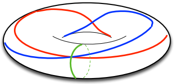
Combinatorics and topology of toric arrangements



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Toblach/Dobbiaco
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The plan

I. **Combinatorics of (toric) arrangements.**

Enumeration and structure theory: posets, polynomials, matroids, semimatroids, and "arithmetic enrichments"

... & questions.

II. **Topology of (toric) arrangements.**

Combinatorial models, minimality, cohomology

... & more questions.

III. **Epilogue: "Equivariant matroid theory".**

... some answers – hopefully – & many more questions.

Cutting a cake

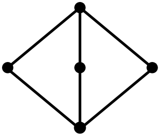
3 “full” cuts.

How many pieces?

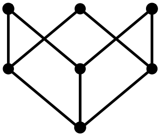
Cutting a cake

6 pieces vs. 7 pieces

Pattern of intersections



vs.



Möbius Functions of posets

Let $\mathcal{P}$ be a locally finite partially ordered set (poset).

The Möbius function of $\mathcal{P}$ is $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$, defined recursively by

$$\begin{cases} \mu(x,y) = 0 & \text{if } x \not\leq y \\ \displaystyle\sum_{x \leq z \leq y} \mu(x,z) = \delta_{x,y} & \text{if } x \leq y \end{cases}$$

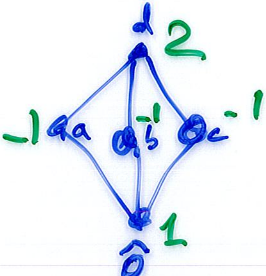
If $\mathcal{P}$ has a minimum $\widehat{0}$ and is ranked*, its characteristic polynomial is

$$\chi_{\mathcal{P}}(t) := \sum_{x \in \mathcal{P}} \mu_{\mathcal{P}}(\widehat{0}, x) t^{\rho(\mathcal{P}) - \rho(x)}$$

---

* i.e., there is $\rho : \mathcal{P} \to \mathbb{N}$ s.t. $\rho(x) =$ length of **any** unrefinable chain from $\widehat{0}$ to x.

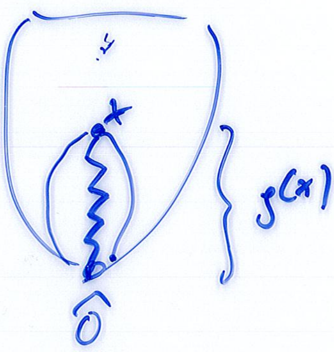
The rank of $\mathcal{P}$ is then $\rho(\mathcal{P}) := \max\{\rho(x) \mid x \in \mathcal{P}\}$



$$\mu(\hat{0},\hat{0}) = 1$$

$$\underbrace{\mu(\hat{0},\hat{0})}_{=1} + \underbrace{\mu(\hat{0},a)}_{-1} = 0$$

$$\underbrace{\mu(\hat{0},\hat{0})}_{=1} + \underbrace{\mu(\hat{0},a)}_{-1} + \underbrace{\mu(\hat{0},b)}_{-1} + \underbrace{\mu(\hat{0},c)}_{-1} + \underbrace{\mu(\hat{0},d)}_{2} = 0$$



$$\chi_P(-1) = \sum \mu(\hat{0},x)(-1)^{\cdots}$$

$$= \sum |\mu(\hat{0},x)| = \underline{\text{nr. of regions}}$$

Topological dissections

Let X be a topological space, $\mathscr{A}$ a finite set of (proper) subspaces of X.

The *dissection* of X by $\mathscr{A}$ gives rise to:

a *poset of intersections*:

$$\mathcal{L}(\mathscr{A}) := \{\cap K \mid K \subseteq \mathscr{A}\} \text{ ordered by reverse inclusion}$$

a *poset of layers* (or connected components of intersections):

$$\mathcal{C}(\mathscr{A}) := \bigcup_{L \in \mathcal{L}(\mathscr{A})} \pi_0(L) \text{ ordered by reverse inclusion.}$$

a collection of *regions*, i.e., the connected components of $X \setminus \cup\mathscr{A}$:

$$\mathcal{R}(\mathscr{A}) := \pi_0(X \setminus \cup\mathscr{A})$$

a collection of *faces*, i.e., regions of dissections induced on intersections.

