Computation of the representation of S_n on the algebra A_n.

We know that on A_n there is a natural S_n-action (where S_n permutes the indices $1, 2, \ldots, n$) and an extended S_{n+1} action.

NOTATION $A_n = A(\mathcal{B}_n)$ the Arnold algebra (= Orlik Solomon algebra for $\mathcal{B}_n$)

Let $S_n = \langle \tau_1, \tau_2, \ldots, \tau_{n-1} \rangle$ where $\tau_i = (i, i+1)$

transposition

and $S_{n+1} = \langle \tau_0, \tau_1, \ldots, \tau_{n-1} \rangle$

transposition $\tau_0 = (0, 1)$.

The extended S_{n+1} action, restricted to $\langle \tau_1, \ldots, \tau_{n-1} \rangle$ coincides with the natural S_n action.

OUR GOAL We want to understand the decomposition of A_n into irreducible representations. We also would like to have some information on the decomposition of A_n^k, the part of A_n of degree k, for every k.

CAUTION: the last problem is still open for general n and k.

Example

$$A_2^0 = \langle 1 \rangle_{\mathbb{C}} \cong \square\square \quad \text{as } S_2 \text{ representation}$$

$$A_2^1 = \langle a_{12} \rangle \cong \square\square \quad \text{as } S_2 \text{ representation}$$

Let us first observe that

$$A_n^k \cong A_{n-1}^k \oplus A_{n-1}^{k-1} \cdot N \qquad (*)$$

where $N = \bigoplus_{s=1}^{n-1} \mathbb{C}\, a_{sn}$.

This easily follows since we know a basis for A_n^k.

The ISOMORPHISM (*) is an iso of $\langle \tau_1, \ , \tau_{n-2} \rangle \cong S_{n-1}$ representations.

Let us consider $k=1$:

$$A_n^1 = A_{n-1}^1 \oplus N$$

We observe that A_{n-1}^1 is a $\langle \tau_0, \ , \tau_{n-2} \rangle$ module since on A_{n-1} there is the extended S_n action.

We can now take a $\langle \tau_0, \ , \tau_{n-2} \rangle$ complement T to A_{n-1}^1:

$$A_n^1 \doteq A_{n-1}^1 \oplus T$$

Now, what is the decomposition of T as $\langle \tau_0, \ldots, \tau_{n-2} \rangle$ module?

We notice that T and N are ISOMORPHIC as $\langle \tau_1, \ldots, \tau_{n-2} \rangle$ modules, since

$$A_{n-1}^1 \oplus T \cong A_{n-1}^1 \oplus N$$ is an ISOMORPHISM of $\langle \tau_1, \ldots, \tau_{n-2} \rangle$ modules.

Now $N = \langle a_{1n}, a_{2n}, \ldots, a_{n-1\,n} \rangle$ that, since $\langle \tau_1, \ldots, \tau_{n-2} \rangle$ permutes the indices $1, \ldots, n-1$, is isomorphic to the natural representation of S_{n-1} on $\mathbb{C}^{n-1}$.

Therefore $T \cong N \cong \overbrace{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array}}^{n-1} \oplus \overbrace{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; \\ \cline{1-1} \end{array}}^{n-2}$ as S_{n-1} rep.

By Pieri's rule we know that there is only one representation of $\langle \tau_0, \ldots, \tau_{n-2} \rangle$ that, restricted to $\langle \tau_1, \ldots, \tau_{n-2} \rangle$ gives $\overbrace{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array}}^{n-1} \oplus \overbrace{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; \\ \cline{1-1} \end{array}}^{n-2}$, that is to say, $\overbrace{\begin{array}{|c|c|}\hline \; & \qquad\qquad \\ \hline \; \\ \cline{1-1} \end{array}}^{n-1}$ the standard representation of S_n.

This proves that $T \cong \overset{n-1}{\boxed{}}$ as $\langle \tau_0, , \tau_{n-2} \rangle$ rep.

