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Classical Solutions of Nonautonomous Riccati Equations Arising in Parabolic Boundary Control Problems, II

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Abstract. An abstract linear-quadratic regulator problem over finite time horizon is considered; it covers a large class of linear nonautonomous parabolic systems in bounded domains, with boundary control of Dirichlet or Neumann type. We give the proof of some result stated in [AT5], and in addition we prove uniqueness of the Riccati operator, provided its final datum is suitably regular.

Key Words. Optimal pair, Boundary control, Parabolic systems.

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0. Introduction

Let H, U be complex Hilbert spaces; for fixed $T > 0$ we consider the following linear-quadratic regulator problem: minimize

$$J(u) := \int_0^T \left\{(M(t)y(t) \mid y(t))_H + (N(t)u(t) \mid u(t))_U\right\} dt + (P_T y(T) \mid y(T))_H \tag{0.1}$$

over all controls $u \in L^2(0, T; U)$ subject to the state equation

$$y(t) = U(t, 0)x - \int_0^t U(t, r)A(r)G(r)u(r)\, dr, \qquad t \in [0, T]. \tag{0.2}$$

Here $\{M(t)\}$ and P_T are positive, bounded, self-adjoint operators in H, $\{N(t)\}$ are positive, bounded, self-adjoint operators in U, x is an element of H, each $A(t)$ generates an analytic semigroup $\{e^{\tau A(t)}\}$ in H, $\{U(t,s)\}$ is the evolution operator associated to $\{A(t)\}$, and $G(t)$ is the "Green map" relative to $A(t)$. More precise assumptions on $\{A(t)\}$, $\{G(t)\}$, $\{M(t)\}$, $\{N(t)\}$, and P_T are listed in Section 1.

The state equation (0.2) represents a large class of linear parabolic nonautonomous initial-boundary value problems, with boundary controls of Dirichlet or Neumann type: see Section 9 of [AT5] for some typical examples. Looking for a pointwise feedback optimal control for problem (0.1)–(0.2), the main step is the study of the associated Riccati equation, whose integral version is

$$\begin{aligned} P(t) = U(T,t)^* P_T U(T,t) + \int_t^T U(r,t)^* \\ \times \left[M(r) - P(r)A(r)G(r)N(r)^{-1}G(r)^*A(r)^*P(r) \right] U(r,t)\, dr, \end{aligned} \tag{0.3}$$

and whose differential version is

$$\begin{cases} P'(t) + A(t)^*P(t) + P(t)A(t) \\ \qquad = -M(t) + P(t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t), \\ P(T) = P_T. \end{cases} \tag{0.4}$$

The Riccati equation and its corresponding control problem in the autonomous case have been widely studied by several people and the whole theory is, more or less, complete: we quote, among others, [B], [LT1], [F1], [DI1], [F2], [F4], [LT3], and [LT4]. Two different approaches are available: (i) the variational method, which starts from the Euler equation for the cost functional and yields explicit formulas which express in terms of the data both the optimal pair and the Riccati operator, and (ii) the dynamic programming method, which solves directly the Riccati equation and obtains, through the Riccati operator, a feedback formula for the optimal control in terms of the optimal state. Both methods are carefully described in the survey papers [LT2] and [BDDM].

Only a few papers deal with the nonautonomous control problem (0.1)–(0.2); the references [Li] and [DS] are based on variational techniques, whereas in [DI2] and [AFT] the dynamic programming approach is used.

In [AFT] it was shown that under certain abstract assumptions, which are naturally fulfilled in the mentioned concrete parabolic problems of Section 9 of [AT5], (0.3) has a unique global solution $P(\cdot)$, where $P(t)$ is a positive, bounded, self-adjoint operator for each $t \in \,]0,T]$, provided the final datum P_T is suitably regular; consequently one gets the existence of an optimal pair $(\hat{u},\hat{y})$ for problem (0.1)–(0.2) in the space $L^2(0,T;U) \times L^2(0,T;H)$. On the other hand, in the autonomous case the minimal assumption on P_T is more general than in [AFT], and in addition the optimal pair turns out to enjoy some regularity properties, as shown in [LT1], [LT3], and [LT4].

Thus the main goal of our previous paper [AT5] was to extend as far as possible to the nonautonomous situation the results of [LT1], [LT3], and [LT4]. To this purpose we were unable to repeat, for a general choice of P_T, the direct proof of existence and uniqueness of mild solutions of (0.3), given in [AFT] by means of the dynamic programming technique; we followed instead the variational approach of [LT1] and [LT3], adapting and refining it according to the nonautonomous situation, through the extensive use of the nonautonomous theory of abstract parabolic equations developed in [AT1], [AT2], [A1], [AT3], [A2], and [AFT]. In this way we generalized to this situation almost all statements of [LT1], [LT3], and [LT4], even improving some of them. In fact in [AT5] we proved that (0.4) has a classical solution $P(\cdot)$, namely,

$$P(t) := \int_t^T U(r,t)^* M(r)\phi(r,t)\, dr + U(T,t)^* P_T \phi(T,t), \tag{0.5}$$

where $\phi(t,s)$ is the state operator, i.e.,

$$\phi(t,s)x := \hat{y}(t,s;x), \qquad \forall x \in H, \tag{0.6}$$

$\hat{y}(\cdot,s;x)$ being the optimal state for the control problem analogous to (0.1)–(0.2), with initial value x and initial time s instead of 0. The optimal control is then given in feedback form by

$$\hat{u}(t,s;x) = N(t)^{-1} G(t)^* A(t)^* P(t)\hat{y}(t,s;x). \tag{0.7}$$

In particular, we showed in [AT5] that $P(\cdot)$ is continuously differentiable in $[0,T[$ as an $\mathcal{L}(H)$-valued function and satisfies (0.4) in the sense of $\mathcal{L}(H)$, provided the operator $A(t)^*P(t) + P(t)A(t)$ is replaced by its bounded extension $\Lambda(t)P(t)$ (see Section 7 of [AT5] for details); the final datum P_T is taken essentially in the largest possible space, as the counterexample in [F3] and the remarks in Section 7 of [LT3] show (see also Remark 8.3 of [AT5]). In the autonomous case this result was first proved in [D] for problems with distributed control, and in [LT4] for boundary control, with some smoothness assumption on the final datum P_T.

This paper is a completion of [AT5]. Indeed, the differentiability result of [AT5] is based on certain related statements whose proofs, being very long and technical, were omitted there; thus we collect here those proofs, showing in addition the uniqueness of the Riccati operator $P(\cdot)$, provided the final datum P_T is suitably regular (exactly as in the autonomous case, see Theorem 6.4 of [LT3]).

We now describe the contents of the following sections. Section 1 contains the list of our assumptions; Section 2 concerns some results about linear Volterra integral equations in certain spaces of singular functions (the same ones as in Appendix A of [AT5]): these spaces were introduced in [AT1] as spaces of maximal regularity for abstract linear parabolic equations, but they were first used in [So]. In Section 3 we introduce the state operator $\phi(t,s)$ and prove its differentiability properties, and finally Section 4 is devoted to the uniqueness of the solution of (0.3). Throughout this paper we use the notations of [AT5].

1. Assumptions

We list here our abstract assumptions.

Hypothesis 1.1. *For each* $t \in [0,T]$, $A(t)$: $D_{A(t)} \subseteq H \to H$ *is a closed linear operator generating an analytic semigroup* $\{e^{\tau A(t)}, \tau \geq 0\}$; *in particular, there exist*

$M > 0$ and $\vartheta \in]\pi/2, \pi[$ such that

$$\left\|[\lambda - A(t)]^{-1}\right\|_{\mathcal{L}(H)} \leq M(1+|\lambda|)^{-1}, \qquad \forall\lambda \in \overline{S(\vartheta)}, \quad \forall t \in [0,T], \tag{1.1}$$

where $S(\vartheta) = \{z \in \mathbb{C}\colon\ |\arg z| < \vartheta\}$.

Hypothesis 1.2. *There exist $N > 0$ and $\rho, \mu \in]0,1]$ with $\delta := \rho + \mu - 1 \in]0,1[$, such that*

$$\begin{aligned}
&\left\|A(t)[\lambda - A(t)]^{-1}\left[A(t)^{-1} - A(s)^{-1}\right]\right\|_{\mathcal{L}(H)} \\
&\qquad + \left\|A(t)^*[\lambda - A(t)^*]^{-1}\left[[A(t)^*]^{-1} - [A(s)^*]^{-1}\right]\right\|_{\mathcal{L}(H)} \\
&\quad \leq N|t-s|^{\mu}(1+|\lambda|)^{-\rho}, \qquad \forall\lambda \in \overline{S(\vartheta)}, \quad \forall t,s \in [0,T].
\end{aligned} \tag{1.2}$$

Hypothesis 1.3. *$\{U(t,s), 0 \leq s \leq t \leq T\}$ is the evolution operator relative to $\{A(t), t \in [0,T]\}$; in particular,*

$$\begin{aligned}
&\left\|[-A(t)]^{\eta}U(t,s)[-A(s)]^{-\gamma}\right\|_{\mathcal{L}(H)} + \left\|[-A(s)^*]^{\eta}U(t,s)^*[-A(t)^*]^{-\gamma}\right\|_{\mathcal{L}(H)} \\
&\quad \leq M_{\eta\gamma}\left[1 + (t-s)^{\gamma-\eta}\right] \qquad \text{for} \quad 0 \leq s < t \leq T, \quad \eta,\gamma \in [0,1].
\end{aligned} \tag{1.3}$$

Hypothesis 1.4. *The number $\delta = \rho + \mu - 1$ is such that*

$$\begin{aligned}
&\left\|[-A(t)]^{\eta}U(t,s)[-A(s)]^{-\gamma} - [-A(\tau)]^{\eta}U(\tau,s)[-A(s)]^{-\gamma}\right\|_{\mathcal{L}(H)} \\
&\quad \leq N_{\gamma\eta}(t-\tau)^{\delta}\left[1 + (\tau-s)^{\gamma-\eta-\delta}\right] \\
&\qquad \text{for} \quad 0 \leq s < \tau \leq t \leq T, \quad \eta,\gamma \in [0,1],
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
&\left\|[-A(\sigma)^*]^{\eta}U(t,\sigma)^*[-A(t)^*]^{-\gamma} - [-A(s)^*]^{\eta}U(t,s)^*[-A(t)^*]^{-\gamma}\right\|_{\mathcal{L}(H)} \\
&\quad \leq N_{\gamma\eta}(\sigma-s)^{\delta}\left[1 + (t-\sigma)^{\gamma-\eta-\delta}\right] \\
&\qquad \text{for} \quad 0 \leq s \leq \sigma < t \leq T, \quad \eta,\gamma \in [0,1],
\end{aligned} \tag{1.5}$$

all operators being strongly continuous with respect to t, τ, σ, s.

