A tropical approach to a generalized Hodge conjecture for positive currents

Farhad Babaee

SNSF/Université de Fribourg

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Are all positive currents with Hodge classes approximable by positive sums of integration currents? (Demailly 1982)
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No! (Joint work with June Huh)
$X$ complex smooth manifold of complex dimension $n$. 

- $\mathcal{D}^k(X) :=$ Space of smooth differential forms of degree $k$, with compact support $=$ test forms
- $\mathcal{D}'^k(X) =$ Space of currents of dimension $k$ $:=$ Topological dual to $\mathcal{D}^k(X)$
- $\langle T, \varphi \rangle \in \mathbb{C}$ (linear continuous action)
- $T \in \mathcal{D}'^k(X)$ current is closed ($= d$-closed), $\langle dT, \varphi \rangle := (-1)^{k+1}\langle T, d\varphi \rangle = 0$, $\forall \varphi \in \mathcal{D}^{k-1}(X)$
• $\mathcal{D}^{p,q}(X)$: Smooth $(p, q)$-forms with compact support

• $\mathcal{D}_{p,q}'(X) := (\mathcal{D}^{p,q}(X))'$

• For currents $(p, q)$-bidimension $= (n - p, n - q)$-bidegree
• $\mathcal{D}^{p,q}(X)$: Smooth $(p, q)$-forms with compact support
• $\mathcal{D}'_{p,q}(X) := (\mathcal{D}^{p,q}(X))'$
• For currents $(p, q)$-bidimension $= (n - p, n - q)$-bidegree
• $T_j \rightarrow T$ in weak limit, if $\langle T_j, \varphi \rangle \rightarrow \langle T, \varphi \rangle \in \mathbb{C}$
Example
Let $Z \subset X$ a smooth submanifold of dimension $p$, define the *integration current along* $Z$, denoted by $[Z] \in D'_p(X)$

$$\langle [Z], \varphi \rangle := \int_Z \varphi, \quad \varphi \in D^{p,p}(X).$$

This definition extends to analytic subsets $Z$, by integrating over the smooth locus.
Definition
A smooth differential $(p, p)$-form $\varphi$ is positive if $\varphi(x)|_S$ is a nonnegative volume form for all $p$-planes $S \subset T_xX$ and $x \in X$.

Definition
A current $T \in \mathcal{D}'_{p,p}(X)$ is called positive if
\[ \langle T, \varphi \rangle \geq 0 \]
for every positive test form $\varphi \in \mathcal{D}_{p,p}(X)$.
Examples of positive currents

- An integration current on an analytic subset is a positive current, with support equal to $Z$
- Convex sum of positive currents
The generalized Hodge conjecture for positive currents \((HC^+)\)

**Question/Conjecture:** Are all the positive closed currents approximable by a convex sum of integration currents along analytic cycles?

\[ \mathcal{I}^+ \leftarrow \sum_{i} \lambda_{ij}^+ [Z_{ij}], \]

Demailly, the superhero, 1982: True for \(p = 0, n-1, n\).
The generalized Hodge conjecture for positive currents \((HC^+)\)

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\[
\mathcal{I}^+ \leftarrow \sum_i \sum_j \lambda_{ij}^+[Z_{ij}],
\]

On a smooth projective variety \(X\), and

\[
\{ \mathcal{I}^+ \} \in \mathbb{R} \otimes_{\mathbb{Z}} \left( H^{2q}(X, \mathbb{Z})/\text{tors} \cap H^{q,q}(X) \right),
\]

where \(q = n - p\).
The generalized Hodge conjecture for positive currents 
\((HC^+ )\)

**Question/Conjecture:** Are all the positive closed currents approximable by a convex sum of integration currents along analytic cycles?

\[
\mathcal{T}^+ \leftarrow \sum_j \lambda_{ij}^+ [Z_{ij}],
\]

On a smooth projective variety \(X\), and

\[
\{ \mathcal{T}^+ \} \in \mathbb{R} \otimes_{\mathbb{Z}} (\mathcal{H}^{2q}(X, \mathbb{Z})/\text{tors} \cap \mathcal{H}^{q,q}(X)),
\]

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The Hodge conjecture (HC)

The Hodge conjecture: The group

\[ \mathbb{Q} \otimes \mathbb{Z} \left( H^{2q}(X, \mathbb{Z})/\text{tors} \cap H^{q,q}(X) \right), \]

consists of classes of \( p \)-dimensional algebraic cycles with rational coefficients.