Zaslavsky's theorem

[Combinatorial analysis of topological dissections, Adv. Math. '77]

Consider the dissection of a topological space X

(connected, Hausdorff, locally compact)

by a family $\mathscr{A}$ of proper subspaces, with $\mathcal{R}(\mathscr{A}) = \{R_1, \ldots, R_m\}$ (finite).

Let $\mathcal{P}$ stand for either $\mathcal{L}(\mathscr{A})$ or $\mathcal{C}(\mathscr{A})$, also assumed to be finite.

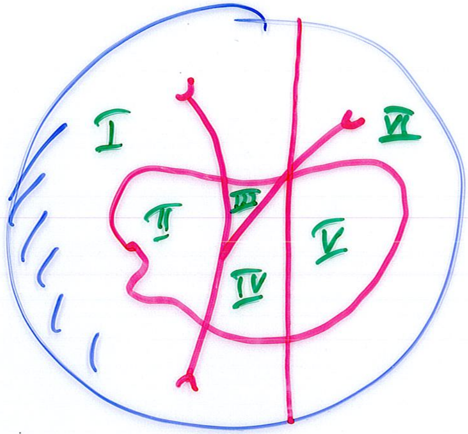
If all faces of this dissection are finite disjoint unions of open balls,

$$\sum_{i=1}^{m} \kappa(R_i) = \sum_{T \in \mathcal{P}} \mu_{\mathcal{P}}(X, T)\kappa(T)$$

where κ denotes the "combinatorial Euler number": $\kappa(T) = \chi(T)$ if T is compact, otherwise $\kappa(T) = \chi(\widehat{T}) - 1$.

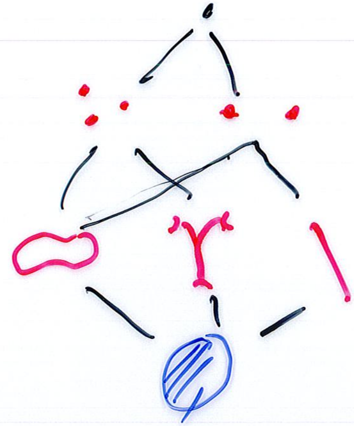
This gives rise to many "region-count formulas".

$X =$

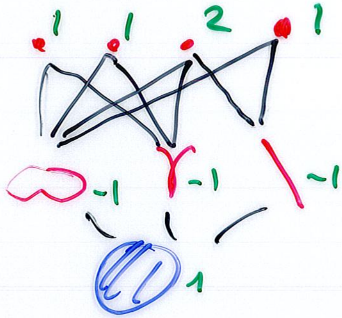


$A = \{ \infty, Y, | \}$

$\mathcal{L}(A)$



$\mathcal{C}(A)$



$\mathcal{R}(A) = \{ I, II, \ldots, VI \}$

$\forall R \in \mathcal{R}(A): \kappa(R) = +1$ ✓

$$\sum_{R \in \mathcal{R}(A)} \kappa(R) = +\binom{\#\text{ of}}{\text{regions}} = \mu(\hat{0},\hat{0})\kappa(\otimes) \quad 1$$

$$\mu(\hat{0},\infty)\kappa(\infty) \quad -1 \cdot 0$$

$$\mu(\hat{0},Y)\kappa(Y) \quad (-1)(-1)$$

$$\mu(\hat{0},\setminus)\kappa(\setminus) \quad (-1)\ 1$$

$$3\ \kappa(\times) \quad 3$$

$$2\ \kappa(\bullet) \quad 2$$

$$6$$ ✓

$\chi(X) = n_0 - n_1 + n_2 \ldots$

Hyperplane arrangements

A hyperplane arrangement in a $\mathbb{K}$-vectorspace V is a locally finite set

$$\mathscr{A} := \{H_i\}_{i \in S}$$

of hyperplanes $H_i = \{v \in V \mid \alpha_i(v) = b_i\}$, where $\alpha_i \in V^*$ and $b_i \in \mathbb{K}$.

The arrangement is called *central* if $b_i = 0$ for all i.