Let us go back to A_n^k. We will prove that

<u>Theorem</u> $A_n^k \cong A_{n-1}^k \oplus A_{n-1}^{k-1} \cdot T$ as $\langle \tau_0, , \tau_{n-2} \rangle$ rep

<u>Proof</u> For every i we write

$$\omega_{in} = \gamma_i^1 + \gamma_i^2 \quad \text{where} \quad \gamma_i^1 \in A_{n-1}^1, \gamma_i^2 \in T$$

Let $z \in A_n^k$. Then $z = z_0 + \sum_{j=1}^{n} z_j \, \omega_{jn}$

since $A_n^k = \underset{\substack{\cup\!\!\!\!\!\!\!\;\;\; \\ z_0}}{A_{n-1}^k} \oplus \underset{\substack{\cup \\ z_j}}{A_{n-1}^{k-1}} \cdot N$

Therefore $z = \underbrace{z_0 + \sum_{j=1}^{n} z_j \gamma_j^1}_{\in A_{n-1}^k} + \underbrace{\sum_{j=1}^{n} z_j \gamma_j^2}_{\in A_{n-1}^{k-1} \cdot T}$

This proves

$$A_n^k = A_{n-1}^k \oplus A_{n-1}^{k-1} \cdot T$$

(an easy dimensional argument shows that the sum is direct). $\square$

__Theorem A__ (Mathieu '96, 6.)

Let us consider A_n^K as a $\langle \tau_1, , \tau_{n-1} \rangle$ representation (this is the natural S_n representation). Moreover, let us consider A_{n-1}^K, A_{n-1}^{K-1} as $\langle \tau_0, , \tau_{n-2} \rangle$ representations (this is the extended S_n representation). Let us consider T as $\langle \tau_0, , \tau_{n-2} \rangle$ representation. Then

$$\underset{\text{natural}}{A_n^K} \cong \underset{\text{extended}}{A_{n-1}^K} \oplus \left(\underset{\text{extended.}}{A_{n-1}^{K-1}} \otimes T \right)$$

as S_n representations.

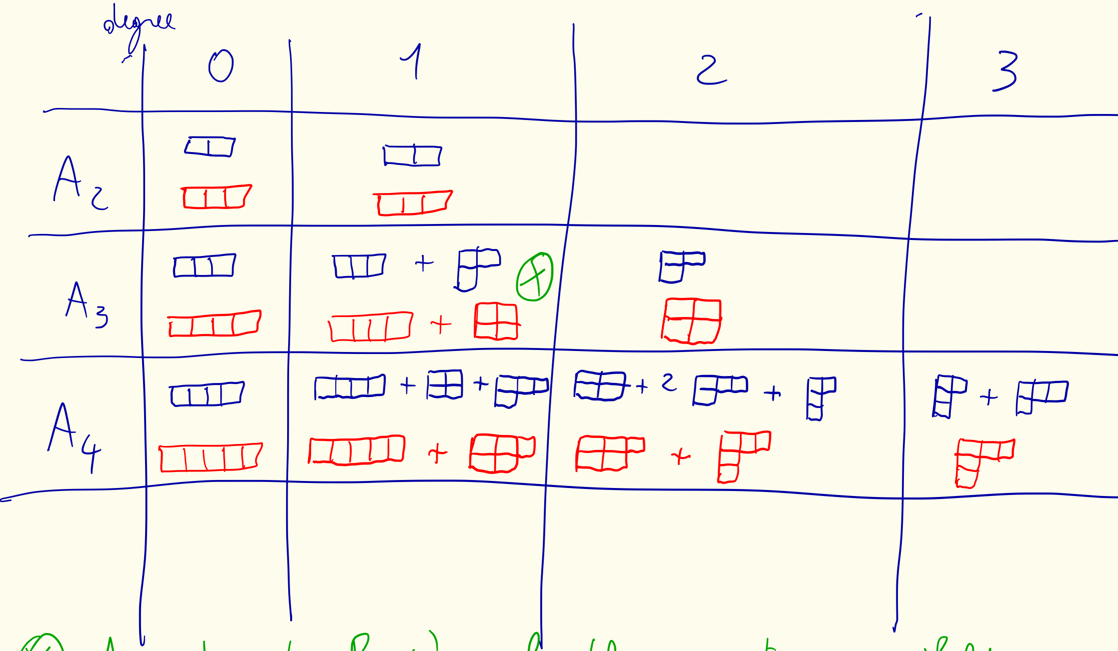
__Proof__ Recall that A_n^K is a $\langle \tau_0, , \tau_{n-1} \rangle \cong S_{n+1}$ representation. We consider the two subgroups of S_{n+1} $H = \langle \tau_0, , \tau_{n-2} \rangle$ and $K = \langle \tau_1, , \tau_{n-1} \rangle$.