Demailly 1982: \( HC^+ \implies HC \).
Hodge conjecture for real currents (HC')

If $\mathcal{T}$ is a $(p, p)$-dimensional real closed current on $X$ with cohomology class

$$\{ \mathcal{T} \} \in \mathbb{R} \otimes \mathbb{Z} \left( H^{2q}(X, \mathbb{Z})/\text{tors} \cap H^{q,q}(X) \right),$$

then $\mathcal{T}$ is a weak limit of the form

$$\mathcal{T} \leftarrow \sum_{i,j} \lambda_{ij} [Z_{ij}],$$

where $\lambda_{ij}$ are real numbers and $Z_{ij}$ are $p$-dimensional subvarieties of $X$.

Demailly 2012: HC’ $\iff$ HC
Theorem (B - Huh)

There is a 4-dimensional smooth projective toric variety $X$ and a $(2, 2)$-dimensional positive closed current $\mathcal{T}^+$ on $X$ with the following properties:

1. The cohomology class of $\mathcal{T}^+$ satisfies
   \[ \{ \mathcal{T}^+ \} \in H^4(X, \mathbb{Z})/\text{tors} \cap H^{2,2}(X). \]

2. The current $\mathcal{T}^+$ is not a weak limit of the form
   \[ \mathcal{T}^+ \leftarrow \sum_i \lambda_{ij}^+ [Z_{ij}], \]
   where $\lambda_{ij}^+ > 0$, $Z_{ij}$ are algebraic surfaces in $X$. 

HC$^+$ not true in general!
Theorem (B - Huh)

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HC$^+$ not true in general!
Extremality in the cone of closed positive currents

Definition
A \((p, p)\)-closed positive current \(T\) is called extremal if for any decomposition \(T = T_1 + T_2\), there exist \(\lambda_1, \lambda_2 \geq 0\) such that \(T = \lambda_1 T_1\) and \(T = \lambda_2 T_2\). (\(T_i\) closed, positive and same bidimension).
Extremality reduces the problem to sequences

**Lemma**

$X$ an algebraic variety, $\mathcal{T}^+$ be a $(p, p)$-dimensional current on $X$ of the form

$$\mathcal{T}^+ \leftarrow_i \sum_j \lambda^+_{ij}[Z_{ij}],$$

where $\lambda^+_{ij} > 0$, $Z_{ij}$ are $p$-dimensional irreducible analytic subsets of $X$. If $\mathcal{T}$ is extremal then

$$\mathcal{T}^+ \leftarrow_i \lambda^+_i[Z_i].$$

for some $\lambda^+_i > 0$ and $Z_i$ irreducible analytic sets.
Proposition

Let \( \{ \mathcal{T} \} \) be a \((2,2)\) cohomology class on the 4 dimensional smooth projective toric variety \( X \). If there are nonnegative real numbers \( \lambda_i \) and 2-dimensional irreducible subvarieties \( Z_i \subset X \) such that

\[
\{ \mathcal{T} \} = \lim_{i \to \infty} \{ \lambda_i [Z_i] \},
\]

then the matrix

\[
[L_{ij}] \{ \mathcal{T} \} = -\{ \mathcal{T} \} \cdot D_{\rho_i} \cdot D_{\rho_j},
\]

has at most one negative eigenvalue.
Our goal

A $(2, 2)$-current on a 4-dimensional smooth projective toric variety which is

- Closed
- Positive
- Extremal, and
- Its intersection form has more than one negative eigenvalues
Tropical currents

\[ \text{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n \]
\[ (z_1, \ldots, z_n) \mapsto (-\log |z_1|, \ldots, -\log |z_n|) \]