Hypothesis 1.5. *For each $t \in [0,T]$, $G(t) \in \mathcal{L}(U,H)$ and there exists $\alpha \in]\delta,1]$ such that*

$$[-A(\cdot)]^{\alpha}G(\cdot) \in C^{\delta}([0,T], \mathcal{L}(U,H)). \tag{1.6}$$

Hypothesis 1.6. *We have $M(\cdot) \in C^{\delta}([0,T], \Sigma^+(H))$, $N(\cdot) \in C^{\delta}([0,T], \Sigma^+(U))$, and there exists $\nu > 0$ such that $N(t) \geq \nu$, i.e., $(N(t)u \mid u)_U \geq \nu\|u\|_U^2$ for each $u \in U$ and $t \in [0,T]$.*

Hypothesis 1.7. *$P_T \in \Sigma^+(H)$ and in addition the linear operator $P_T^{1/2} L_{0T} \colon D(L_{0T}) \subseteq L^2(0,T;U) \to H$ is closed, where the operator L_{0T} is defined by*

$$\begin{cases} D(L_{0T}) := \Big\{ u \in L^2(0,T,U) \colon \\ \qquad \displaystyle\int_0^T [-A(T)]^{-\eta} U(T,r) A(r) G(r) u(r)\, dr \in D([-A(T)]^{\eta}) \Big\}, \\ L_{0T}(u) := -[-A(T)]^{\eta} \displaystyle\int_0^T [-A(T)]^{-\eta} U(T,r) A(r) G(r) u(r)\, dr. \end{cases} \tag{1.7}$$

Comments and remarks on this set of assumptions can be found in Remark 1.8 of [AT5].

2. Linear Volterra Integral Equations

Let X be a Banach space. We introduce some spaces of X-valued functions, having a singularity at an endpoint of their interval of definition.

Definition 2.1. Let $[a,b]$ be a real interval.

(i) If $\gamma \geq 0$, $B_\gamma([a,b[, X)$ (resp. $B_\gamma(]a,b], X)$) is the Banach space of Bochner measurable functions $u \colon [a,b[\to X$ (resp. $u \colon]a,b] \to X$) such that $\|u\|_\gamma < \infty$, where

$$\|u\|_\gamma := \sup_{s \in [a,b[} (b-s)^\gamma \|u(s)\|_X \qquad \left(\text{resp. } \sup_{s \in]a,b]} (s-a)^\gamma \|u(s)\|_X \right).$$

(ii) If $\gamma \geq 0$, $C_\gamma([a,b[, X)$ (resp. $C_\gamma(]a,b], X)$) is the space of continuous functions belonging to $B_\gamma([a,b[, X)$ (resp. $B_\gamma(]a,b], X)$), endowed by the same norm.

(iii) If $\eta \in]0,1]$ and $\gamma \geq 0$, $Z_{\gamma,\eta}([a,b[, X)$ (resp. $Z_{\gamma,\eta}(]a,b], X)$) is the space of functions $u \in C_\gamma([a,b[, X)$ (resp. $C_\gamma(]a,b], X)$) such that $[u]_{\gamma,\eta} < \infty$, where

$$[u]_{\gamma,\eta} := \sup_{s \in [a,b[} \left\{ (b-s)^{\gamma+\eta} \sup_{s \leq p < q \leq (s+b)/2} (q-p)^{-\eta} \|u(q) - u(p)\|_X \right\}$$
$$\left(\text{resp. } \sup_{s \in]a,b]} \left\{ (s-a)^{\gamma+\eta} \sup_{(s+a)/2 \leq p < q \leq s} (q-p)^{-\eta} \|u(q) - u(p)\|_X \right\} \right).$$

(iv) If $\eta \in]0,1]$ and $\gamma \in [-\eta, 0[$, $Z_{\gamma,\eta}([a,b[, X)$ (resp. $Z_{\gamma,\eta}(]a,b], X)$) is the space of functions $u \in C^{|\gamma|}([a,b], X)$ such that $[u]_{\gamma,\eta} < \infty$, where $[u]_{\gamma,\eta}$ is defined as before.

The spaces $Z_{\gamma,\eta}$ are Banach spaces with their obvious norms, i.e.,

$$\|u\|_{Z_{\gamma,\eta}} := \begin{cases} \|u\|_\gamma + [u]_{\gamma,\eta} & \text{if} \quad \gamma \geq 0, \\ \|u\|_\infty + [u]_{|\gamma|} + [u]_{\gamma,\eta} & \text{if} \quad \gamma \in [-\nu, 0[; \end{cases}$$

they will be useful in describing the weighted Hölder continuity of the optimal pair of problem (0.1)–(0.2).

The following characterization of the spaces $Z_{\gamma,\eta}$ holds true:

Proposition 2.2. *If $\eta \in]0,1]$ and $\gamma \ge -\eta$, $\gamma \ne 0$, then we have $w \in Z_{\gamma,\eta}([a,b[,X)$ if and only if $w\colon [a,b[\to X$ fulfills*

$$\|w(p)-w(q)\|_X \le c(p-q)^{\eta}(b-p)^{-\gamma-\eta} \qquad \text{for} \quad a \le q \le p < b. \tag{2.1}$$

Proof. Assume $w \in Z_{\gamma,\eta}([a,b[,X)$. Fix p,q with $a \le q \le p < b$. If $p \le (a+b)/2$ we have $a \le q \le p \le (a+b)/2$, so that, by definition of $Z_{\gamma,\eta}$,

$$\|w(p)-w(q)\|_X \le c(p-q)^{\eta}(b-a)^{-\gamma-\eta} \le c(p-q)^{\eta}(b-p)^{-\gamma-\eta}.$$

If $p > (a+b)/2$, then two cases may occur: $p-q < b-p$ or $p-q \ge b-p$. In the first case, we have $q \le p < (q+b)/2$, so that

$$\|w(p)-w(q)\|_X \le c(p-q)^{\eta}(b-q)^{-\gamma-\eta} \le c(p-q)^{\eta}(b-p)^{-\gamma-\eta};$$

in the second case we have $b-q \ge 2(b-p)$, so that

$$\begin{aligned}\|w(p)-w(q)\|_X &\le \|w(p)\|_X + \|w(q)\|_X \le c(b-p)^{-\gamma} + c(b-q)^{-\gamma}\\ &\le 2c(b-p)^{-\gamma} \le 2c(p-q)^{\eta}(b-p)^{-\eta-\gamma}.\end{aligned}$$

Note that this inclusion holds for $\gamma = 0$ too.

Assume conversely that w fulfills (2.1); we must prove that $w \in Z_{\gamma,\eta}([a,b[,X)$. It is obvious that if $a \le s \le q \le p \le (s+b)/2 < b$, then

$$\|w(p)-w(q)\|_X \le c(p-q)^{\eta}(b-p)^{-\gamma-\eta} \le 2^{\gamma+\eta}c(p-q)^{\eta}(b-s)^{-\gamma-\eta},$$

so that $[w]_{\gamma,\eta}$ is finite. Next, we have to show that

$$\begin{aligned}&\|w(p)\| \le c(b-p)^{-\gamma} \qquad \text{for} \quad a \le p < b \quad \text{if} \quad \gamma > 0,\\ &\|w(p)-w(q)\|_X \le c(p-q)^{-\gamma} \qquad \text{for} \quad a \le q \le p \le b \quad \text{if} \quad -\eta \le \gamma < 0.\end{aligned}$$

Suppose $\gamma > 0$. If $a \le p \le (a+b)/2$ we have

$$\begin{aligned}\|w(p)\|_X &\le \|w(p)-w(a)\|_X + \|w(a)\|_X \le c(p-a)^{\eta}(b-a)^{-\eta-\gamma} + \|w(a)\|_X\\ &\le c(b-a)^{-\gamma} + \|w(a)\|_X \le c(b-p)^{-\gamma} + \|w(a)\|_X,\end{aligned}$$

whereas if $(a+b)/2 < p < b$, i.e., $b-p < (b-a)/2$, then there exists $n \in \mathbb{N}^+$ such that

$$2^{-n-1}(b-a) < b-p \le 2^{-n}(b-a),$$

so that we have, denoting by a_k the point of $[a,b[$ whose distance from b is $2^{-k}(b-a)$,

$$\|w(p)\|_X \le \|w(p)-w(a_n)\|_X + \sum_{k=0}^{n-1}\|w(a_{k+1})-w(a_k)\|_X + \|w(a_0)\|_X$$

$$
\begin{aligned}
&\leq c(p-a_n)^{\eta}(b-p)^{-\eta-\gamma} \\
&\qquad + c\sum_{k=0}^{n-1}(a_{k+1}-a_k)^{\eta}(b-a_{k+1})^{-\eta-\gamma} + \|w(a_0)\|_X \\
&\leq c\sum_{k=0}^{n}(a_{k+1}-a_k)^{\eta}(b-a_{k+1})^{-\eta-\gamma} + \|w(a)\|_X \\
&= c(b-a)^{-\gamma}\sum_{k=0}^{n}2^{(k+1)\gamma} + \|w(a)\|_X \\
&\leq c'(b-a)^{-\gamma}2^{(n+2)\gamma} + \|w(a)\|_X \leq c''(b-p)^{-\gamma} + \|w(a)\|_X.
\end{aligned}
$$

This shows that $\|w\|_\gamma$ is finite if $\gamma > 0$.