Combinatorial objects

Poset of intersections. $\mathcal{L}(\mathscr{A})$ $(= \mathcal{C}(\mathscr{A}))$

– "Geometry"

Rank function. $\mathrm{rk} : 2^S \to \mathbb{N}$, $\mathrm{rk}(I) := \dim_{\mathbb{K}}(\mathrm{span}\{\alpha_i \mid i \in I\})$

– "Algebra"

CENTRAL EXAMPLE (SAY $\mathbb{K} = \mathbb{R}$)

$$A := [\alpha_1, \alpha_2, \alpha_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad \mathrm{rk}(\emptyset) = 0,\ \mathrm{rk}(I) = \begin{cases} 1 \text{ if } |I| = 1, \\ 2 \text{ if } |I| > 1. \end{cases}$$

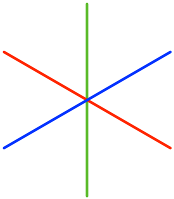
$$(\mathrm{R}) \begin{cases} - \quad I \subseteq J \text{ implies } \mathrm{rk}(I) \leq \mathrm{rk}(J) \\ - \quad \mathrm{rk}(I \cap J) + \mathrm{rk}(I \cup J) \leq \mathrm{rk}(I) + \mathrm{rk}(J) \\ - \quad 0 \leq \mathrm{rk}(I) \leq |I| \\ - \quad \text{For every } I \subseteq S \text{ there is a finite } J \subseteq I \text{ with } \mathrm{rk}(J) = \mathrm{rk}(I) \end{cases}$$

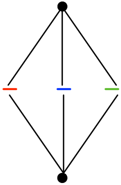
A **matroid** is any function $\mathrm{rk} : 2^S \to \mathbb{N}$ satisfying (R).

Its *characteristic "polynomial"* is $\chi_{\mathrm{rk}}(t) = \sum_{I \subseteq S} (-1)^{|T|} t^{\mathrm{rk}(S) - \mathrm{rk}(I)}$

Central example (say $\mathbb{K} = \mathbb{R}$)

$$A := [\alpha_1, \alpha_2, \alpha_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad \mathrm{rk}(\emptyset) = 0,\ \mathrm{rk}(I) = \begin{cases} 1 \text{ if } |I| = 1, \\ 2 \text{ if } |I| > 1. \end{cases}$$

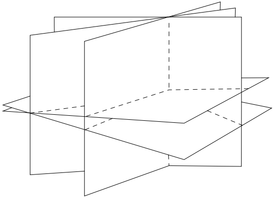
$\mathscr{A}$: 

$\mathcal{L}(\mathscr{A})$: 

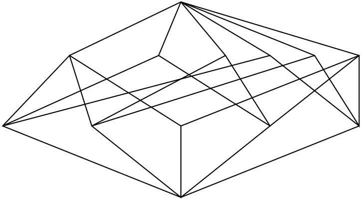
Setting $X_I := \bigcap_{i \in I} H_i$,

$$\mathrm{rk}(I) = \mathrm{codim}(X_I) = \rho(X_I), \text{ the rank function on } \mathcal{L}(\mathscr{A})$$

Central example (say $\mathbb{K} = \mathbb{R}$)



$\mathscr{A}$:



$\mathcal{L}(\mathscr{A})$:

$\mathcal{L}(\mathscr{A})$ is a *lattice* with $\widehat{0} = V$. Moreover,

(G) $x \lessdot y$ if and only if there is an atom p with $p \not\leq x$ and $y = x \vee p$.

A **geometric lattice** is a chain-finite lattice satisfying (G).

