An application of Pieri's rule

In Lesson 1A we gave the following statement for Pieri's rule:

Theorem Let λ be a partition of n.

Then
$$\operatorname{Ind}_{S_n}^{S_{n+1}} V_\lambda = \sum_{\nu} V_\nu$$

where ν varies among all the partitions of $n+1$ obtained adding a square to the Young diagram of λ.

Actually the following more general statement is still called Pieri's rule (and is a particular case of Littlewood-Richardson rule):

Theorem Let λ be a partition of n.

Then
$$\operatorname{Ind}_{S_n \times S_m}^{S_{n+m}} \Big(V_\lambda \times \overbrace{\boxed{}}^{m} \Big) = \sum_{\nu} V_\nu$$

where ν varies among all the partitions of $n+m$ obtained adding m squares to the Young diagram of λ, NO TWO BOXES IN THE SAME COLUMN.

Lemma (Hemmer 2011)

Let V be a representation of S_n, and let $m \geq n$. Let us consider the decomposition:

$$\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} \left(V \otimes \boxed{} \right) = \sum_{\substack{\tau \text{ partition} \\ \text{of } n+m}} d_\tau V_\tau$$

Then

$$\mathrm{Ind}_{S_n \times S_{m+1}}^{S_{n+m+1}} \left(V \otimes \boxed{}\boxed{} \right) = \sum_{\substack{\tau \text{ partition} \\ \text{of } n+m}} d_\tau V_{\tilde{\tau}}$$

where $\tilde{\tau}$ is obtained from τ by adding a box in the first row.

<u>Proof</u> This is an application of Pieri's rule, since $m \geq n$ implies that the only way to add $m+1$ boxes (no two in the same column) is to add one box in the first row of a diagram where m boxes have already been added. $\square$

Theorem (Hemmer 2011).

Let $H \leq S_n$ and let V be a representation of H.

Suppose $m \geq n$ and consider the decomposition

$$\mathrm{Ind}_{H\times S_m}^{S_{n+m}}\left(V\otimes \boxed{}\right) = \sum_{\tau \text{ part. of } n+m} d_\tau V_\tau$$

Then

$$\mathrm{Ind}_{H\times S_{m+1}}^{S_{n+m+1}}\left(V\otimes \boxed{}\!\boxed{}\right) \doteq \sum_{\tau \text{ part. of } n+m} d_\tau V_{\tilde{\tau}}$$

where the notation for $\tilde{\tau}$ is as before.

Proof We notice that:

$$\mathrm{Ind}_{H\times S_m}^{S_{n+m}}\left(V\times \boxed{}\right) \simeq \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}\left(\mathrm{Ind}_{H\times S_m}^{S_n\times S_m} V\times \boxed{}\right)$$

$$= \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}\left(\left(\mathrm{Ind}_{H}^{S_n} V\right)\times \boxed{}\right).$$

At the same way,

$$\mathrm{Ind}_{H\times S_{m+1}}^{S_{n+m+1}}\left(V\otimes \boxed{}\right) = \mathrm{Ind}_{S_n\times S_{m+1}}^{S_{n+m+1}}\left(\mathrm{Ind}_{H}^{S_n} V\otimes \boxed{}\right).$$

Now we can apply the lemma.

This means that if we consider $H < S_k$ and V a representation of H, the sequence of representations

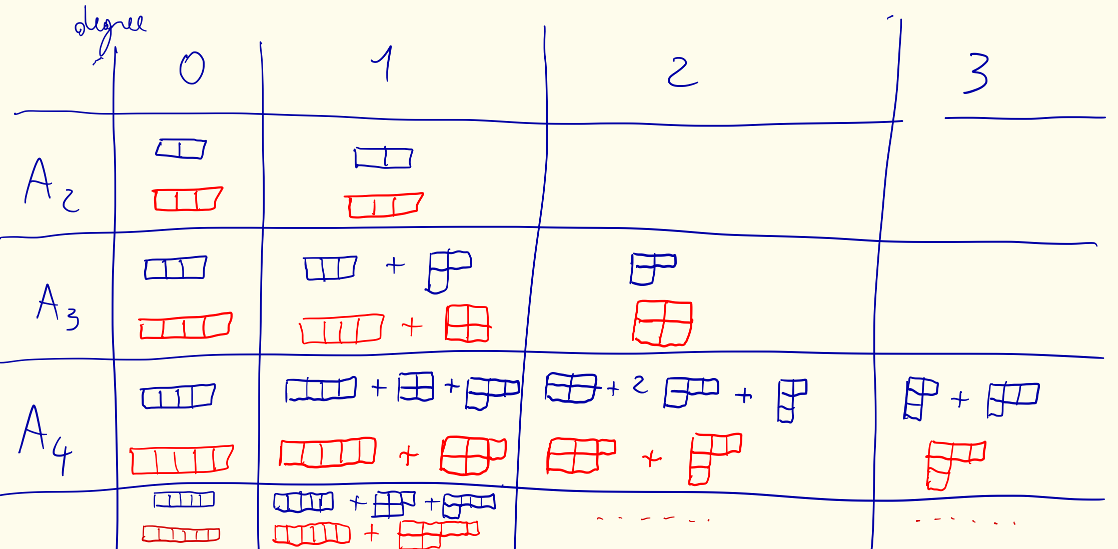
$$\mathrm{Ind}_{H \times S_{n-k}}^{S_n}\left(V \otimes \underbrace{\boxed{}}_{n-k}\right)$$

$n-k \geq k$

$n \geq 2k$

"stabilizes" (is "uniformly representation stable"); for every $n \geq 2k$ the decomposition into irreducibles in the case $n+1$ can be obtained from the decomposition in the case n by adding a box in the first row of all the diagrams that appear.