Suppose finally $-\eta \leq \gamma < 0$. We proceed similarly: let $a \leq q \leq p \leq b$; if $p-q \leq b-p$, then recalling that $\gamma+\eta \geq 0$ we get

$$
\begin{aligned}
-\|w(p)-w(q)\|_X &\leq c(p-q)^{\eta}(b-p)^{-\eta-\gamma} \\
&= c(p-q)^{\eta+\gamma-\gamma}(b-p)^{-\eta-\gamma} \leq c(p-q)^{-\gamma},
\end{aligned}
$$

whereas if $p-q > b-p$, i.e., $b-q > 2(b-p)$, there exists $n \in \mathbb{N}^+$ such that

$$
2^{-n-1}(b-q) < b-p \leq 2^{-n}(b-q),
$$

and consequently, denoting by q_k the point in $[q,b]$ whose distance from b is $2^{-k}(b-q)$, we obtain (since $-\gamma > 0$)

$$
\begin{aligned}
\|w(p)-w(q)\|_X &\leq \|w(p)-w(q_n)\|_X + \sum_{k=0}^{n-1}\|w(q_{k+1})-w(q_k)\|_X \\
&\leq c(p-q_n)^{\eta}(b-p)^{-\eta-\gamma} + c\sum_{k=0}^{n-1}(q_{k+1}-q_k)^{\eta}(b-q_{k+1})^{-\eta-\gamma} \\
&\leq c(b-q)^{-\gamma}\sum_{k=0}^{n}2^{(k+1)\gamma} \leq c'(b-q)^{-\gamma} \\
&\leq c'(b-p)^{-\gamma} + c'(p-q)^{-\gamma} \leq 2c'(p-q)^{-\gamma}.
\end{aligned}
$$

This shows that $w \in C^{|\gamma|}([a,b],X)$. □

Remark 2.3. When $\gamma = 0$, a function may satisfy (2.1) without being bounded at the point b, as the scalar example $\log((b-x)/(b-a))$ shows; such a function cannot be in the space $Z_{0,\eta}([a,b[,X)$. However, if u: $[a,b[\, \to X$ satisfies property (2.1), then necessarily $u \in \mathrm{BMO}(a,b;X)$ (the space of functions with bounded mean oscillation).

Remark 2.4. In view of Definition 2.1, Hypotheses 1.3 and 1.4 just say that, for each $\gamma, \eta \geq 0$,

$$
\begin{cases}
t \to [-A(t)]^{\eta}U(t,s)[-A(s)]^{-\gamma} \in Z_{(\eta-\gamma)\vee 0,\delta}(]s,T],\mathcal{L}(H)), \\
s \to [-A(s)^*]^{\eta}U(t,s)^*[-A(t)^*]^{-\gamma} \in Z_{(\eta-\gamma)\vee 0,\delta}([0,t[,\mathcal{L}(H)).
\end{cases}
\tag{2.2}
$$

We introduce other spaces of functions having nonintegrable singularities.

Definition 2.5. Let $[a, b]$ be a real interval.

(i) If $\gamma \geq 1$, $I_\gamma([a, b[, X)$ (resp. $I_\gamma(]a, b], X)$) is the space of functions $u \in C_\gamma([a, b[, X)$ (resp. $u \in C_\gamma(]a, b], X)$) such that the limit

$$\lim_{h \to 0^+} \int_a^{b-h} u(t)\, dt \qquad \left(\text{resp. } \lim_{h \to 0^+} \int_{a+h}^{b} u(t)\, dt\right)$$

exists in the norm of X.

(ii) If $\gamma \geq 1$ and $\eta \in]0, 1]$, we set

$$Z^*_{\gamma,\eta}([a, b[, X) := Z_{\gamma,\eta}([a, b[, X) \cap I_\gamma([a, b[, X),$$
$$Z^*_{\gamma,\eta}(]a, b], X) := Z_{\gamma,\eta}(]a, b], X) \cap I_\gamma(]a, b], X).$$

The spaces $I_\gamma([a, b[, X)$ (resp. $I_\gamma(]a, b], X)$) are Banach spaces with the norm

$$\|u\|_{I_\gamma} := \|u\|_\gamma + \|u\|_*,$$

where

$$\|u\|_* := \sup\left\{\left\|\int_c^d u(t)\, dt\right\|_X : a \leq c \leq d < b\right\}$$
$$\left(\text{resp. } \|u\|_* := \sup\left\{\left\|\int_c^d u(t)\, dt\right\|_X : a < c \leq d \leq b\right\}\right);$$

for a proof see Lemma 1.7 of [AT1]. The spaces $Z^*_{\gamma,\eta}([a, b[, X)$, $Z^*_{\gamma,\eta}(]a, b], X)$ are Banach spaces with the norm

$$\|u\|_{Z^*_{\gamma,\eta}} := \|u\|_\gamma + [u]_{\gamma,\eta} + \|u\|_*.$$

We now recall some results concerning linear Volterra integral equations in the spaces B_γ, C_γ, $Z_{\gamma,\eta}$, I_γ, and $Z^*_{\gamma,\eta}$ introduced above.

Proposition 2.6. *Let $a, b \in \mathbb{R}$ with $a < b$, let X be a Banach space, let $0 < \delta < \alpha \leq 1$. Let $Q(t, s)$ be a bounded operator in X, continuous for $a \leq s < t \leq b$, and consider for $s \in [a, b[$ the integral operators*

$$(Q_s\varphi)(t) := \int_s^t Q(t, \sigma)\varphi(\sigma)\, d\sigma, \qquad t \in [s, b].$$

(i) *If*

$$\|Q(t, \sigma)\|_{\mathcal{L}(X)} \leq B(t - \sigma)^{\alpha-1} \qquad \textit{for} \quad a \leq \sigma < t \leq b, \tag{2.3}$$

then $Q_s \in \mathcal{L}(B_\gamma(]s, b], X))$ and $(1 - Q_s)^{-1} \in \mathcal{L}(B_\gamma(]s, b], X))$ for each $\gamma \in [0, 1[$.

(ii) *If* (2.3) *holds, and in addition*

$$\|Q(t, s) - Q(\tau, s)\|_{\mathcal{L}(X)} \leq B(t - \tau)^\delta(\tau - s)^{\alpha-1-\delta}$$
$$\textit{for} \quad a \leq s < \tau \leq t \leq b, \tag{2.4}$$

then $Q_s \in \mathcal{L}(B_\gamma(]s, b], X), Z_{(\gamma-\alpha)\vee(-\delta),\delta}(]s, b], X))$ for each $\gamma \in [0, 1[$.

(iii) *If* (2.3) *and* (2.4) *hold, and in addition*

$$\|Q(t,\tau)-Q(t,s)\|_{\mathcal{L}(H)} \le B(\tau-s)^{\delta}(t-\tau)^{\alpha-1-\delta} \quad \textit{for} \quad a \le s \le \tau < t \le b, \tag{2.5}$$

then $Q_s \in \mathcal{L}(I_\gamma(]s,b],X))$ *and* $(1-Q_s)^{-1} \in \mathcal{L}(I_\gamma(]s,b],X))$ *for each* $\gamma \in [1,1+\delta[$.

(iv) *Finally if* (2.3), (2.4), *and* (2.5) *hold, and in addition*

$$\|Q(t,q)-Q(\tau,q)-Q(t,s)-Q(\tau,s)\|_{\mathcal{L}(H)} \le B(t-\tau)^{\delta}(q-s)^{\delta}(\tau-q)^{\alpha-1-2\delta} \quad \textit{for} \quad a \le s \le q < \tau \le t \le b, \tag{2.6}$$

then $Q_s \in \mathcal{L}(I_\gamma(]s,b],X), Z_{\gamma-\alpha,\delta}(]s,b],X))$ *for each* $\gamma \in [1,1+\delta[$.

Moreover, in all cases the corresponding norms are bounded by constants depending only on $b-a$, B, α, δ.

Proof. All results follow by adapting the proof of Propositions 2.4 and 2.6 of [AT1]. However, we explicitly prove that $(1-Q_s)^{-1} \in \mathcal{L}(I_\gamma(]s,b],X))$ when $\gamma \in [1,1+\delta[$ since the corresponding proof in Proposition 2.6(ii) of [AT1] is not completely correct.

First it is tedious but easy to verify by induction that the iterated operator Q_s^m, $m \in \mathbb{N}^+$, is given by

$$Q_s^m\varphi(t) = \int_s^t Q_m(t,\sigma)\varphi(\sigma)\,d\sigma, \qquad t \in [s,b],$$

where

$$Q_1(t,\sigma) := Q(t,\sigma),$$
$$Q_m(t,\sigma) := \int_\sigma^t Q_{m-1}(t,q)Q(q,\sigma)\,dq, \qquad \forall m \in \mathbb{N}, \quad a \le \sigma < t \le b,$$

and in addition, for each $m \in \mathbb{N}^+$,

$$\|Q_m(t,\sigma)\|_{\mathcal{L}(X)} \le B_m(t-\sigma)^{m\alpha-1} \qquad \text{for} \quad a \le \sigma < t \le b, \tag{2.7}$$

$$\|Q_m(t,\sigma)-Q_m(\tau,\sigma)\|_{\mathcal{L}(H)} \le B_m(t-\tau)^{\delta}(t-\tau)^{m\alpha-1-\delta} \quad \text{for} \quad a \le s \le \tau < t \le b, \tag{2.8}$$

$$\|Q_m(t,\tau)-Q_m(t,s)\|_{\mathcal{L}(H)} \le B_m(\tau-s)^{\delta}(t-\tau)^{m\alpha-1-\delta} \quad \text{for} \quad a \le s \le \tau < t \le b, \tag{2.9}$$

$$\|Q_m(t,\sigma)-Q_m(\tau,\sigma)-Q_m(t,s)-Q_m(\tau,s)\|_{\mathcal{L}(H)} \le B_m(t-\tau)^{\delta}(\sigma-s)^{\delta}(\tau-\sigma)^{\alpha-1+2\delta} \quad \text{for} \quad a \le s \le \sigma < \tau \le t \le b, \tag{2.10}$$

where B_m is a constant depending only on $b-a$, B, α, δ and such that

$$\lim_{m\to\infty} B_m T^m = 0, \qquad \forall T > 0. \tag{2.11}$$

We show now that Q_s^m maps boundedly $I_\gamma(]s,b],X)$ into itself with norm less than $\frac{1}{2}$ provided m is sufficiently large. This will show that $(1-Q_s^m)^{-1}$, and hence $(1-Q_s)^{-1}$ too, belongs to $\mathcal{L}(I_\gamma(]s,b],X))$. Indeed we have

$$Q_s^m\varphi(t) = \int_s^t [Q_m(t,\sigma) - Q_m(t,s)]\varphi(\sigma)\, d\sigma + Q_m(t,s)\int_s^t \varphi(\sigma)\, d\sigma,$$

so that by (2.9) and (2.7) we get

$$\begin{aligned}(t-s)^\gamma \|Q_s^m\varphi(t)\|_X &\le B_m(t-s)^\gamma \int_s^t (\sigma-s)^\delta (t-\sigma)^{m\alpha-1-\delta}(\sigma-s)^{-\gamma}\, d\sigma[\varphi]_\gamma \\ &\quad + B_m(t-s)^{m\alpha+\gamma-1}[\varphi]_* \\ &\le cB_m(b-a)^{m\alpha}\|\varphi\|_{I_\gamma}, \qquad \forall t\in\,]s,b],\end{aligned}$$

i.e.,

$$\left[Q_s^m\varphi\right]_\gamma \le cB_m(b-a)^{m\alpha}\|\varphi\|_{I_\gamma}. \tag{2.12}$$