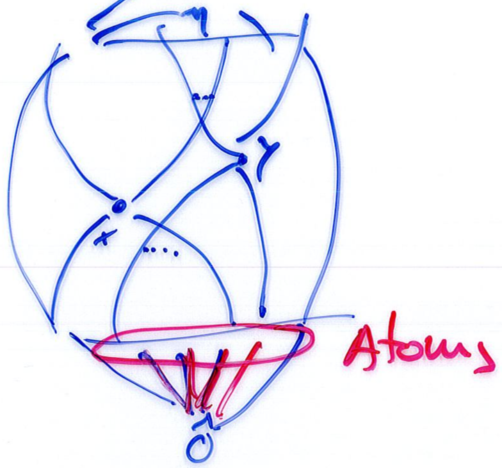
Poset P, $x, y \in P$

$$x \wedge y := \max \left\{ z \in P \;\middle|\; \begin{matrix} z \leq x \\ z \leq y \end{matrix} \right\}$$

"exists" if $|x \wedge y| = 1$

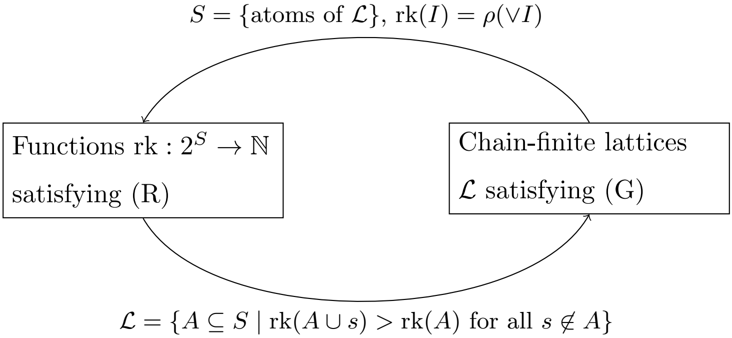
$$x \vee y := \min \left\{ z \in P \;\middle|\; \begin{matrix} z \geq x \\ z \geq y \end{matrix} \right\}$$

"exists" if $|x \vee y|$



If both exist, for all x, y: **P lattice**

Cryptomorphisms



$$\chi_{\mathrm{rk}}(t) \overset{\text{thm.}}{=} \chi_{\mathcal{L}}(t)$$

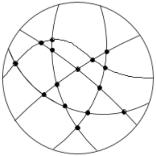
$(S$ finite, $\mathrm{rk} > 0)$

Finite matroids

Rank functions / intersection posets

... of central hyperplane arrangements

Representable m.



Orientable m.

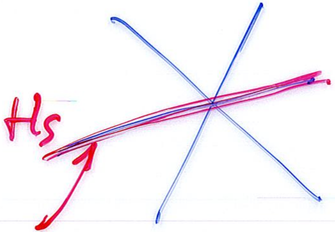
...of pseudosphere arrangements

$|\mathcal{R}(\mathscr{A})| = \chi_{\mathrm{rk}}(-1)$

matroids / geometric lattices

(tropical linear spaces, matroids over the hyperfield $\mathbb{K}$)

Infinite example: set of all subspaces of V.



$\mathcal{A}$

$\mathcal{A} \setminus H_s$

$$\mathcal{A}/H_s := \{ H \cap H_s \mid H \in \mathcal{A} \}$$

$$|\mathcal{R}(\mathcal{A})| = |\mathcal{R}(\mathcal{A} \setminus H_s)| + |\mathcal{R}(\mathcal{A}/H_s)|$$

New matroids from old

Let (S, rk) be a matroid and let $s \in S$

Notice: it could be that $\mathrm{rk}(s) = 0$ – in this case s is called a *loop*.

An *isthmus* is any $s \in S$ with $\mathrm{rk}(I \cup s) = \mathrm{rk}(I \setminus s) + 1$ for all $I \subseteq S$.

The *contraction* of s is the matroid defined by the rank function

$$\mathrm{rk}_{/s} : 2^{S \setminus s} \to \mathbb{N}, \quad \mathrm{rk}_{/s}(I) := \mathrm{rk}(I \cup s) - \mathrm{rk}(s)$$

The *deletion* of s is the matroid defined by the rank function

$$\mathrm{rk}_{\setminus s} : 2^{S \setminus s} \to \mathbb{N}, \quad \mathrm{rk}_{\setminus s}(I) := \mathrm{rk}(I)$$

The *restriction* to s is the one-element matroid given by

$$\mathrm{rk}_{[s]} : 2^{\{s\}} \to \mathbb{N}, \quad \mathrm{rk}_{[s]}(I) = \mathrm{rk}(I).$$

The Tutte polynomial

The Tutte polynomial of a finite matroid (S, rk) is

$$T_{\mathrm{rk}}(x,y) := \sum_{I \subseteq S} (x-1)^{\mathrm{rk}(S)-\mathrm{rk}(I)} (y-1)^{|I|-\mathrm{rk}(I)}$$

(first introduced by W. T. Tutte as the "dichromate" of a graph).

