \mathbb{R}_{\geq 0} \times ([-\pi, \pi] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C})$$

and, if $t > 0$, the corresponding point in C_n is

$$[(i, i + e^{i\theta_2} + ti, i + e^{i\theta_2} + ti + te^{i\theta_4}, i + e^{i\theta_4} + tu_5, i + e^{i\theta_4} + tu_6, i + e^{i\theta_4} + tu_7)]$$

Therefore we have the following $\mathbb{C}^\infty$ transition function from U_T to U_S:

$$(\gamma_2, s, \gamma_4, q_5, q_6, \nu, \gamma_8) \mapsto (\gamma_2, s, \gamma_4, q_5, q_6 + \nu i, q_6 + \nu i + \nu e^{i\gamma_8})$$

and from U_S to U_T

$$(\theta_2, t, \theta_4, u_5, u_6, u_7) \mapsto (\theta_2, t, \theta_4, u_5, u_6 - |u_7 - u_6|i, |u_7 - u_6|, Arg(u_7 - u_6))$$

Remark. This transition function is smooth since $(\theta_4, u_5, u_6, u_7)$ is a parametrization of a manifold C_5, hence we have $u_6 \neq u_7$.

The case of the manifold $C_{n,m}$ can be treated in a similar way. In [5] the following appropriate new definition of "standard position" for the points belonging to a finite subset S of $\mathcal{H} \cup \mathbb{R}$ is given.

Definition 7.1 *Let S be as above. Then the elements of S are said to be in "standard position" if*

1. *the projection of the convex hull of S to the horizontal line $\mathbb{R}$ is either 0 or an interval with center 0,*
2. *the maximum of the diameter of S and of the distance from S to $\mathbb{R}$ is equal to 1.*

Note that any configuration of n points in $\mathcal{H}$ and m points in $\mathbb{R}$ can be put uniquely in standard position using the group G_1. The trees associated to the strata of $C_{n,m}$ have now two different types of leaves (n leaves corresponding to points in $\mathcal{H}$ and m leaves corresponding to points in $\mathbb{R}$). As a consequence, the strata are isomorphic to the product of manifolds of type C_j and $C_{r,s}$. Then $C_{n,m}$ is given the structure of $\mathbb{C}^\infty$ manifold in the same way explained for C_n.

We can now pass to describe low dimensional spaces $\overline{C}_{n,m}$ and $\overline{C}_n$. It is immediate to check that, by construction, $C_{1,0} = \overline{C}_{1,0}$ is a single point.

The space $C_{0,2} = \overline{C}_{0,2}$ is a two element set (it corresponds to the configuration in standard position provided by the points $+\frac{1}{2}$ and $-\frac{1}{2}$).

The space $C_{1,1}$ is an open interval. In fact, it can be viewed as the subset $(0, \pi)$ of S^1 described by the angle formed with the real line by a rigid edge of lenght 1 which has one vertex on the real line and the other on the half-line $\{\lambda i \,|\lambda > 0\}$. As a consequence, $\overline{C}_{1,1}$ is isomorphic to the segment $[0, \pi] \subset S^1$.

The space $C_{2,0}$ is isomorphich to $\mathcal{H} - \{i\}$ since we can put the first point of the configuration to be equal to i, while C_2 is isomorphic to S^1 (the embedding $\phi_2 : C_2 \mapsto S^1$ is easily seen to be surjective).

Then the manifold $\overline{C}_{2,0}$ can be identified with $\mathcal{H} - \{i\}$ plus two distinct boundary components. The first one is isomorphic to C_2 (hence to S^1) and corresponds to the case when the points p_1 and p_2 of the configuration come close to each other in the half-plane $\mathcal{H}$. The other one is isomorphic to two copies of $\overline{C}_{1,1}$ with the boundary points pairwise identified:

It corresponds to the case when one or both of the points of the configuration come close to the real line. Hence we have the following picture ("The Eye") for $\overline{C}_{2,0}$:

8 Angle maps

In this short section we will give the definition of angle maps and a relevant example. Such maps in [5] constitute an essential ingredient for the construction of the explicit universal formula which gives the star product for arbitrary Poisson structure in an open domain of $\mathbb{R}^n$.