Similarly, we easily obtain, for each p,r with $s<p<r\le b$,

$$\begin{aligned}\left\|\int_p^r Q_s^m\varphi(t)\, dt\right\|_X &\le cB_m\int_p^r (t-s)^{m\alpha-\gamma}\, dt[\varphi]_\gamma \\ &\quad + B_m\int_p^r (t-s)^{m\alpha-1}\, dt[\varphi]_* \le cB_m(b-a)^{m\alpha}\|\varphi\|_{I_\gamma},\end{aligned}$$

i.e.,

$$\left[Q_s^m\varphi\right]_* \le cB_m(b-a)^{m\alpha}\|\varphi\|_{I_\gamma}. \tag{2.13}$$

The result follows by (2.11), (2.12), and (2.13). □

Remark 2.7. From Proposition 2.6 we also deduce an integral representation of the operator $(1-Q_s)^{-1}$: Indeed we have, for $t\in[s,T-\varepsilon]$,

$$[(1-Q_s)^{-1}\varphi](t) = \varphi(t) + \sum_{m=1}^{\infty}[Q_s^m\varphi](t) = \varphi(t) + \sum_{m=1}^{\infty}\int_s^t Q_m(t,\sigma)\varphi(\sigma)\, d\sigma$$

and by (2.7) we get

$$[(1-Q_s)^{-1}\varphi](t) = \varphi(t) + \int_s^t R(t,s)\varphi(\sigma)\, d\sigma, \tag{2.14}$$

where the kernel $R(t,\sigma)$ is given by

$$R(t,\sigma) := \sum_{m=1}^{\infty} Q_m(t,\sigma) \tag{2.15}$$

and satisfies estimates similar to (2.7), (2.8), (2.9), and (2.10).

3. Differentiability of $s \to \Phi(t, s)$

We collect the main properties of the state operator $\phi(t, s)$ defined by (0.6). By the results of Sections 4 and 5 of [AT5] we have $\phi(t, s) \in \mathcal{L}(H)$ for $0 \le s \le t < T$ and

$$\phi(t,t) = 1_H, \qquad \phi(t,s) = \phi(t,r)\phi(r,s) \qquad \text{for} \quad 0 \le s \le r \le t \le T, \tag{3.1}$$

$$\|\phi(\cdot, s)\|_{\mathcal{L}(H, L^2(s,T,H))} \le c, \qquad \forall s \in [0, T[, \tag{3.2}$$

$$t \to \phi(t,s)x \in C_{1-2\alpha}([s, T[, H), \qquad \forall x \in H, \quad \forall s \in [0, T[, \tag{3.3}$$

$$s \to \phi(t,s)x \in C([0,t], H) \qquad \forall x \in H, \quad \forall t \in]0, T[, \tag{3.4}$$

$$\left\| P_T^{1/2} \phi(T,s) \right\|_{\mathcal{L}(H)} \le c, \qquad \forall s \in [0, T[. \tag{3.5}$$

Moreover, $\phi(t, s)$ satisfies the following integral equation for $0 \le s \le t < T$:

$$\begin{aligned}\phi(t,s) &= U(t,s) \\ &\qquad - \int_s^t U(t,\tau)A(\tau)G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau)\phi(\tau,s)\, d\tau.\end{aligned} \tag{3.6}$$

This equation will be the starting point in order to prove differentiability of $s \to \phi(t, s)$ for $0 \le s < t \le T$.

Proposition 3.1 [AT5, Proposition 6.1]. *Under Hypotheses* 1.1–1.7 *let* $\phi(t, s)$ *be the operator defined by* (0.6). *Then for* $0 \le s < t < T$ *we have*

$$\begin{aligned}&\lim_{h \to 0} \left(\frac{\phi(t+h,s) - \phi(t,s)}{h} x \mid y \right)_H \\ &\quad = \left(\left[1_H - G(t)N(t)^{-1}G(t)^* A(t)^* P(t) \right] \phi(t,s)x \mid A(t)^* y \right)_H \\ &\qquad \forall x \in H, \quad \forall y \in D_{A(t)^*}.\end{aligned}$$

Proof. By (3.6) we get, for $0 < h < (T - t)/2$,

$$\begin{aligned}&\left(\frac{\phi(t+h,s) - \phi(t,s)}{h} x \mid y \right)_H \\ &\quad = \left(\frac{U(t+h,s) - U(t,s)}{h} x \mid y \right)_H \\ &\qquad + \left(\int_t^{t+h} [-A(\tau)]^\alpha G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau)\phi(\tau,s)x \, d\tau \mid \right. \\ &\qquad\qquad \left. \times \left[[-A(\tau)^*]^{1-\alpha} U(t+h,\tau)^* - [-A(t)^*]^{1-\alpha} U(t+h,t)^* \right] y \right)_H \\ &\qquad + \left(\int_t^{t+h} [-A(\tau)]^\alpha G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau)\phi(\tau,s)x \, d\tau \mid \right. \\ &\qquad\qquad \left. \times [-A(t)^*]^{1-\alpha} U(t+h,t)^*]y \right)_H\end{aligned}$$

$$+\left(\int_s^t [-A(\tau)]^\alpha G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau)\phi(\tau,s)x \, d\tau \mid \right.$$
$$\left. \times [-A(\tau)^*]^{1-\alpha} \frac{U(t+h,\tau)^* - U(t,\tau)^*}{h} y \right)_H .$$

Letting $h \to 0^+$, by Hypothesis 1.4 we obtain, in a straightforward way,

$$\left(\frac{\phi(t+h,s)-\phi(t,s)}{h}x \mid y\right)_H \to (A(t)U(t,s)x \mid y)_H + 0$$
$$+ \left([-A(t)]^\alpha G(t)N(t)^{-1}G(t)^* A(t)^* P(t)\phi(t,s)x \mid [-A(t)^*]^{1-\alpha} y\right)_H$$
$$+ \left(\int_s^t [-A(\tau)]^\alpha G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau)\phi(\tau,s)x \, d\tau \mid \right.$$
$$\left. [-A(\tau)^*]^{1-\alpha} U(t,\tau)^* y \right)_H .$$

A similar computation when $(s-t)/2 < h < 0$ yields the same conclusion as $h \to 0^-$, and the result follows. □

We are unable to get a better result, i.e., differentiability of $t \to \phi(t,s)x$ in a stronger sense: see Remark 6.2 of [AT5].

We now examine the differentiability properties of $s \to \phi(t,s)$. Fix a number $\varepsilon \in]0, T[$. For $0 \le q < T - \varepsilon$ we introduce the integral operator

$$[K_q\varphi](t) := \int_q^t U(t,\tau)A(\tau)G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau)\varphi(\tau)\, d\tau,$$
$$t \in [q, T-\varepsilon], \tag{3.7}$$

whose kernel is

$$K(t,\tau) := U(t,\tau)A(\tau)G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau), \qquad 0 \le \tau < t < T, \tag{3.8}$$

and satisfies, by Hypotheses 1.4–1.6 and by (5.9) of [AT5],

$$\|K(t,\tau)\|_{\mathcal{L}(H)} \le c(t-\tau)^{\alpha-1}(T-\tau)^{\alpha-1} \qquad \text{for} \quad 0 \le \tau < t < T; \tag{3.9}$$

in particular,

$$\|K(t,\tau)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^{\alpha-1} \qquad \text{for} \quad 0 \le \tau < t \le T-\varepsilon. \tag{3.10}$$

We are going to show that the results of Proposition 2.6 are applicable to the operators K_q in the interval $[0, T-\varepsilon]$ and in the space H. To start with, by (3.10) we see that Proposition 2.6(i) holds, so that $(1+K_q)^{-1}$ belongs to $\mathcal{L}(B_\gamma(]q, T-\varepsilon], \mathcal{L}(H)))$ for each $\gamma \in [0,1[$ and

$$\|(1+K_q)^{-1}\|_{\mathcal{L}(B_\gamma(]q,T-\varepsilon],\mathcal{L}(H)))} \le c_\varepsilon, \qquad \forall q \in [0, T-\varepsilon], \quad \forall \gamma \in [0,1[$$

(the space B_γ is introduced in Definition 2.1). Moreover, by Remark 2.7

$$[(1+K_q)^{-1}\phi](t) = \phi(t) + \int_q^t R(t,\sigma)\phi(\sigma)\, d\sigma,$$
$$\forall \phi \in B_\gamma(]q, T-\varepsilon], H), \quad \forall t \in [q, T-\varepsilon], \tag{3.11}$$

where the kernel $R(t, \sigma)$ is given (compare with (2.14) and (2.15)) by

$$R(t, \sigma) := \sum_{m=1}^{\infty} (-1)^m K_m(t, \sigma), \tag{3.12}$$

with

$$K_1(t, \sigma) := K(t, \sigma), \qquad K_{m+1}(t, \sigma) := \int_{\sigma}^{t} K_m(t, q) K(q, \sigma)\, dq, \qquad \forall m \in \mathbb{N}^+.$$

It also satisfies

$$\|R(t, \tau)\|_{\mathcal{L}(H)} \le c_\varepsilon (t - \tau)^{\alpha - 1} \qquad \text{for} \quad 0 \le \tau < t \le T - \varepsilon. \tag{3.13}$$

Hence we can rewrite (3.6), for $0 \le s < t \le T - \varepsilon$, as

$$\begin{aligned} \phi(t, s) &= \left[(1 + K_s)^{-1}(U(\cdot, s))\right](t) \\ &= U(t, s) + \int_s^t R(t, \sigma) U(\sigma, s)\, d\sigma, \qquad t \in [s, T - \varepsilon], \end{aligned} \tag{3.14}$$

and for small $h > 0$ we easily obtain

$$\begin{aligned} &\frac{\phi(t, s + h) - \phi(t, s)}{h} \\ &\quad = \left[(1 + K_{s+h})^{-1} \left(\frac{U(\cdot, s + h) - U(\cdot, s)}{h} + \frac{1}{h} \int_s^{s+h} K(\cdot, \tau) \phi(\tau, s)\, d\tau \right)\right](t) \\ &\quad \text{for} \quad s < s + h \le t \le T - \varepsilon, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\frac{\phi(t, s - h) - \phi(t, s)}{-h} \\ &\quad = \left[(1 + K_s)^{-1} \left(\frac{U(\cdot, s - h) - U(\cdot, s)}{-h} + \frac{1}{h} \int_{s-h}^{s} K(\cdot, \tau) \phi(\tau, s - h)\, d\tau \right)\right](t) \\ &\quad \text{for} \quad 0 \le s - h < s \le t \le T - \varepsilon; \end{aligned} \tag{3.16}$$

now we have to let $h \to 0^+$. We need some lemmas.