It turns out that there is an important stabilization phenomenon in the table:



Representation stability for the algebras A_n

We will illustrate a result of Church and Farb (2013), as an example of the rich field of research called "representation stability". We recall from a previous lesson the following result for the Orlik-Solomon algebra:

Theorem For $X \in L(A)$, we put $A_X = \varphi(E_X)$

Then

$$A = \bigoplus_{X \in L(A)} A_X$$

In the case of the algebra $A_n = A(\mathcal{B}_n)$ this can be rewritten in the following way

We denote by $S = \{S_1, S_2, \dots, S_k\}$ a PARTITION of the set $\{1, , n\}$.

For every S_i we denote by A_{S_i} the subalgebra of A_n given by the generators a_{ij} with $\{i, j\} \subset S_i$.

For example $A_{\{1,3,4\}}$ is the subalgebra of A_5 generated by a_{13}, a_{14}, a_{34}.

Furthermore, we denote by $A^{top}_{S_i}$ the component of top degree of A_{S_i} (i.e. of degree $|S_i|-1$).

Finally, given the partition $S=\{S_1, \dots, S_k\}$ we denote by A_n^S the product

$$A_n^S = A^{top}_{S_1} \times A^{top}_{S_2} \times \dots \times A^{top}_{S_k}$$

which is a subspace of A_n^{n-k} ← degree $n-k$.

Then we can write

$$A_n = \bigoplus_{S} A_n^S$$

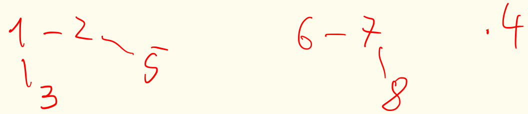
← this ranges among all the partitions of $\{1, \dots, n\}$

(this is the translation of the theorem on O-S algebras, and it can be obtained also considering the basis of A_n)

For instance, if $n=8$

$$w_{12}\, w_{13}\, w_{25}\, w_{67}\, w_{78}$$

belongs to A_8^S where $S=\{\{1,2,3,5\},\{4\},\{6,7,8\}\}$



We notice that the symmetric group S_n permutes the summands in $\bigoplus_S A_n^S$ according to its action on the partitions of $\{1, \dots, n\}$.

Let us denote by $\lambda_S = (\lambda_1, \ , \lambda_k)$ the partition of n determined by the numbers $|S_1|, |S_2|, \dots, |S_k|$.

For any partition $\mu = (\mu_1, \ , \mu_k)$ of n, let S_μ be the "canonical" partition of $\{1, \ , n\}$ defined by

$$S_\mu = (\{1, \ , \mu_1\}, \{\mu_1 + 1, \ , \mu_1 + \mu_2\}, \dots \dots).$$

We have by definition $\lambda_{S_\mu} = \mu$.

Now we observe that, once a partition μ of n is fixed, then $\bigoplus_{\substack{S \text{ s.t} \\ \lambda_S = \mu}} A_n^S$ is a subrepresentation of A_n, since S_n permutes the summands A_n^S such that $\lambda_S = \mu$.

By definition of the induced representation,

$$\bigoplus_{\substack{S \text{ s.t} \\ \lambda_S = \mu}} A_m^S = \mathrm{Ind}_{\mathrm{Stab}(S_\mu)}^{S_n} A_m^{S_\mu}$$

Let us introduce another notation. If S is a partition of $\{1, , n\}$, we define $S\langle n+1\rangle$ as the partition of $\{1, , n+1\}$ whose parts are the parts of S and $\{n+1\}$.

For instance, if $S = \{\{1,2,4\},\{3,5\}\}$, then
$S\langle 6\rangle = \{\{1,2,4\},\{3,5\},\{6\}\}$.

Moreover, for every $m \geq n+1$ we define $S\langle m\rangle$ as the partition of $\{1, , m\}$ whose parts are the parts of S and $\{n+1\}, \{n+2\}, \dots, \{m\}$.

