The extended S_{n+1} action on $H^*(C_n(\mathbb{C}), \mathbb{C})$

Let us consider the configuration space

$$C_n(\mathbb{C}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{C}^n \;\middle|\; \sum x_i = 0,\ x_i \neq x_j \right\}$$

It can be viewed as the complement of the essential braid arrangement $\mathcal{B}_n$, whose hyperplanes are

$$H_{ij} = \{ x_i - x_j = 0 \} \text{ in } V = \mathbb{C}^n \Big/ \mathbb{C}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

We can identify V with $\left\{ (x_1, \ldots, x_n) \in \mathbb{C}^n \;\middle|\; \sum x_i = 0 \right\}$

After this identifications, the hyperplanes of $\mathcal{B}_n$ can still be written as $H_{ij} = \{ x_i - x_j = 0 \}$.

Now we construct an S_n-equivariant map

$$\begin{aligned} C_n(\mathbb{C}) &\xrightarrow{\ \gamma\ } \mathbb{P}(C_n(\mathbb{C})) \times \mathbb{C}^* \\ p &\longrightarrow \left(\pi(p),\ Q^2(p) \right) \end{aligned}$$

where $\pi : C_n(\mathbb{C}) \to \mathbb{P}(C_n(\mathbb{C}))$ is the standard projection, and $Q = \prod_{i<j} (x_i - x_j)$ is the defining polynomial of the arrangement $\mathcal{B}_n$ and S_n acts trivially on $\mathbb{C}^*$!

Since $\pi(p) = \pi(q) \iff p = \lambda q$ for $\lambda \in \mathbb{C}^*$

and $Q^2(q) = Q^2(\lambda q) \iff Q^2(q) = \lambda^{n(n-1)} Q^2(q)$

$$\iff \lambda^{n(n-1)} = 1$$

the fibers of γ are the orbits of the action of the cyclic group

$$\Gamma = \{ z \in S^1 \mid z^{n(n-1)} = 1 \}$$

<u>Proposition</u> The map γ is a covering map.

Now we recall the following Transfer Theorem (see for instance Hatcher, Algebraic Topology, Prop 3G.1) in a general version:

<u>Theorem</u> Let G be a finite group, X a paracompact, Hausdorff, locally euclidean space where G acts, and $\mathbb{F}$ a field with char $\mathbb{F} = 0$ or char $\mathbb{F} > 0$ but char $\mathbb{F} \nmid |G|$. Then

$$H^*(X/G, \mathbb{F}) \cong H^*(X, \mathbb{F})^G$$

In our setting we have:

$$H^*(C_n(\mathbb{C}))^{\Gamma} \overset{\text{TRANSFER}}{\cong} H^*\left(C_n(\mathbb{C})/\Gamma\right) \cong$$

$$\cong H^*(\mathbb{P}(C_n(\mathbb{C})) \times \mathbb{C}^*) \underset{\text{KÜNNETH FORMULA}}{\cong} H^*(\mathbb{P}(C_n(\mathbb{C}))) \otimes H^*(\mathbb{C}^*)$$

where the cohomology is with complex coefficients.

Now we observe that the action of Γ on $C_n(\mathbb{C})$ is the restriction of the continuous action of S^1 on $C_n(\mathbb{C})$. Since S^1 is a path-connected topological group, the action of S^1 on $H^*(C_n(\mathbb{C}))$ is trivial (every element ϑ of S^1 acts by a map homotopic to the identity). Therefore $H^*(C_n(\mathbb{C}))^{\Gamma} = H^*(C_n(\mathbb{C}))$.

It is easy to check that the map γ is S_n equivariant (where S_n acts in the natural way on $C_n(\mathbb{C})$ and $\mathbb{P}(C_n(\mathbb{C}))$ and acts trivially on $\mathbb{C}^*$).