Lemma 3.2 [AT5, Lemma 6.3]. *Under Hypotheses* 1.1–1.7 *let* $\phi(t, s)$ *be defined by* (0.6). *Then*

$$\|\phi(t, \tau) - \phi(t, s)\|_{\mathcal{L}(H)} \le c_\varepsilon (\tau - s)^\delta (t - \tau)^\delta \qquad \textit{for} \quad 0 \le s \le \tau < t \le T - \varepsilon.$$

Proof. From (3.14) we deduce

$$\begin{aligned} \phi(t, \tau) - \phi(t, s) &= (1 + K_\tau)^{-1} \left(U(\cdot, \tau) - U(\cdot, s) + \int_s^\tau K(\cdot, q) \phi(q, s)\, dq \right)(t) \\ &= U(t, \tau) - U(t, s) + \int_s^\tau K(t, q) \phi(q, s)\, dq + \int_\tau^t R(t, \sigma) \\ &\quad \times \left(U(\sigma, \tau) - U(\sigma, s) + \int_s^\tau K(\sigma, q) \phi(q, s)\, dq \right) d\sigma. \end{aligned} \tag{3.17}$$

Now by Hypothesis 1.4 we have

$$\|U(r,\tau)-U(r,s)\|_{\mathcal{L}(H)} \le c(\tau-s)^{\delta}(r-\tau)^{-\delta} \quad \text{for} \quad 0 \le s \le \tau < r \le T, \tag{3.18}$$

whereas by (3.9) and (3.3) we get, for $0 \le s \le \tau < r < T$,

$$\left\| \int_s^\tau K(r,q)\phi(q,s)\,dq \right\|_{\mathcal{L}(H)} \le c\int_s^\tau (T-q)^{3\alpha-2}(r-q)^{\alpha-1}\,dq \le c(T-\tau)^{3\alpha-2}(\tau-s)^{\alpha},$$

and, in particular,

$$\left\| \int_s^\tau K(r,q)\phi(q,s)\,dq \right\|_{\mathcal{L}(H)} \le c_\varepsilon(\tau-s)^{\alpha} \qquad \text{for} \quad 0 \le s \le \tau < r \le T-\varepsilon;$$

hence, by (3.18), (3.13), (3.3), and (3.17),

$$\|\phi(t,\tau)-\phi(t,s)\|_{\mathcal{L}(H)} \le c(\tau-s)^{\delta}(t-\tau)^{-\delta} + c_\varepsilon \int_s^\tau (t-q)^{\alpha-1}\,dq + c_\varepsilon \int_\tau^t (t-\sigma)^{\alpha-1}\left[(\tau-s)^{\delta}(\sigma-\tau)^{-\delta}+(\tau-s)^{\alpha}\right]d\sigma,$$

and the result follows. □

Lemma 3.3 [AT5, Lemma 6.4]. *Under Hypotheses* 1.1–1.7 *let* $P(t)$ *be defined by* (0.5). *Then*

$$\|[-A(\tau)^*]^{1-\alpha}P(\tau)-[-A(s)^*]^{1-\alpha}P(s)\|_{\mathcal{L}(H)} \le c_\varepsilon(\tau-s)^{\delta} \quad \textit{for} \quad 0 \le s \le \tau \le T-\varepsilon.$$

Proof. Starting from (0.5) we split

$$\begin{aligned}
&[-A(\tau)^*]^{1-\alpha}P(\tau)-[-A(s)^*]^{1-\alpha}P(s) \\
&\quad = \int_\tau^T \left[[-A(\tau)^*]^{1-\alpha}U(q,\tau)^* - [-A(s)^*]^{1-\alpha}U(q,s)^*\right] M(q)\phi(q,\tau)\,dq \\
&\qquad + \int_\tau^T [-A(s)^*]^{1-\alpha}U(q,s)^*M(q)[\phi(q,\tau)-\phi(q,s)]\,dq \\
&\qquad - \int_s^\tau [-A(s)^*]^{1-\alpha}U(q,s)^*M(q)\phi(q,s)\,dq \\
&\qquad + \left[[-A(\tau)^*]^{1-\alpha}U(T,\tau)^* - [-A(s)^*]^{1-\alpha}U(t,s)^*\right] P_T\phi(T,\tau) \\
&\qquad + [-A(s)^*]^{1-\alpha}U(T,s)^*]P_T[\phi(T,\tau)-\phi(T,s)] =: \sum_{i=1}^{5} I_i.
\end{aligned}$$

Now we have

$$\|I_1\|_{\mathcal{L}(H)} \le c\int_\tau^T (\tau-s)^\delta (q-\tau)^{\delta+\alpha-1}(T-q)^{2\alpha-1}\,dq \le c_\varepsilon(\tau-s)^\delta$$

(by Hypotheses 1.4, 1.6, and by (3.3));

$$\begin{aligned}
\|I_2\|_{\mathcal{L}(H)} &= \Bigg\| \int_\tau^{T-\varepsilon/2} [-A(s)^*]^{1-\alpha} U(q,s)^* M(q)[\phi(q,\tau)-\phi(q,s)]\,dq \\
&\quad + \int_{T-\varepsilon/2}^T [-A(s)^*]^{1-\alpha} U(q,s)^* M(q)\phi\left(q, T-\frac{\varepsilon}{2}\right) \\
&\quad \times \left[\phi\left(T-\frac{\varepsilon}{2},\tau\right) - \phi\left(T-\frac{\varepsilon}{2},s\right)\right]\,dq \Bigg\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon \int_\tau^{T-\varepsilon/2} (q-s)^{\alpha-1}(\tau-s)^\delta (q-\tau)^{-\delta}\,dq \\
&\quad + c_\varepsilon \int_{T-\varepsilon/2}^T (q-s)^{\alpha-1}(T-q)^{2\alpha-1}(\tau-s)^\delta \left(T-\frac{\varepsilon}{2}-\tau\right)^{-\delta}\,dq \\
&\le c_\varepsilon(\tau-s)^\delta
\end{aligned}$$

(by Hypotheses 1.3, 1.6, by (3.1), (3.3), and by Lemma 3.2);

$$\|I_3\|_{\mathcal{L}(H)} \le c\int_s^\tau (q-s)^{\alpha-1}(T-q)^{2\alpha-1}\,dq \le c_\varepsilon(\tau-s)^\alpha$$

(by Hypotheses 1.3, 1.6, and by (3.3));

$$\|I_4\|_{\mathcal{L}(H)} \le c(\tau-s)^\delta (T-t)^{\delta+\alpha-1} \le c_\varepsilon(\tau-s)$$

(by Hypothesis 1.4 and by (3.5));

$$\begin{aligned}
\|I_5\|_{\mathcal{L}(H)} &= \big\| [-A(s)^*]^{1-\alpha} U(t,s)^*] P_T \phi(T, T-\varepsilon) \\
&\quad \times [\phi(T-\varepsilon,\tau) - \phi(T-\varepsilon,s)] \big\|_{\mathcal{L}(H)} \\
&\le c(T-s)^{\alpha-1}(\tau-s)^\delta \varepsilon^{-\delta} \le c_\varepsilon(\tau-s)^\delta
\end{aligned}$$

(by Hypothesis 1.3, by (3.1), (3.5), and by Lemma 3.2),

and the result follows. □

Lemma 3.4 [AT5, Lemma 6.5]. *Under Hypotheses* 1.1–1.7 *let* $K(t,\tau)$ *be defined by* (3.8). *Then*

(i) $\|K(t,s)-K(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^\delta (\tau-s)^{\alpha-1-\delta}$ *for* $0 \le s < \tau \le t \le T-\varepsilon$,
(ii) $\|K(t,\tau)-K(t,s)\|_{\mathcal{L}(H)} \le c_\varepsilon (\tau-s)^\delta (t-\tau)^{\alpha-1-\delta}$ *for* $0 \le s \le \tau < t \le T-\varepsilon$,
(iii) $\|K(t,q)-K(\tau,q)-K(t,s)-K(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^\delta (q-s)^\delta (\tau-q)^{\alpha-1-2\delta}$ *for* $0 \le s \le q < \tau \le t \le T-\varepsilon$.

Proof. (i) By (3.8), (3.18), and (3.10),

$$\|K(t,s)-K(\tau,s)\|_{\mathcal{L}(H)} = \left\| \left[U\left(t, \frac{\tau+s}{2}\right) - U\left(\tau, \frac{\tau+s}{2}\right)\right] K\left(\frac{\tau+s}{2}, s\right)\right\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^\delta (\tau-s)^{\alpha-1-\delta}.$$

(ii) We have, by (3.8),

$$\begin{aligned} K(t,\tau) = {}& \left[[-A(\tau)^*]^{1-\alpha} U(t,\tau)^*\right]^* \left[[-A(\tau)]^\alpha G(\tau)\right] N(\tau)^{-1} \\ & \times \left[[-A(\tau)]^\alpha G(\tau)\right]^* \left[[-A(\tau)^*]^{1-\alpha} P(\tau)\right], \qquad 0 \le \tau < t < T, \end{aligned} \tag{3.19}$$

so that by Hypotheses 1.4–1.6, by (5.8) of [AT5], and by Lemma 3.3 we obtain in a straightforward way

$$\|K(t,\tau)-K(t,s)\|_{\mathcal{L}(H)} \le c_\varepsilon \left[(\tau-s)^\delta (t-\tau)^{\alpha-1-\delta} + (\tau-s)^\delta (t-\tau)^{\alpha-1}\right],$$

and the result follows.

(iii) We have, by (3.8), Hypothesis 1.3, and (ii),

$$\begin{aligned} &\|K(t,q)-K(\tau,q)-K(t,s)-K(\tau,s)\|_{\mathcal{L}(H)} \\ &\quad = \left\| \int_\tau^t A(p) U\left(p, \frac{\tau+q}{2}\right) \left[K\left(\frac{\tau+q}{2}, q\right) - K\left(\frac{\tau+q}{2}, s\right)\right] dp \right\|_{\mathcal{L}(H)} \\ &\quad \le c_\varepsilon \int_\tau^t \left(p - \frac{\tau+q}{2}\right)^{-1} (q-s)^\delta (\tau-q)^{\alpha-1-\delta}\, dp \\ &\quad \le c_\varepsilon (t-\tau)^\delta (\tau-q)^{-\delta} (q-s)^\delta (\tau-q)^{\alpha-1-\delta}, \end{aligned}$$

and the result follows. □

Remark 3.5 [AT5, Remark 6.6]. (i) In view of the results of Proposition 2.6, Lemma 3.4 tells us that the operators K_q and $(1+K_q)^{-1}$ beong to $\mathcal{L}(I_\gamma(]q, T-\varepsilon], H))$, $\gamma \in [1, 1+\delta[$, for each $q \in [0, T-\varepsilon[$, with norms bounded independently of q (the space I_γ is introduced in Definition 2.5).