Let us now consider $A_m^{\mathcal{S}}$ where $\boxed{n \geq 2\mathcal{S}}$

We have:

$$A^J_n = \bigoplus_{\substack{\mu \text{ partition of } n \\ \ell(\mu) = n-J}} \left(\bigoplus_{\substack{S \text{ such that} \\ \lambda_S = \mu}} A^S_n \right)$$

number of nonzero terms in the partition: if $\mu = (\mu_1, , \mu_K)$ then $\ell(\mu) = K$ $\mu_K \neq 0$

$$= \bigoplus_{\substack{\mu \text{ part. of } n \\ \ell(\mu) = n-J}} \mathrm{Ind}^{S_n}_{\mathrm{Stab}\, S_\mu} A^{S_\mu}_n$$

Now we consider A^J_{n+1}:

$$A^J_{n+1} = \bigoplus_{\substack{\gamma \text{ partition of } n+1 \\ \ell(\gamma) = n+1-J}} \mathrm{Ind}^{S_{n+1}}_{\mathrm{Stab}\, S_\gamma} A^{S_\gamma}_{n+1}$$

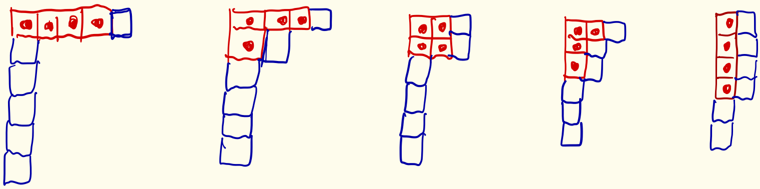
$2n+2-2J \geq$
$\geq n+2$ ASSURDO

A crucial remark is that, under our ypothesis $n \geq 2J$ every partition γ of $n+1$ with $\ell(\gamma) = n+1-J$ must be of the form $\gamma = (\gamma_1 \text{-----}, 1)$. Therefore γ is obtained by a partition of n adding 1 in the end.

We can then write

$$A^{j}_{n+1} = \bigoplus_{\substack{\nu \text{ partition of } n \\ \ell(\nu) = n-j}} \mathrm{Ind}^{S_{n+1}}_{\mathrm{Stab}\, S_{\nu\langle n+1\rangle}} A^{S_{\nu\langle n+1\rangle}}_{n+1}$$

Remark: Since $n \geq 2j$,
there is a bijection between the partitions ν of n with $\ell(\nu) = n-j$ and the partitions of j: one can subtract a box from every row of the diagram of ν. For instance, if $n=10$ and $j=4$



Hence, the range of the sum $\bigoplus$ does not depend on n for $n \geq 2j$.

This means that we are interested in considering

$$\mathrm{Ind}^{S_m}_{\mathrm{Stab}(S_{\nu\langle m\rangle})} A^{S_{\nu\langle m\rangle}}_{m}$$

For every ν that appears in the expression above, we denote by ν' the partition obtained by deleting the entries equal to 1

For instance, if $\nu = (4,2,2,1,1)$ then

$$\nu' = (4,2,2).$$

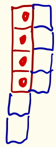
We notice that $|\nu'| - \ell(\nu') = J$

NOTATION: $|\nu'| = \nu_1 + \quad + \nu_{\ell(\nu')}$

The maximal value for $\ell(\nu')$ is J, obtained when

$$\nu = (\underbrace{2,2,\cdots,2}_{J \text{ times}},1,\cdots,1)$$

$\longleftarrow$ recall the picture $\longrightarrow$



We notice that $\operatorname{Stab} S_{\nu<m>} = \operatorname{Stab} S_{\nu'<m>}$

which is $= \left(\operatorname{Stab} S_{\nu'}\right) \times S_{m-|\nu'|}$

in the symmetric group $S_{|\nu'|}$

Therefore

$$\operatorname{Ind}_{\operatorname{Stab} S_{\nu<m>}}^{S_m} A_m^{S_{\nu<m>}} \cong \operatorname{Ind}_{(\operatorname{Stab} S_{\nu'}) \times S_{m-|\nu'|}}^{S_m} \left(A_{|\nu'|}^{S_{\nu'}} \times \boxed{\qquad} \right)$$

By the Hemmer Theorem we know that this sequence stabilizes when $m - |\nu'| \geq |\nu'|$, that is to say

$$m \geq 2|\nu'| = 2J + 2\ell(\nu')$$

Since the maximum value for $\ell(\nu')$ is J, we have that for $m \geq 4J$ the sequence stabilizes.

RIASSUMENDO:

Teorema (Church-Farb 2013) Fissato J, le rappresentazioni A_m^J si stabilizzano per $m \geq 4J$.

Some references

Church T., Farb. B., Representation theory and homological stability, Adv. Math, 2013

Here you can also find the general definition of rep. stability and many crucial results.

Hersh P., Reiner V., Representation stability for cohomology of configuration spaces in $\mathbb{R}^d$, IMRN, 2017

Here you can find an improvement of the stability bound for A_m^J: it stabilizes at $m = 3J+1$.

READ MORE

Hemmer, D., Stable decomposition for some symmetric group characters arising in braid group cohomology, J. of Comb. Theory, series A, 2011.

Bibby, C. Representation stability for the cohomology of arrangements associated to root systems, ARXIV 2016.

Wilson, J. FI_W - modules and costraints on classical Weyl group characters, MATH. Z., 2015.

Here you can find generalizations of the stability results to arrangements associated to other root systems and their Weyl groups.