Immediately: $\chi_{\mathrm{rk}}(t) = (-1)^{\mathrm{rk}(S)} T_{\mathrm{rk}}(1-t, 0)$

Deletion - contraction recursion: there are numbers σ, τ s.t.

$$\text{(DC)} \qquad T_{\mathrm{rk}}(x,y) = \begin{cases} T_{\mathrm{rk}_{[s]}}(x,y) T_{\mathrm{rk}_{\setminus s}}(x,y) & \text{if } s \text{ isthmus or loop} \\ \sigma T_{\mathrm{rk}\,/s}(x,y) + \tau T_{\mathrm{rk}\,\setminus s}(x,y) & \text{otherwise.} \end{cases}$$

(in fact, $\sigma = \tau = 1$).

The Tutte polynomial - universality

Let $\mathscr{M}$ be the class of all isomorphism classes of nonempty finite matroids, and R be a commutative ring.

Every function $f : \mathscr{M} \to R$ for which there are $\sigma, \tau \in R$ such that, for every matroid rk on the set $|S| \geq 2$

$$\text{(DC)} \qquad f(\mathrm{rk}) = \begin{cases} f(\mathrm{rk}_{[s]}) f(\mathrm{rk}_{\setminus s}) & \text{if } s \text{ isthmus or loop} \\ \sigma f(\mathrm{rk}\,/s) + \tau f(\mathrm{rk}\setminus s) & \text{otherwise,} \end{cases}$$

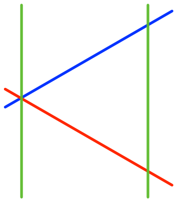
is an evaluation of the Tutte polynomial. [Brylawski '72]

(More precisely, if you really want to know: $f(\mathrm{rk}) = T_{\mathrm{rk}}(f(\mathbf{i}), f(\mathbf{l}))$, where $\mathbf{i}$, resp. $\mathbf{l}$ is the single-isthmus, resp. single-loop, one-element matroid.

AFFINE EXAMPLE ($\mathbb{K} = \mathbb{R}$)

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \ (b_1, b_2, b_3, b_4) = (0, 0, 0, 1)$$

$\mathscr{A}$:



I such that $\cap_{i \in I} H_i \neq \emptyset$

$\{\}, \{1\}, \{2\}, \{3\}, \{4\}$

$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}$

$\{1,2,3\}$

These are the *central sets.*

The family $\mathcal{K}$ is an abstract simplicial complex on the set of vertices S.

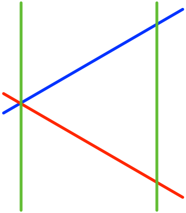
The function $\mathrm{rk} : \mathcal{K} \to \mathbb{N}$, $\mathrm{rk}(I) := \dim \mathrm{span}_{\mathbb{K}}\{\alpha_i \mid i \in I\}$ satisfies [...]

Any such triple $(S, \mathcal{K}, \mathrm{rk})$ is called a **semimatroid**.

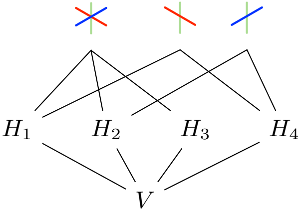
[Kawahara '04, Ardila '07]

Affine example ($\mathbb{K} = \mathbb{R}$)

$\mathscr{A}$:



$\mathcal{L}(\mathscr{A})$:



The poset of intersections $\mathcal{L}(\mathscr{A})$

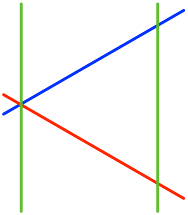
– is *not* a lattice; it is a *meet-semilattice* (i.e., only $x \wedge y$ exists)

– every interval satisfies (G), thus it is ranked by codimension

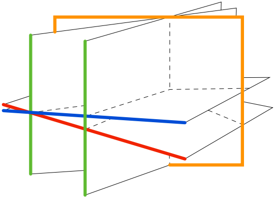
... what kind of posets are these?