Also the map

$$C_n(\mathbb{C})/\Gamma \xrightarrow{\sim} \mathbb{P}(C_n(\mathbb{C})) \times \mathbb{C}^*$$

is S_n equivariant

Therefore:

<u>Theorem A</u> There is an isomorphism of S_n representations

$$H^*(C_n(\mathbb{C})) \cong H^*(\mathbb{P}(C_n(\mathbb{C}))) \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$$

This is the cohomology ring $H^*(\mathbb{C}^*)$, where ϵ has degree 1, and S_n acts trivially.

Our plan is to point out an S_{n+1} action on $H^*(\mathbb{P}(C_n(\mathbb{C})))$, that restricted to S_n gives the natural action. Using Theorem A we will "lift" the S_{n+1} action to $H^*(C_n(\mathbb{C}))$.

The moduli space $\mathcal{M}_{0,n+1}$

For $n \geq 2$ consider the configuration space

$$C_{n+1}(\mathbb{P}^1(\mathbb{C})) = \{ (p_0, p_1, \dots, p_n) \in (\mathbb{P}^1(\mathbb{C}))^{n+1} \mid p_i \neq p_j \}$$

Definition The moduli space of $n+1$ pointed curves of genus 0 is

$$\mathcal{M}_{0,n+1} = C_{n+1}(\mathbb{P}^1(\mathbb{C})) \Big/ \mathbb{P}SL(2,\mathbb{C})$$

i.e. the group of projective automorphisms of $\mathbb{P}^1(\mathbb{C})$.

Let $p = [p_0, p_1, \dots, p_n] \in \mathcal{M}_{0,n+1}$. There is a unique projective automorphism $F \in \mathbb{P}SL(2,\mathbb{C})$ such that

$$F(p_0) = (0,1) = \infty \ , \quad F(p_1) = (1,0) \ , \quad F(p_2) = (1,1)$$

SIMPLIFIED NOTATION; it should be $[(0,1)]$

In particular, every point $p \in \mathcal{M}_{0,n+1}$ has a UNIQUE representative of this type:

$$p = [P_0, \quad , P_n] = [(0,1), (1,0), (1,1), (x_2,y_2), \quad , (x_{n-1},y_{n-1})]$$

with $x_i \neq 0 \ \forall i$, $y_j \neq 0 \ \forall j$, $x_i \neq y_i \ \forall i$

and $\dfrac{y_i}{x_i} \neq \dfrac{y_j}{x_j} \quad \forall \ i \neq j$.

Now we choose in V^* the basis
$$\begin{aligned} \alpha_1 &= x_2 - x_1 \\ \alpha_2 &= x_i - x_1 \\ \alpha_{n-1} &= x_n - x_1 \end{aligned}$$

and we denote by $v_1, \ , v_{n-1}$ its dual basis in V.

Notice that if we write an element $q \in C_n(\mathbb{C})$ using this basis we have: $q = (\gamma_1, \ , \gamma_{n-1})$ with $\gamma_i \neq 0 \ \forall i$ (otherwise $q \in H_{1,i+1}$) and $\gamma_i \neq \gamma_j \ \forall i \neq j$ (otherwise $q \in H_{i+1,j+1}$).

We now define the map (using the basis described above)

$$\phi : \mathbb{P}(C_n(\mathbb{C})) \longrightarrow \mathcal{M}_{0,n+1}$$

$$[(\gamma_1, \ , \gamma_{n-1})] \longrightarrow [(0,1), (1,0), (1,1), (\gamma_1,\gamma_2), \cdots, (\gamma_1,\gamma_{n-1})]$$

Remark ϕ is well defined since, as remarked above, $\gamma_i \neq 0 \ \forall i$, $\gamma_i \neq \gamma_j \ \forall i \neq j$.

Proposition The map ϕ is bijective.

Proof The map

$$\psi : M_{0,n+1} \longrightarrow \mathbb{P}(C_n(\mathbb{C}))$$

$$[(0,1),(1,0),(1,1),(1,x_2),\dots,(1,x_{n-1})] \longrightarrow [(1,x_2,\ ,x_{n-2})]$$

is the inverse of ϕ. $\square$

Let us now consider S_{n+1} as the group that permutes $\{0,1,\ ,n\}$. We identify S_n with the subgroup that fixes 0.