Since they are conjugate in $\langle \tau_0, , \tau_{n-1} \rangle$ the representation $\mathrm{Res}_H^{\langle \tau_0, , \tau_{n-1} \rangle} A_n^K$ and $\mathrm{Res}_K^{\langle \tau_0 \; \tau_{n-1} \rangle} A_n^K$ are isomorphic.

WHY? Think of their characters... They coincide since we are considering traces of conjugate operators.

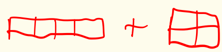
Now the claim follows from the preceding theorem. □

Let us write a table :



⊗ According to Pieri's rule, there are two possibilities here :

By the geometric construction of the extended action we know that $H^*(C_n(\mathbb{C})) \cong H^*(\mathbb{P}(C_n(\mathbb{C}))) \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$ as S_{n+1} modules, so, informally: "every irreducible representation appears together with another copy shifted in degree +1". So the right choice is



Remark

To fill the table, we also used the computation

$$\ydiagram{2,2} \otimes \ydiagram{3,1} = \ydiagram{2,1,1} + \ydiagram{3,1}$$

that follows from the exercise

$$U \otimes \mathrm{Ind}_{S_{n-1}}^{S_n} \text{trivial} \cong \mathrm{Ind}_{S_{n-1}}^{S_n} \left(\mathrm{Res}_{S_{n-1}}^{S_n} U \right)$$

that is to say

$$U \otimes \left(\ydiagram{4} + \ydiagram{3,1} \right) \cong \mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{Res}_{S_{n-1}}^{S_n} U$$

Exercise Fill the row "A_5" in the table.

Let us now deal with the full representation A_n (i.e, we consider all the degrees together).

From Theorem A we have the isomorphism of S_n modules

$$\underset{\text{natural } S_n \text{ action}}{\underset{\uparrow}{A_n}} \cong \underset{\uparrow}{A_{n-1}} \oplus \left(\underset{\uparrow}{A_{n-1}} \otimes \begin{array}{|c|c|}\hline \ & \qquad\qquad \\ \hline \ \\ \cline{1-1}\end{array} \right)$$

extended S_n action

This can be rewritten as

$$\underset{\text{natural } S_n \text{ action}}{\underset{\uparrow}{A_n}} \cong \underset{\text{extended } S_n \text{ action}}{\underset{\uparrow}{A_{n-1}}} \otimes \operatorname{Ind}_{S_{n-1}}^{S_n} \boxed{}$$

Using the exercise $\quad U \otimes \operatorname{Ind}_{S_{n-1}}^{S_n} \boxed{} \cong \operatorname{Ind}_{S_{n-1}}^{S_n}\left(\operatorname{Res}_{S_{n-1}}^{S_n} U\right)$

we have

$$A_n \cong \operatorname{Ind}_{S_{n-1}}^{S_n} \underbrace{\left(\operatorname{Res}_{S_{n-1}}^{S_n} A_{n-1}\right)}_{\text{this is the natural } S_{n-1} \text{ action on } A_{n-1}} \cong \operatorname{Ind}_{S_{n-1}}^{S_n} A_{n-1}$$

Inductively we find $\quad A_n \cong \operatorname{Ind}_{S_2}^{S_n} A_2 .$

This, since $A_2 \cong 2\ \boxed{}\!\boxed{}$ as S_2 representation,

gives a quick proof of the following result of Lehrer:

Theorem (Lehrer 1987)

$$A_n \cong 2\ \mathrm{Ind}_{S_2}^{S_n} \boxed{}\!\boxed{}$$

Remark A generalization is provided by the algebra

$$H^*\left(C_n(\mathbb{R}^d), \mathbb{C}\right)$$

where $C_n(\mathbb{R}^d) = \left\{ (v_1, \quad, v_n) \in (\mathbb{R}^d)^n \;\middle|\; v_i \neq v_j \text{ if } i \neq j \right\}$

The case $d=2$ (in fact d even) gives A_n.
The case d odd has been studied by several authors (F. Cohen, Lehrer, Mathieu) and can also be studied using the same methods illustrated above (extended S_{n+1} action).