Coning

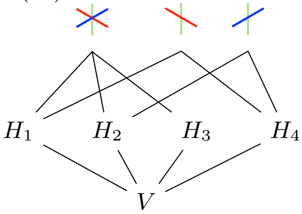
$\mathscr{A}$:



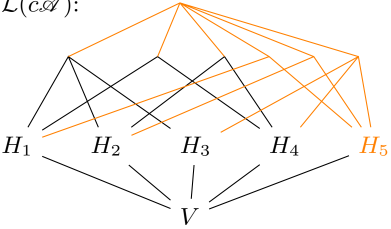
$c\mathscr{A}$:



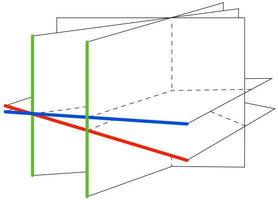
$\mathcal{L}(\mathscr{A})$:



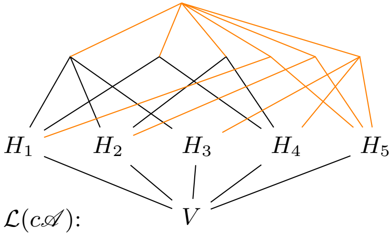
$\mathcal{L}(c\mathscr{A})$:



Geometric semilattices



$c\mathscr{A}$:

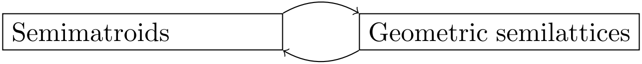


$\mathcal{L}(c\mathscr{A})$:

A **geometric semilattice** is any poset of the form $\mathcal{L}_{\not\geq x}$,

where $\mathcal{L}$ is a geometric lattice and $\widehat{0} \lessdot x$.

Cryptomorphism



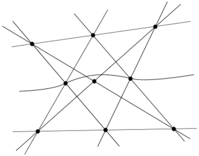
[Wachs-Walker ‘86, Ardila ‘06, D.-Riedel ‘15]

Abstract theory

Semimatroid $(S, \mathcal{K}, \mathrm{rk})$ / intersection posets $\mathcal{L}$

of affine hyperplane arrangements

of "pseudoarrangements"



[Baum-Zhu '15, D.-Knauer '17+]

semimatroids / geometric semilattices

(Q: abstract tropical manifolds?)

Tutte polynomials

If $(S, \mathcal{K}, \mathrm{rk})$ is a finite semimatroid, the associated Tutte polynomial is

$$T_{\mathrm{rk}}(x, y) = \sum_{I \in \mathcal{K}} (x-1)^{\mathrm{rk}(S) - \mathrm{rk}(I)} (y-1)^{|I| - \mathrm{rk}(I)}$$

Exercise: Define contraction and deletion for semimatroids (analogously as for matroids) and prove that $T_{\mathrm{rk}}(x, y)$ satisfies (DC) with $\sigma = \tau = 1$.

[Ardila '07]

Toric arrangements

A toric arrangement in the complex torus $T := (\mathbb{C}^*)^d$ is a set

$$\mathscr{A} := \{K_1, \ldots, K_n\}$$

of 'hypertori' $K_i = \{z \in T \mid z^{a_i} = b_i\}$ with $a_i \in \mathbb{Z}^d \setminus \{0\}$, $b_i \in \mathbb{C}^* / = 1 / \in S^1$

The arrangement is called *centered* if all $b_i = 0$, *complexified* if all $b_i \in S^1$.

For simplicity assume that the matrix $[a_1, \ldots, a_n]$ has rank d.

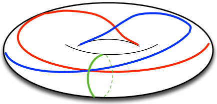
Note: Arrangements in the *discrete torus* $(\mathbb{Z}_q)^d$ or in the *compact torus* $(S^1)^d$ are defined accordingly, by suitably restricting the b_is.