(ii) The kernel $R(t,\sigma)$ introduced in (3.12) satisfies the same estimates as $K(t,\sigma)$ does, i.e., (3.13) and

$$\|R(t,s)-R(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^\delta (\tau-s)^{\alpha-1-\delta} \quad \text{for} \quad 0 \le s < \tau \le t \le T-\varepsilon, \tag{3.20}$$

$$\|R(t,\tau)-R(t,s)\|_{\mathcal{L}(H)} \le c_\varepsilon (\tau-s)^\delta (t-\tau)^{\alpha-1-\delta} \quad \text{for} \quad 0 \le s \le \tau < t \le T-\varepsilon, \tag{3.21}$$

$$\|R(t,q)-R(\tau,q)-R(t,s)-R(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^\delta (q-s)^\delta (\tau-q)^{\alpha-1-2\delta} \quad \text{for} \quad 0 \le s \le q < \tau \le t \le T-\varepsilon. \tag{3.22}$$

This is proved in Remark 6.6(ii) of [AT5].

Lemma 3.6 [AT5, Lemma 6.7]. *Under Hypotheses* 1.1–1.4 *there is an operator* $V(t, s) \in \mathcal{L}(H)$, *continuous for* $0 \le s < t \le T$, *such that for* $0 \le s \le \sigma < t \le T$ *we have*:

(i) $(d/ds)U(t,s) = V(t,s)$, $V(t,s)x = -U(t,s)A(s)x$, $\forall x \in D_{A(s)}$;
(ii) $\|V(t,s)\|_{\mathcal{L}(H)} \le c(t-s)^{-1}$;
(iii) $\|V(t,s) + A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(H)} \le c(t-s)^{\delta-1}$;
(iv) $\|V(t,\sigma) - V(t,s)\|_{\mathcal{L}(H)} \le c_\eta(\sigma-s)^\eta(t-\sigma)^{-1-\eta}$, $\forall \eta \in]0,\delta[$;
(v) $\|V(t,\sigma) + A(\sigma)e^{(t-\sigma)A(\sigma)} - V(t,s) - A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(H)} \le c_\eta(\sigma-s)^\eta (t-\sigma)^{\delta-1-\eta}$, $\forall \eta \in]0,\delta[$;
(vi) $\|[V(t,\sigma) - V(t,s)]A(s)^{-1}\|_{\mathcal{L}(H)} \le c_\eta[(\sigma-s)^\eta(t-\sigma)^{-\eta} + (\sigma-s)^\mu(t-\sigma)^{\rho-1}]$, $\forall \eta \in]0,\delta[$.

Proof. Statements (i)–(iv) follow by Theorems 6.4 and 6.5 of [AT3]; part (v) is implicitly proved there, taking into account also Lemma 2.2 of [A1]. Finally concerning part (vi) we have

$$\begin{aligned}
&\left\|[V(t,\sigma) - V(t,s)]A(s)^{-1}\right\|_{\mathcal{L}(H)} \\
&\quad\le \left\|[V(t,\sigma) - V(t,s)]\left[1_H - e^{(t-\sigma)A(s)}\right]A(s)^{-1}\right\|_{\mathcal{L}(H)} \\
&\qquad + \left\|V(t,\sigma)\left[e^{(t-\sigma)A(s)} - e^{(t-\sigma)A(\sigma)}\right]A(s)^{-1}\right\|_{\mathcal{L}(H)} \\
&\qquad + \left\|U(t,\sigma)\left[A(\sigma)e^{(t-\sigma)A(\sigma)} - A(s)e^{(t-s)A(s)}\right]A(s)^{-1}\right\|_{\mathcal{L}(H)} \\
&\qquad + \left\|[U(t,\sigma) - U(t,s)]e^{(t-s)A(s)}\right\|_{\mathcal{L}(H)};
\end{aligned}$$

the result then follows in a straightforward way, using (iv), (ii), (3.18), and the estimate

$$\begin{aligned}
&\|[A(q)^m e^{(\sigma-q)A(q)} - A(s)^m e^{(\sigma-s)A(s)}]A(s)^{-1}\|_{\mathcal{L}(H)} \\
&\quad\le c\left[(q-s)^\mu(\sigma-q)^{\rho-m} + \int_{\sigma-q}^{\sigma-s} \xi^{-m}\,d\xi\right] \\
&\qquad\qquad \text{for} \quad 0 \le s \le q < \sigma \le T, \qquad m \in \mathbb{N},
\end{aligned} \tag{3.23}$$

which is a consequence of Lemma 1.10(i)–(ii) of [AT1]. □

Remark 3.7. As shown in Remark 6.8 of [AT5], it follows by Lemma 3.6 that for each $x \in H$ the function $V(\cdot, s)x$ belongs to the space $I_1(]s, T], H)$, and in addition, for each $\eta \in]0, \delta[$,

$$V(\cdot, s) \in Z_{1,\eta}(]s,T], \mathcal{L}(H)), \qquad V(\cdot,s)x \in Z^*_{1,\eta}(]s,T], H), \quad \forall x \in H, \tag{3.24}$$

$$V(\cdot,s) + A(s)e^{(\cdot-s)A(s)} \in Z_{1-\delta,\eta}(]s,T], \mathcal{L}(H)). \tag{3.25}$$

Now we return to the integral equations (3.15) and (3.16). By (3.11) we can rewrite them as

$$\frac{\phi(t, s+h) - \phi(t,s)}{h}$$

$$= \frac{U(t,s+h)-U(t,s)}{h} + \frac{1}{h}\int_s^{s+h} K(t,\tau)\phi(\tau,s)\,d\tau + \int_{s+h}^t R(t,\sigma)$$
$$\times \left(\frac{U(\sigma,s+h)-U(\sigma,s)}{h} + \frac{1}{h}\int_s^{s+h} K(\sigma,\tau)\phi(\tau,s)\,d\tau\right) d\sigma,$$
$$t \in]s+h, T-\varepsilon], \tag{3.26}$$

$$\frac{\phi(t,s-h)-\phi(t,s)}{-h}$$
$$= \frac{U(t,s-h)-U(t,s)}{-h} + \frac{1}{h}\int_{s-h}^{s} K(t,\tau)\phi(\tau,s-h)\,d\tau + \int_s^t R(t,\sigma)$$
$$\times \left(\frac{U(\sigma,s-h)-U(\sigma,s)}{-h} + \frac{1}{h}\int_{s-h}^{s} K(\sigma,\tau)\phi(\tau,s-h)\,d\tau\right) d\sigma,$$
$$t \in]s, T-\varepsilon]. \tag{3.27}$$

Proceeding formally, letting $h \to 0^+$ we find

$$\phi_s(t,s) = V(t,s) + K(t,s) + \int_s^t R(t,\sigma)[V(\sigma,s)+K(\sigma,s)]\,d\sigma,$$

i.e.,

$$\phi_s(t,s) = \left[(1+K)s)^{-1}[V(\cdot,s)+K(\cdot,s)]\right](t), \qquad t \in]s, T-\varepsilon]. \tag{3.28}$$

Notice that this formula has no meaning in $\mathcal{L}(H)$, since the operator $(1+K_s)^{-1}$ acts in $I_1(]s,T-\varepsilon],\mathcal{L}(H))$ but does not operate in $Z_{1,\eta}(]s,T-\varepsilon],\mathcal{L}(H))$, whereas, by (3.24), $V(\cdot,s)$ is in the latter space but is not in the former one. However, $V(\cdot,s)x \in I_1(]s,T-\varepsilon],H)$ for each $x \in X$, so that instead of (3.28) we may write

$$\frac{d}{ds}[\phi(t,s)x] = \left[(1+K_s)^{-1}[V(\cdot,s)x+K(\cdot,s)x]\right](t),$$
$$\forall t \in]s,T-\varepsilon], \forall x \in H. \tag{3.29}$$

Nevertheless, we are going to show that $\phi_s(t,s)$ exists in the sense of $\mathcal{L}(H)$; in fact we have:

Theorem 3.8 [AT5, Theorem 6.9]. *Under Hypotheses* 1.1–1.7, *let $\phi(t,s)$ be the operator defined by* (0.6). *For $0 \le s < t < T$ it holds, in the sense of $\mathcal{L}(H)$, that*

$$\frac{d}{ds}\phi(t,s) = V(t,s) + \int_s^t R(t,\sigma)\left[V(\sigma,s)+A(s)e^{(\sigma-s)A(s)}\right] d\sigma$$
$$- \int_s^t [R(t,\sigma)-R(t,s)]A(s)e^{(\sigma-s)A(s)}\,d\sigma - R(t,s)e^{(t-s)A(s)},$$

where $V(t,s) = (d/ds)U(t,s)$ and $R(t,s)$ is defined by (3.12).

We remark that this formula reduces to (3.29) when applied to any $x \in H$, as is easily seen.