Theorem The map ϕ is S_n equivariant

$$\phi : \underset{\substack{\uparrow \\ \text{natural action of } S_n}}{\mathbb{P}(C_n(\mathbb{C}))} \longrightarrow \underset{\substack{\uparrow \\ \text{here } S_{n+1} \text{ acts, and } S_n \text{ is} \\ \text{identified with the subgroup} \\ \text{that fixes } 0.}}{M_{0,n+1}}$$

Proof Let us denote by $\tau_i = (i, i+1)$ the transposition so S_n is generated by $\tau_1, \tau_2, \ , \tau_{n-1}$.

It is sufficient to check our claim on $\tau_1, \ , \tau_{n-1}$.

Let us for instance consider τ_1.

Let $q = [(1, \gamma_2, \ , \gamma_{n-1})] \in \mathbb{P}(C_n(\mathbb{C}))$. Recall that $(1, \gamma_2, \ , \gamma_{n-1})$ is written with respect to the basis $v_1, \ v_{n-1}$ and represents a point $(a_1, \ , a_n) \in V$ such that

$$a_2 - a_1 = 1$$
$$a_3 - a_1 = \gamma_2$$
$$a_n - a_1 = \gamma_{n-1}$$

Since $\tau_1(a_1, \dots, a_n) = (a_2, a_1, \dots, a_n)$, this vector with respect to the basis $v_1, \ v_{n-1}$ is $(-1, \gamma_2 - 1, \dots, \gamma_{n-1} - 1)$

Therefore $\tau_1 q = \tau_1 [(1, \gamma_2, \dots, \gamma_{n-1})] = [(1, 1-\gamma_2, \dots, 1-\gamma_{n-1})]$

On the other hand, let us focus on the action of τ_1 on $\phi(q) = [(0,1), (1,0), (1,1), (1,\gamma_2), \ , (1, \gamma_{n-1})]$.

τ_1 permutes

$$\tau_1 \phi(q) = [(0,1), (1,1), (1,0), (1,\gamma_2), \ , (1,\gamma_{n-1})]$$

To put this in canonical form we act by the projectivity $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, i.e. the projectivity that

sends $\infty \to \infty$, $(1,1) \to (1,0)$, $(1,0) \to (1,1)$.

So $\tau_1 \phi(q) = [(0,1),(1,0),(1,1),(1,1-\gamma_2),\dots,(1,1-\gamma_{n-1})]_\sim$

And the two actions of τ_1 agree.

Similarly one checks that $\forall i$ the actions of τ_i agree $\square$

One observes that ϕ is an isomorphism of algebraic varieties

$$\phi : \mathbb{P}(C_n(\mathbb{C})) \cong \underbrace{M_{0,n+1}}_{\text{here } S_{n+1} \text{ acts}}$$

From this isomorphism one has a S_{n+1} action on $\mathbb{P}(C_n(\mathbb{C}))$ and therefore on $H^*(\mathbb{P}(C_n(\mathbb{C})))$.

<u>Remark 1</u> Let $q = [(1,\gamma_2, \ ,\gamma_{n-1})] \in \mathbb{P}(C_n(\mathbb{C}))$.

Using ϕ one can compute

$$s_0 q = [(1,\gamma_2^{-1}, \ , \gamma_{n-1}^{-1})]_\sim$$ (EXERCISE!)

Remark 2

In the end, via Theorem A, we have a S_{n+1} action on $H^*(C_n(\mathbb{C}))$. One can prove (Mathieu '96) that this action does not come from an S_{n+1} action on the space $C_n(\mathbb{C})$.

Some references

G - The actions of S_{n+1} and S_n on the cohomology ring of a Coxeter arrangement of type A_{n-1}, Manuscripta Mathematica '96

Mathieu, O., Hidden S_{n+1} actions, Comm. Math. Phys. '96