Example: Identify $\mathbb{Z}^d$ with the coroot lattice of a crystallographic Weyl system, and let the a_is denote the vectors corresponding to positive roots.

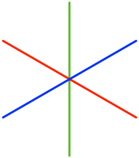
Example - centered, in $(S^1)^2$

$$A := [\alpha_1, \alpha_2, \alpha_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad \mathrm{rk}(\emptyset) = 0,\ \mathrm{rk}(I) = \begin{cases} 1 \text{ if } |I| = 1, \\ 2 \text{ if } |I| > 1. \end{cases}$$

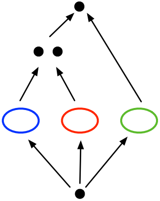
$\mathscr{A}$:



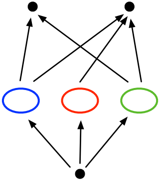
$\mathscr{A}_0$:



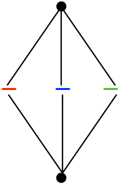
$\mathcal{L}(\mathscr{A})$:



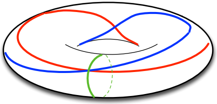
$\mathcal{C}(\mathscr{A})$:

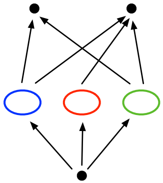


$\mathcal{L}(\mathscr{A}_0)$:



Example - centered, in $(S^1)^2$

$\mathscr{A}$: 

$\mathcal{C}(\mathscr{A})$: 

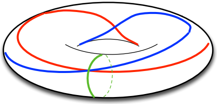
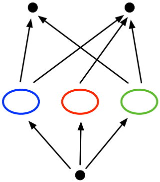
Since A has maximal rank, every region is an open d-ball. Thus

$$\sum_j \kappa(R_j) = \sum_j (-1)^d = (-1)^d |\mathcal{R}(\mathscr{A})|$$

Since $\kappa((S^1)^d) = 0$ for $d > 0$, $\kappa(*) = 1$, and $\mathcal{C}(\mathscr{A})$ is ranked,

$$|\mathcal{R}(\mathscr{A})| = (-1)^d \chi_{\mathcal{C}(\mathscr{A})}(0)$$

Example - centered, in $(S^1)^2$

$\mathscr{A}$:  $\mathcal{C}(\mathscr{A})$: 

What kind of posets are these?

What structural properties do they have?

What natural class of abstract posets do these belong to?

Example - centered, in $(S^1)^2$

$$A := [\alpha_1, \alpha_2, \alpha_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad \mathrm{rk}(\emptyset) = 0,\ \mathrm{rk}(I) = \begin{cases} 1 \text{ if } |I| = 1, \\ 2 \text{ if } |I| > 1. \end{cases}$$

For $I \subseteq [n]$ let

$$m(I) := \text{product of the invariant factors of the matrix } A(I) = [\alpha_i : i \in I],$$

$$\chi_{\mathrm{rk},m}(t) := \sum_{I \subseteq [n]} m(I)(-1)^{|I|} t^{d-\mathrm{rk}(I)}$$

Then,

$$m(I) = |\pi_0(\textstyle\bigcap_{i \in I} K_i)|, \qquad \chi_{\mathrm{rk},m}(t) = \chi_{\mathcal{C}(\mathscr{A})}(t)$$

[Ehrenborg-Readdy-Slone ‘09, Lawrence ‘11, Moci ‘12]

The triple $([n], \mathrm{rk}, m)$ satisfies the axioms of an **arithmetic matroid**

[d’Adderio-Moci ‘13, Brändén-Moci ‘14]

Arithmetic Tutte polynomial

The "arithmetic tutte polynomial" associated to a toric arrangement is

$$T_{\mathrm{rk}\,.m}(x,y) := \sum_{I\subseteq S} m(I)(x-1)^{\mathrm{rk}(S)-\mathrm{rk}(I)}(y-1)^{|I|-\mathrm{rk}(I)}$$

[Moci '12]