Proof. We need the following, tedious computation. Fix $\varepsilon \in\,]0, T[$, take $s \in [0, T-\varepsilon[$ and $t \in\,]s, T-\varepsilon]$, and assume that $0 < h \le (t-s)/2$. Denote by T_i, with $i = 1, 2, 3, 4$, the terms appearing on the right-hand side of formula (3.26); then we have

$$\begin{aligned}
T_1 &:= \frac{U(t, s+h) - U(t,s)}{h} \\
&\quad \text{(by Lemma 3.6(i))} \\
&= V(t,s) + o(1) \qquad \text{as} \quad h \to 0^+; \\
T_2 &:= \frac{1}{h}\int_s^{s+h} K(t,\tau)\phi(\tau,s)\,d\tau \\
&\quad \text{(by (3.6))} \\
&= \frac{1}{h}\int_s^{s+h} [K(t,\tau) - K(t,s)]\,\phi(\tau,s)\,d\tau \\
&\quad + K(t,s)\frac{1}{h}\int_s^{s+h} \left[U(\tau,s) - e^{(\tau-s)A(s)}\right]\,d\tau \\
&\quad + K(t,s)\frac{1}{h}\int_0^h e^{pA(s)}\,dp + K(t,s)\frac{1}{h}\int_s^{s+h}\int_s^\tau K(\tau,q)\phi(q,s)\,dq\,d\tau \\
&:= K(t,s)\frac{1}{h}\int_0^h e^{pA(s)}\,dp + \sum_{i=1}^3 T_{2i}. \qquad (3.30)
\end{aligned}$$

Now, by Lemma 3.4(ii) and (3.3),

$$\begin{aligned}
\|T_{21}\|_{\mathcal{L}(H)} &= \left\| \frac{1}{h}\int_s^{s+h} [K(t,\tau) - K(t,s)]\,\phi(\tau,s)\,d\tau \right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon \frac{1}{h}\int_s^{s+h} (\tau-s)^\delta (t-\tau)^{\alpha-1-\delta}(T-\tau)^{2\alpha-1}\,d\tau \le c(\varepsilon,t,s)h^\delta;
\end{aligned}$$

next, recalling that

$$\left\|U(\tau,s) - e^{(\tau-s)A(s)}\right\|_{\mathcal{L}(H)} \le c(\tau-s)^\delta \qquad \text{for} \quad 0 \le s \le \tau \le T \qquad (3.31)$$

(see (2.6) and Lemma 2.2(i) of [A1]), we get, by (3.10),

$$\begin{aligned}
\|T_{22}\|_{\mathcal{L}(H)} &= \left\| K(t,s)\frac{1}{h}\int_s^{s+h} \left[U(\tau,s) - e^{(\tau-s)A(s)}\right]\,d\tau \right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon (t-s)^{\alpha-1} h^\delta = c(\varepsilon,t,s)h^\delta,
\end{aligned}$$

whereas, by (3.10) and (3.3),

$$\begin{aligned}
\|T_{23}\|_{\mathcal{L}(H)} &= \left\| K(t,s)\frac{1}{h}\int_s^{s+h}\int_s^\tau K(\tau,q)\phi(q,s)\,dq\,d\tau \right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon (t-s)^{\alpha-1} h^\alpha = c(\varepsilon,t,s)h^\alpha.
\end{aligned}$$

Thus we deduce that

$$T_2 = K(t,s)\frac{1}{h}\int_0^t e^{pA(s)}\,dp + o(1) \qquad \text{as} \quad h \to 0^+. \tag{3.32}$$

Concerning T_3, using Lemma 3.6 we split it as follows:

$$\begin{aligned}
T_3 &:= \int_{s+h}^t R(t,\sigma)\frac{U(\sigma,s+h)-U(\sigma,s)}{h}d\sigma \\
&= \int_{s+h}^t R(t,\sigma)\frac{1}{h}\int_s^{s+h} V(\sigma,q)\,dq\,d\sigma \\
&= \int_{s+h}^t R(t,\sigma)\frac{1}{h} \\
&\quad \times \int_s^{s+h}\left[V(\sigma,q)+A(q)e^{(\sigma-q)A(q)} - V(\sigma,s) - A(s)e^{(\sigma-s)A(s)}\right]\,dq\,d\sigma \\
&\quad - \int_s^{s+h} R(t,\sigma)\left[V(\sigma,s)+A(s)e^{(\sigma-s)A(s)}\right]\,d\sigma \\
&\quad + \int_s^t R(t,\sigma)\left[V(\sigma,s)+A(s)e^{\sigma-s)A(s)}\right]\,d\sigma \\
&\quad - \int_{s+h}^t [R(t,\sigma)-R(t,s+h)]\,\frac{1}{h} \\
&\quad \times \int_s^{s+h}\left[A(q)e^{(\sigma-q)A(q)} - A(s)e^{(\sigma-s)A(s)}\right]\,dq\,d\sigma \\
&\quad + [R(t,s+h)-R(t,s)]\int_{s+h}^t A(s)e^{(\sigma-s)A(s)}\,d\sigma \\
&\quad + \int_s^{s+h}[R(t,\sigma)-R(t,s)]A(s)e^{(\sigma-s)A(s)}\,d\sigma \\
&\quad - \int_s^t [R(t,\sigma)-R(t,s)]A(s)e^{(\sigma-s)A(s)}\,d\sigma \\
&\quad - [R(t,s+h)-R(t,s)]\frac{1}{h}\int_s^{s+h}\left[e^{(t-q)A(q)} - e^{(s+h-q)A(q)}\right]\,dq \\
&\quad - R(t,s)\frac{1}{h}\int_s^{s+h}\left[e^{(t-q)A(q)} - e^{(t-s)A(s)}\right]\,dq + R(t,s)e^{(t-s)A(s)} \\
&\quad + R(t,s)\frac{1}{h}\int_s^{s+h}\left[e^{(s+h-q)A(q)} - e^{(s+h-q)A(s)}\right]\,dq \\
&\quad + R(t,s)\frac{1}{h}\int_s^{s+h} e^{pA(s)}\,dp \\
&=: \int_s^t R(t,\sigma)\left[V(\sigma,s)+A(s)e^{(\sigma-s)A(s)}\right]\,d\sigma \\
&\quad - \int_s^t [R(t,\sigma)-R(t,s)]A(s)e^{(\sigma-s)A(s)}\,d\sigma \\
&\quad - R(t,s)e^{(t-s)A(s)} + R(t,s)\frac{1}{h}\int_s^{s+h} e^{pA(s)}\,dp + \sum_{j=1}^{8} T_{3j}.
\end{aligned}$$

Now we have

$$
\begin{aligned}
&\|T_{31}\|_{\mathcal{L}(H)} \\
&\quad = \left\| \int_{s+h}^{t} R(t,\sigma)\frac{1}{h}\int_{s}^{s+h} \left[V(\sigma,q) + A(q)e^{(\sigma-q)A(q)} \right.\right. \\
&\qquad\qquad \left.\left. -V(\sigma,s) - A(s)e^{(\sigma-s)A(s)}\right]\, dq\, d\sigma \right\|_{\mathcal{L}(H)} \\
&\qquad\qquad \text{(by (3.13) and Lemma 3.6(v))} \\
&\quad \le c_\varepsilon \int_{s+h}^{t} (t-\sigma)^{\alpha-1}\frac{1}{h}\int_{s}^{s+h} (q-s)^{\delta/2}(\sigma-q)^{\delta/2-1}\, dq\, d\sigma \\
&\quad \le c_\varepsilon (t-s-h)^{\alpha+\delta/2-1}h^\delta \le c(\varepsilon,t,s)h^\delta;
\end{aligned}
$$

next, by (3.13) and Lemma 3.6(iii),

$$
\begin{aligned}
\|T_{32}\|_{\mathcal{L}(H)} &= \left\| \int_{s}^{s+h} R(t,\sigma)\left[V(\sigma,s) + A(s)e^{(\sigma-s)A(s)}\right]\, d\sigma \right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon \int_{s}^{s+h} (t-\sigma)^{\alpha-1}(\sigma-s)^{\delta-1}\, d\sigma \le c(\varepsilon,t,s)h^\delta.
\end{aligned}
$$

Now we recall the following property, similar to (3.23) and true for $0 \le s \le q < \sigma \le T$ and $m \in \mathbb{N}$, which follows easily by Lemma 1.10(i)–(ii) of [AT1]:

$$
\begin{aligned}
&\left\| A(q)^m e^{(\sigma-q)A(q)} - A(s)^m e^{(\sigma-s)A(s)} \right\|_{\mathcal{L}(H)} \\
&\quad \le c\left[(q-s)^\mu(\sigma-q)^{\rho-1-m} + \int_{\sigma-q}^{\sigma-s} \xi^{-1-m}\, d\xi \right].
\end{aligned} \tag{3.33}
$$

As $\delta = \mu + \rho - 1$, (3.33) implies that

$$
\begin{aligned}
&\|T_{33}\|_{\mathcal{L}(H)} \\
&\quad = \left\| \int_{s+h}^{t} [R(t,\sigma) - R(t,s+h)]\frac{1}{h} \right. \\
&\qquad\quad \left. \times \int_{s}^{s+h} \left[A(q)e^{(\sigma-q)A(q)} - A(s)e^{(\sigma-s)A(s)}\right]\, dq\, d\sigma \right\|_{\mathcal{L}(H)} \\
&\quad \le c_\varepsilon \int_{s+h}^{t} (\sigma-s-h)^\delta(t-\sigma)^{\alpha-1-\delta}\frac{1}{h} \\
&\qquad \times \int_{s}^{s+h} \left[(q-s)^\mu(\sigma-q)^{\rho-2} + \int_{\sigma-q}^{\sigma-s} \xi^{-2}\, d\xi \right] dq\, d\sigma \\
&\quad \le c_\varepsilon \int_{s+h}^{t} (\sigma-s-h)^{\delta-1}(t-\sigma)^{\alpha-1-\delta}\frac{1}{h}\int_{s}^{s+h} (q-s)^\mu(s+h-q)^{\rho-1}\, dq \\
&\qquad + c_\varepsilon \int_{s+h}^{t} (\sigma-s-h)^\delta(t-\sigma)^{\alpha-1-\delta}\frac{1}{h} \\
&\qquad \times \int_{s}^{s+h} (q-s)(\sigma-s)^{-1}(\sigma-q)^{-1}\, dq\, d\sigma
\end{aligned}
$$

$$\leq c_\varepsilon (t-s-h)^{\alpha-1} h^\delta + c_\varepsilon \int_{s+h}^t (\sigma - s - h)^{\delta-1} (t-\sigma)^{\alpha-1-\delta} h^{\delta/2} (\sigma - s)^{-\delta/2}\, d\sigma$$
$$\leq c(\varepsilon, t, s) h^\delta + c(\varepsilon, t, s) h^{\delta/2}.$$

Concerning T_{34}, T_{35}, and T_{36} we have, by (3.21),

$$\begin{aligned}\|T_{34}\|_{\mathcal{L}(H)} &= \left\| [R(t, s+h) - R(t,s)] \int_{s+h}^t A(s) e^{(\sigma - s)A(s)}\, d\sigma \right\|_{\mathcal{L}(H)} \\ &\leq c_\varepsilon h^\delta (t-s-h)^{\alpha-1-\delta} \left\| e^{(t-s)A(s)} - e^{hA(s)} \right\|_{\mathcal{L}(H)} \leq c(\varepsilon, t, s) h^\delta,\end{aligned}$$

$$\begin{aligned}\|T_{35}\|_{\mathcal{L}(H)} &= \left\| \int_s^{s+h} [R(t,\sigma) - R(t,s)] A(s) e^{(\sigma - s)A(s)}\, d\sigma \right\|_{\mathcal{L}(H)} \\ &\leq c_\varepsilon \int_s^{s+h} (\sigma - s)^{\delta-1} (t-\sigma)^{\alpha-1-\delta}\, d\sigma \leq c(\varepsilon, t, s) h^\delta;\end{aligned}$$

$$\begin{aligned}\|T_{36}\|_{\mathcal{L}(H)} &= \left\| [R(t, s+h) - R(t,s)] \frac{1}{h} \int_s^{s+h} \left[e^{(t-q)A(q)} - e^{(s+h-q)A(q)} \right] dq \right\|_{\mathcal{L}(H)} \\ &\leq c_\varepsilon h^\delta (t-s-h)^{\alpha-1-\delta} \leq c(\varepsilon, t, s) h^\delta.\end{aligned}$$

