Basics on the Orlik-Solomon algebra

Let $\mathcal{A} = \{H_1, \quad, H_N\}$ be a central hyperplane arrangement in a vector space $\mathbb{K}^n$. Let R be a commutative ring. We denote by $\overline{E} = \bigoplus_{H \in \mathcal{A}} R e_H$.

This is a free R-module where the elements e_H are in bijection with the hyperplanes of $\mathcal{A}$.

Let $E = \Lambda(\overline{E})$ be the exterior algebra, that is naturally graded:

$$E_0 = R$$

$$E_1 = \langle e_{H_1}, \quad, e_{H_n} \rangle_R$$

$$E_p = \langle \cdots\cdots, e_{H_{i_1}} \cdots\cdots e_{H_{i_p}}, \cdots\cdots \rangle_R$$

We notice that $E_p = 0 \quad \forall\, p > N = |\mathcal{A}|$.

<u>Definition</u> We define a R-linear map $\partial_E : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1 \quad \forall\, H \in \mathcal{A}$ and, for $p \geq 2$

$$\partial\left(e_{H_1} \cdots\cdots e_{H_p}\right) = \sum_{K=1}^{p} (-1)^{K-1} e_{H_1} \cdots\cdots \widehat{e_{H_K}} \cdots\cdots e_{H_p}$$

with $H_1, \quad, H_p \in \mathcal{A}$

Given a p-tuple of hyperplanes $S = (H_1, \dots, H_p)$ we denote $|S| = p$ and

$$e_S = e_{H_1} \cdots\cdots e_{H_p} \in E \qquad \cap S = H_1 \cap \quad \cap H_p$$

<u>NOTATION</u> If $p = 0$, then $S = (\ )$ and

$$e_S = 1 \qquad \cap S = \mathbb{K}^n$$

Given a subspace U in $\mathbb{K}^n$, we denote by $r(U)$ the CODIMENSION of U.

<u>Definition</u> We call S <u>independent</u> if $r(\cap S) = |S|$ and <u>dependent</u> if $r(\cap S) < |S|$

Let S_p denote the set of all the p-tuples and let

$$S = \bigcup_{p \geq 0} S_p$$

Definition Let $\mathcal{A}$ be a central hyperplane arrangement as above. We denote by $I = I(\mathcal{A})$ the ideal of E generated by the elements ∂e_S for all dependent $S \in \mathcal{S}$.

We notice that I is generated by homogeneous elements, hence it is a graded ideal:

$$I = \bigoplus_{p=0}^{N} I \cap E_p$$

Orlik and Solomon introduced in 1980 the following algebra.

Definition Let $\mathcal{A}$ be a central hyperplane arrangement as above. We define

$$A = A(\mathcal{A}) = E / I$$

We denote by $\varphi : E \to A$ the projection and we write

$$\varphi(e_H) = a_H \quad \forall\, H \in \mathcal{A}$$

$$\varphi(e_S) = a_S \quad \forall\, S \in \mathcal{S}$$

$$\varphi(E_p) = A_p$$

Exercise If $S \in \mathcal{S}$ and $H \in S$, we have

$$e_S = e_H \, \partial e_S .$$

We observe that A is a graded anticommutative algebra, and $A_0 = R$, $A_1 = \bigoplus_{H \in \mathcal{A}} R a_H$

the only dependent elements of $\mathcal{S}_2$ are $S = (H, H)$ for $H \in \mathcal{A}$ and $\partial e_S = e_H^2 = 0$, so $I \cap E_1 = \{0\}$.

From the exercise we have that $A_p = 0 \quad \forall p > n$ since if $p > n$ every p-tuple in $\mathcal{S}_p$ is dependent, n being the dimension of the ambient vector space.

Let $L(\mathcal{A})$ be the poset of all the intersections of the hyperplanes in $\mathcal{A}$, ordered by reverse inclusion.

For $X \in L(\mathcal{A})$ let $\mathcal{S}_X = \{ S \in \mathcal{S} \mid \cap S = X \}$

and let

$$E_X = \sum_{S \in \mathcal{S}_X} R e_S$$

We observe that

$$E = \bigoplus_{X \in L} E_X .$$

<u>Theorem</u> For $X \in L(A)$, we put $A_X = \varphi(E_X)$.

Then

$$A = \bigoplus_{X \in L(A)} A_X$$

<u>Remark</u> The Orlik-Solomon algebra can be defined also for affine hyperplane arrangements. The main difference is that one takes the ideal I generated by ∂e_S (S dependent) and by e_S (with $\cap S = \emptyset$).

<u>Theorem</u> (Orlik-Solomon, 1980).

Let $\mathcal{A}$ be an hyperplane arrangement in the complex vector space V. Let $\mathcal{M}(\mathcal{A}) = V - \bigcup_{H \in \mathcal{A}} H$.