Definition 8.1 *An angle map is a smooth map $\phi : \overline{C}_{2,0} \mapsto S^1$ such that the restriction of ϕ to $C_2 \cong S^1$ is the angle measured in the anti-clockwise direction from the vertical line, and ϕ maps the whole upper interval $\overline{C}_{1,1}$ of $\overline{C}_{2,0}$ to a point in S^1.*

Let us consider a point $[(p,q)] \in C_{2,0}$ and call by $\phi^h : C_{2,0} \mapsto S^1$ the function which associates to $[(p,q)]$ the angle at p formed by the two lines (in the Lobacevsky metric) $l(p,q)$ (passing through p and q) and $l(p,\infty)$ (passing through p and ∞). The angle is measured anti-clockwise from $l(p,\infty)$ to $l(p,q)$. Note that the map is well defind since the value $\phi^h([(p,q)])$ is independent from the representative element in the fiber.

If we consider $C_{2,0}$ immersed in $S^1 \times S^1$ via $\phi_{2,0}$, which sends $[(p,q)]$ in $(Arg(q-p), Arg(q-\overline{p}))$, we see that the map ϕ^h can be written in the following way: $\phi^h(\theta,\mu) = \theta - \mu$. In fact, looking at the picture

we have that the angles $Arg(q-\overline{p}) - \dfrac{\pi}{2} = q\widehat{\overline{p}}p$ and $q\widehat{p}a$ are equal since they determine the same arc on the circumference. Therefore $Arg(q-p) - Arg(q-\overline{p}) = b\widehat{p}q - q\widehat{p}a$ which is equal by definition to $\phi^h([p,q])$.

It follows that the function $\phi^h : C_{2,0} \subset S^1 \times S^1 \mapsto S^1$, $\phi^h(\theta,\mu) = \theta - \mu$, can be defined by continuity also on the whole space $S^1 \times S^1$. In particular, ϕ^h is defined and smooth on $\overline{C}_{2,0}$: if we read the upper interval $\overline{C}_{1,1} \subset \overline{C}_{2,0}$ as the interval where $p \in \mathbb{R}$, it follows that ϕ^h restricted to this upper interval is constant (in fact, in this upper interval we have $(Arg(q-p) = Arg(q-\overline{p}))$.

Reading C_2 as the boundary component in $\overline{C}_{2,0}$ where $q \to p = i$ we note that ϕ^h restricted to $C_2 \cong S^1$ measures the angle in the anticlockwise direction from the vertical line; in fact

$$\phi^h_{|C_2}([p,q]) = \phi^h_{|C_2}(Arg(q-p), \frac{\pi}{2}) = Arg(q-p) - \frac{\pi}{2}$$

Therefore ϕ^h is an angle map: in Section 2 of [5] ϕ^h is used to write explicitely the universal formula in the case $\mathbb{R}^2$.

References

[1] C. De Concini, C. Procesi, Wonderful models of subspace arrangements. Selecta Mathematica, New series, 1 (1995), 459-494.

[2] C. De Concini, C. Procesi, Hyperplane arrangements and holonomy equations. Ibid, 495-536.

[3] W. Fulton, R. MacPherson, A compactification of configuration space. Annals of Math., 139 (1994), 183-225.

[4] S. Keel, Intersection theory of moduli space of stable N-pointed curves of genus 0. T.A.M.S. 330 (1992), 545-574.

[5] M. Kontsevich, Deformation quantization of Poisson manifolds, I. q-alg/9709040.

[6] S. Yuzvinsky, Cohomology bases for De Concini - Procesi models of hyperplane arrangements and sums over trees. Invent. Math. 127 (1997), 319-335.