Immediately: $\chi_{\mathrm{rk},m}(t) = (-1)^{\mathrm{rk}(S)} T_{\mathrm{rk},m}(1-t,0)$. Also:

$$\text{(NRDC)}\quad T_{\mathrm{rk}}(x,y) = \begin{cases} (x-1)T_{\mathrm{rk}_{\setminus s},m_{\setminus s}}(x,y) + T_{\mathrm{rk}_{/s},m_{/s}}(x,y) & s \text{ isthmus} \\ T_{\mathrm{rk}_{\setminus s},m_{\setminus s}}(x,y) + (y-1)T_{\mathrm{rk}_{\setminus s}}(x,y) & s \text{ loop} \\ T_{\mathrm{rk}_{/s},m_{/s}}(x,y) + T_{\mathrm{rk}_{\setminus s},m_{\setminus s}}(x,y) & \text{otherwise.} \end{cases}$$

[d'Adderio-Moci '13]

(NRDC) holds whenever $([n],\mathrm{rk},m)$ is an **arithmetic matroid**

[d'Adderio-Moci '13, Brändén-Moci '14]

Abstract theory?

– Arithmetic matroids

axioms for (S, rk, m) with

- a duality theory,
- a "Tutte" polynomial $T_A(x, y)$ satisfying NRDC
- No cryptomorphisms
- No natural nonrealizable examples

– Matroids over rings [Fink-Moci '15]

Axioms for $\{\mathbb{Z}^d/\langle\alpha_i\rangle_{i\in I}\}_{I\subseteq[n]}$

("even more algebraic")

– $\chi_{\mathcal{C}(\mathscr{A})}(t)$ enumerates points/faces in the compact and discrete torus. [Lawrence '08 ans '11, E-R-S '09]

– "**ab**/**cd** index" for $\mathcal{C}(\mathscr{A})$ [Ehrenborg-Readdy-Slone '09]

– $\mathcal{C}(\mathscr{A})$ via "marked" partitions for

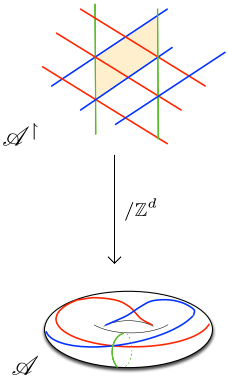
- $\mathscr{A}$ "graphical" [Aguiar-Chan]
- $\mathscr{A}$ from root system [Bibby '16], shellable in type ABC [Girard '17+]

- No abstract characterization

(More about arithmetic matroids on Friday)

Towards a comprehensive abstract theory

Ansatz: "periodic arrangements"



$\mathcal{L}(\mathscr{A}^\upharpoonright)$

Poset of intersections

$\downarrow$ $/\mathbb{Z}^d$(as posets)

$\mathcal{C}(\mathscr{A})$

$\mathcal{F}(\mathscr{A}^\upharpoonright)$

Poset (category) of polyhedral faces

$\downarrow$ $/\mathbb{Z}^d$ (as acyclic categories)

$\mathcal{F}(\mathscr{A})$

Characterize axiomatically the involved posets and the group actions.

Let $A = [a_1, \ldots, a_n] \in M_{d\times n}(\mathbb{Z})$

(Central) hyperplane arrangement

$$\lambda_i : \mathbb{C}^d \to \mathbb{C}, \quad \underline{z} \mapsto \sum_j a_{ji} z_j$$

$H_i := \ker \lambda_i$

$\mathscr{A} = \{H_1, \ldots, H_n\}$

$M(\mathscr{A}) := \mathbb{C}^d \setminus \cup\mathscr{A}$

(Centered) toric arrangement

$$\lambda_i : (\mathbb{C}^*)^d \to \mathbb{C}^*, \quad \underline{z} \mapsto \prod_j z_j^{a_{ji}}$$

$K_i := \ker \lambda_i$

$\mathscr{A} = \{K_1, \ldots, K_n\}$

$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \cup\mathscr{A}$

(Centered) elliptic arrangement

$$\lambda_i : \mathbb{E}^d \to \mathbb{E}, \quad \underline{z} \mapsto \sum_j a_{ji} z_j$$

$L_i := \ker \lambda_i$

$\mathscr{A} = \{L_1, \ldots, L_n\}$

$M(\mathscr{A}) := \mathbb{E}^d \setminus \cup\mathscr{A}$