Then

$$H^*(\mathcal{M}(\mathcal{A}), \mathbb{C}) \cong A(\mathcal{A})$$

coefficients in $R = \mathbb{C}$

Remark This theorem holds also with integer coefficients:

$$H^*(\mathcal{M}(\mathcal{A}), \mathbb{Z}) \cong A(\mathcal{A})$$

coefficients in $R = \mathbb{Z}$.

Remark Let $\mathcal{A}$ be a central arrangement.

For every $H \in \mathcal{A}$, let α_H be a functional in V^* such that $H = \{ v \in V \mid \alpha_H(v) = 0 \}$.

Then the map

$$a_H \longleftrightarrow \left[\frac{d\alpha_H}{\alpha_H} \right]$$

gives the isomorphism in De Rham cohomology.

See the book of Orlik and Terao "Arrangements of Hyperplanes", 1992 for proofs and further properties of $A(\mathcal{A})$.
(CHAPTER 3, SECTIONS 1 and 2)

The Arnold's algebra

Let us consider the braid arrangement $\mathcal{B}_n$.
It is the arrangement in $V = \mathbb{C}^n / \mathbb{C}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ given by
the hyperplanes $H_{ij} = \{ x_i - x_j = 0 \}$
(they are well defined in V).
Let us identify V with $\{ (x_1, \ , x_n) \in \mathbb{C}^n \mid \sum x_i = 0 \}$
Then the complement of the arrangement $\mathcal{B}_n$ is the Configuration space

$$C_n(\mathbb{C}) = \{ (x_1, \ , x_n) \in \mathbb{C}^n \mid \sum x_i = 0, \ x_i \neq x_j \ \forall i \neq j \}$$

In 1969 Arnold proved that the cohomology algebra $H^*(C_n(\mathbb{C}), \mathbb{C})$ is isomorphic to the skew commutative algebra with generators

$$\{ a_{ij} \mid 1 \leq i < j \leq n \}$$

and relations

$$(*) \quad a_{ij} a_{ik} - a_{ij} a_{jk} + a_{ik} a_{jk} = 0 \qquad \forall \ 1 \leq i < j < k \leq n$$

Remark Actually it was proved for integer coefficients.
From now on we focus on complex coefficients.

From Orlik and Solomon result it follows that the Arnold algebra is isomorphic to $A(\mathcal{B}_n)$.

complex coefficients from now on

The following exercises show a proof.

<u>Exercise 1</u> Let us consider the braid arrangement $\mathcal{B}_n$. Let $S = (H_1, \ldots, H_p)$ be a p-tuple. Then S is dependent if and only if it contains a subsequence of the form $(H_{i_1 j_1}, H_{i_2 j_2}, \ldots, H_{i_p j_p})$ with $j_h = i_{h+1}$ $\forall\ h = 1, \ldots, p-1$ and $j_p = i_1$ (i.e., the sequence is "a cycle").

<u>Exercise 2</u> Arnold relations $\circledast$ can be written as

$$\partial e_S = 0 \quad \text{for all } S \in \mathcal{S}_3 \text{ minimally dependent}$$

this means that if we delete an hyperplane from the triple, what remains is independent

<u>Exercise 3</u> For the braid arrangement $\mathcal{B}_n$, the ideal I in E coincides with the ideal generated by

$$\partial e_S \quad \text{for every } S \in \mathcal{S}_3 \text{ dependent.}$$

A basis for $A(\mathcal{B}_n)$.

Let us consider the following array

$$
\begin{array}{cccccc}
1 & & & & & \\
a_{12} & 1 & & & & \\
a_{13} & a_{23} & 1 & & & \\
\vdots & \vdots & a_{34} & 1 & & \\
 & & \vdots & & \ddots & \\
 & & & & 1 & \\
a_{1n} & a_{2n} & a_{3n} & & a_{n-1,n} & 1
\end{array}
$$

Theorem A basis for $A(\mathcal{B}_n)_p$ is

$$
\left\{ a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_p j_p} \;\middle|\; \begin{array}{l} 2 \leq j_1 < \quad < j_p \leq n \\ i_h < j_h \quad \forall\, h = 1, .., p \end{array} \right\}
$$

This means that a basis for $A(\mathcal{B}_n)$ is given by all the monomials obtained taking one element from each row of the array above.

Exercise Using Arnold's relations $*$, prove that the monomials above are a set of generators for $A(\mathcal{B}_n)$.

We notice that on the algebra $A(\mathcal{B}_n)$ there is a natural S_n action, that permutes the indices in a_{ij} (NOTATION $a_{ij} = a_{ji}$).

One of the goals of this course is to illustrate some properties of this interesting representation of S_n that comes from the geometry of configuration spaces.