- \( \text{Log}^{-1}(\{pt\}) \simeq (S^1)^n \),
- \( \dim_{\mathbb{R}} \text{Log}^{-1}(\text{rational p-plane}) = n + p \)
- \( \text{Log}^{-1}(\text{rational p-plane}) \) has a natural fibration over \( (S^1)^{n-p} \) with fibers of complex dimension \( p \)
- Similarly for any \( p \)-cell \( \sigma \), \( \text{Log}^{-1}(\sigma) \) has a natural fibration over \( (S^1)^{n-p} \)
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- Similarly for any \(p\)-cell \(\sigma\), \(\text{Log}^{-1}(\sigma)\) has a natural fibration over \((S^1)^{n-p}\)
\( n = 2, \ p = 1 \)

\[ \mathbb{R}^2 \longrightarrow (\mathbb{C}^\times)^2 \]

Support \( \mathcal{F}_\mathcal{C} = \text{Log}^{-1}(\mathcal{C}), \ \mathcal{F}_\mathcal{C} = \sum_\sigma w_\sigma \int_{S^{n-p}} [\text{fibers of } \text{Log}^{-1}(\sigma)] \ d\mu \)
\( \mathcal{C} \subset \mathbb{R}^n, \dim(\mathcal{C}) = p \) \( \mathcal{T}_\mathcal{C} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n) \), Support \( \mathcal{T}_\mathcal{C} = \log^{-1}(\mathcal{C}) \)

\( \{ \overline{\mathcal{T}}_\mathcal{C} \} = \text{rec}(\mathcal{C}) \in H^{n-p,n-p}(X_\Sigma) \) \( \overline{\mathcal{T}}_\mathcal{C} \in \mathcal{D}'_{p,p}(X_\Sigma) \)
\( \mathcal{C} \subset \mathbb{R}^n, \dim(\mathcal{C}) = p \quad \mathcal{T}_\mathcal{C} \in \mathcal{D}_{p,p}^\prime((\mathbb{C}^*)^n), \text{Support } \mathcal{T}_\mathcal{C} = \log^{-1}(\mathcal{C}) \)

\[ \{ \overline{\mathcal{T}}_\mathcal{C} \} = \text{rec}(\mathcal{C}) \in H^{n-p,n-p}(X_\Sigma) \quad \overline{\mathcal{T}}_\mathcal{C} \in \mathcal{D}_{p,p}^\prime(X_\Sigma) \]
A $(2, 2)$-current on a 4-dimensional smooth projective toric variety which is

- Closed
  - Balanced complex
- Positive
  - Positive weights
- Extremal
  - ?
- Its intersection form has more than one negative eigenvalues
  - ?
Extremality of tropical currents in any dimension/codimension

Weights unique up to a multiple + Not contained in any proper affine subspace
Examples of extremal currents

Lelong 1973: Integration currents along irreducible analytic subsets are extremal. Is that all?
Demailly 1982: $i\pi \partial \bar{\partial} \log \max\{|z_0|, |z_1|, |z_2|\}$ is extremal on $\mathbb{P}^2$, and its support has real dimension 3, thus cannot be an integration current along any analytic set.

Dynamical systems (usually with fractal supports, thus non-analytic):
**Higher Codimension:** Dinh and Sibony 2005, Guedj 2005, Dinh and Sibony 2013

Complicated structures, easily seen to be approximable!
Extremal if: weights unique up to a multiple \( + \) Not contained in any proper affine subspace
Manipulation of signatures for 2-cells in dimension 4

The operation $F \mapsto -F_{ij}$ produces one new positive and one new negative eigenvalue for its intersection matrix.

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A $(2, 2)$-current on a 4-dimensional smooth projective toric variety which is

- Closed
  - Balanced complex
- Positive
  - Positive weights
- Extremal
  - Non-degenerate + weights unique up to a multiple
- Its intersection form has more than one negative eigenvalues
  - The operation on two cells provides one new negative and one new positive eigenvalue
A concrete example

Consider \( G \subseteq \mathbb{R}^4 \setminus \{0\} \)

where \( e_1, e_2, e_3, e_4 \) are the standard basis vectors of \( \mathbb{R}^4 \) and \( f_1, f_2, f_3, f_4 \) the rows of

\[
M := \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 \\
1 & 1 & 0 & -1 \\
1 & -1 & 1 & 0
\end{pmatrix}.
\]

The weights of solid (resp. dashed) edges are +1 (resp. −1).
Thank you for your attention, indeed!