Finally by (3.13) and (3.33) we obtain

$$\begin{aligned}\|T_{37}\|_{\mathcal{L}(H)} &= \left\| R(t,s) \frac{1}{h} \int_s^{s+h} \left[e^{(t-q)A(q)} - e^{(t-s)A(s)} \right] dq \right\|_{\mathcal{L}(H)} \\ &\leq c_\varepsilon (t-s)^{\alpha-1} \frac{1}{h} \int_s^{s+h} \left[(q-s)^\mu (t-q)^{\rho-1} + \int_{t-q}^{t-s} \xi^{-1}\, d\xi \right] dq \\ &\leq c_\varepsilon (t-s)^{\alpha-1} \frac{1}{h} \int_s^{s+h} \left[(q-s)^\mu (s+h-q)^{\rho-1} + \log\left[1 + \frac{q-s}{t-q} \right] \right] dq \\ &\leq c_\varepsilon (t-s)^{\alpha-1} h^\delta + c_\varepsilon (t-s)^{\alpha-1} (t-s-h)^{-\delta} h^\delta \\ &\leq c(\varepsilon, t, s) h^\delta,\end{aligned}$$

$$\begin{aligned}\|T_{38}\|_{\mathcal{L}(H)} &= \left\| R(t,s) \frac{1}{h} \int_s^{s+h} \left[e^{(s+h-q)A(q)} - e^{(s+h-q)A(s)} \right] dq \right\|_{\mathcal{L}(H)} \\ &\leq c_\varepsilon (t-s)^{\alpha-1} \frac{1}{h} \int_s^{s+h} (q-s)^\mu (s+h-q)^{\rho-1} dq \leq c_\varepsilon (t-s)^{\alpha-1} h^\delta \\ &\leq c(\varepsilon, t, s) h^\delta.\end{aligned}$$

Summing up, we get

$$\begin{aligned} T_3 &= \int_s^t R(t,\sigma) \left[V(\sigma, s) + A(s) e^{(\sigma - s)A(s)} \right] d\sigma \\ &\quad - \int_s^t [R(t,\sigma) - R(t,s)]\, A(s) e^{(\sigma - s)A(s)}\, d\sigma - R(t,s) e^{(t-s)A(s)} \\ &\quad + R(t,s) \frac{1}{h} \int_s^{s+h} e^{pA(s)}\, dp + o(1) \qquad \text{as} \quad h \to 0^+. \end{aligned} \tag{3.34}$$

We finally consider T_4. Taking into account (3.6) we split it as follows:

$$\begin{aligned}
T_4 &= \int_{s+h}^{t} R(t,\sigma)\frac{1}{h}\int_s^{s+h} K(\sigma,\tau)\phi(\tau,s)\,d\tau\,d\sigma \\
&= \int_{s+h}^{t} R(t,\sigma)\frac{1}{h}\int_s^{s+h} [K(\sigma,\tau)-K(\sigma,s)]\phi(\tau,s)\,d\tau\,d\sigma \\
&\quad - \int_s^{s+h} R(t,\sigma)K(\sigma,s)\frac{1}{h}\int_s^{s+h} \phi(\tau,s)\,d\tau\,d\sigma \\
&\quad + \int_s^{t} R(t,\sigma)K(\sigma,s)\frac{1}{h}\int_s^{s+h} \left[U(\tau,s)-e^{(\tau-s)A(s)}\right]\,d\tau\,d\sigma \\
&\quad + \int_s^{t} R(t,\sigma)K(\sigma,s)\,d\sigma\frac{1}{h}\int_0^{h} e^{pA(s)}\,dp \\
&\quad - \int_s^{t} R(t,\sigma)K(\sigma,s)\frac{1}{h}\int_s^{s+h}\int_s^{\tau} K(\tau,q)\phi(q,s)\,dq\,d\tau\,d\sigma \\
&=: \int_s^{t} R(t,\sigma)K(\sigma,s)\,d\sigma\frac{1}{h}\int_0^{h} e^{pA(s)}\,dp + \sum_{h=1}^{4} T_{4h}.
\end{aligned}$$

We have, by (3.13), (3.8), Lemma 3.4(ii), and (3.3),

$$\begin{aligned}
\|T_{41}\|_{\mathcal{L}(H)} &= \left\|\int_{s+h}^{t} R(t,\sigma)\frac{1}{h}\int_s^{s+h}[K(\sigma,\tau)-K(\sigma,s)]\phi(\tau,s)\,d\tau\,d\sigma\right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon\int_{s+h}^{t}(t-\sigma)^{\alpha-1}\frac{1}{h}\int_s^{s+h}(\tau-s)^{\delta}(\sigma-\tau)^{\alpha-1-\delta}(T-\tau)^{2\alpha-1}\,d\tau\,d\sigma \\
&\le c_\varepsilon\int_{s+h}^{t}(t-\sigma)^{\alpha-1}(\sigma-s-h)^{\alpha-1-\delta}\,d\sigma h^{\delta} \le c(\varepsilon,t,s)h^{\delta},
\end{aligned}$$

$$\begin{aligned}
\|T_{42}\|_{\mathcal{L}(H)} &= \left\|\int_s^{s+h} R(t,\sigma)K(\sigma,s)\frac{1}{h}\int_s^{s+h}\phi(\tau,s)\,d\tau\,d\sigma\right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon\int_s^{s+h}(t-\sigma)^{\alpha-1}(\sigma-s)^{\alpha-1}(T-s-h)^{2\alpha-1}\,d\sigma \le c(\varepsilon,t,s)h^{\alpha},
\end{aligned}$$

$$\begin{aligned}
\|T_{44}\|_{\mathcal{L}(H)} &= \left\|\int_s^{t} R(t,\sigma)K(\sigma,s)\frac{1}{h}\int_s^{s+h}\int_s^{\tau}K(\tau,q)\phi(q,s)\,dq\,d\tau\,d\sigma\right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon\int_s^{t}(t-\sigma)^{\alpha-1}(\sigma-s)^{\alpha-1}\frac{1}{h}\int_s^{s+h}\int_s^{\tau}(\tau-q)^{\alpha-1}(T-q)^{2\alpha-1}\,dq\,d\tau\,d\sigma \\
&\le c(\varepsilon,t,s)h^{\alpha},
\end{aligned}$$

whereas, by (3.13), (3.8), and (3.31),

$$\begin{aligned}
\|T_{43}\|_{\mathcal{L}(H)} &= \left\|\int_s^{t} R(t,\sigma)K(\sigma,s)\frac{1}{h}\int_s^{s+h}\left[U(\tau,s)-e^{(\tau-s)A(s)}\right]\,d\tau\,d\sigma\right\|_{\mathcal{L}(H)} \\
&\le c_\varepsilon\int_s^{t}(t-\sigma)^{\alpha-1}(\sigma-s)^{\alpha-1}\frac{1}{h}\int_s^{s+h}(\tau-s)^{\delta}\,d\tau\,d\sigma \le c(\varepsilon,t,s)h^{\delta}.
\end{aligned}$$

Thus we deduce that

$$T_4 = \int_s^t R(t,\sigma)K(\sigma,s)\,d\sigma \frac{1}{h}\int_0^h e^{pA(s)}\,dp + o(1) \qquad \text{as} \quad h \to 0^+. \tag{3.35}$$

By (3.30), (3.32), (3.34), and (3.35) we finally obtain

$$\begin{aligned}
&\frac{\phi(t,s+h)-\phi(t,s)}{h}\\
&\quad = V(t,s) + \int_s^t R(t,\sigma)\left[V(\sigma,s) + A(s)e^{(\sigma-s)A(s)}\right]\,d\sigma\\
&\qquad - \int_s^t [R(t,\sigma)-R(t,s)]A(s)e^{(\sigma-s)A(s)}\,d\sigma - R(t,s)e^{(t-s)A(s)}\\
&\qquad + \left[K(t,s) + R(t,s) + \int_s^t R(t,\sigma)K(\sigma,s)\,d\sigma\right]\frac{1}{h}\int_0^h e^{pA(s)}\,dp\\
&\qquad + o(1) \qquad \text{as} \quad h \to 0^+;
\end{aligned}$$

but observing that

$$\begin{aligned}
&K(t,s) + R(t,s) + \int_s^t R(t,\sigma)K(\sigma,s)\,d\sigma\\
&\quad K_1(t,s) + \sum_{m=1}^{\infty}(-1)^m K_m(t,s) + \sum_{m=1}^{\infty}(-1)^m K_{m+1}(t,s) = 0,
\end{aligned}$$

we can conclude that

$$\begin{aligned}
\frac{\phi(t,s+h)-\phi(t,s)}{h} &= V(t,s) + \int_s^t R(t,\sigma)\left[V(\sigma,s) + A(s)e^{(\sigma-s)A(s)}\right]\,d\sigma\\
&\quad - \int_s^t [R(t,\sigma)-R(t,s)]A(s)e^{(\sigma-s)A(s)}\,d\sigma\\
&\quad - R(t,s)e^{(t-s)A(s)} + o(1) \qquad \text{as} \quad h \to 0^+,
\end{aligned} \tag{3.36}$$

i.e., the right derivative of $\phi(t,s)$ with respect to s exists in $\mathcal{L}(H)$ for each $t \in\,]s, T-\varepsilon]$.

In a completely similar way, starting from formula (3.27), we obtain that the left derivative of $\phi(t,s)$ with respect to s also exists in $\mathcal{L}(H)$ for each $t \in\,]s, T-\varepsilon]$, and equals (3.36); hence the proof is complete. □

Corollary 3.9 [AT5, Corollary 6.10]. *Under Hypotheses* 1.1–1.7, *let* $\phi(t,s)$ *be the operator defined by* (0.6). *Then for* $0 \le s < t < T$ *we have, in the sense of* $\mathcal{L}(H)$,

$$\left[\frac{d}{ds}\phi(t,s)\right]A(s)^{-1} = -U(t,s) - \int_s^t R(t,\sigma)U(\sigma,s)\,d\sigma - R(t,s)A(s)^{-1}.$$

Proof. It is an easy consequence of Theorem 3.8 and Lemma 3.6(i). □

Remark 3.10. (i) As shown in Remark 6.11 of [AT5], the result of Theorem 3.8 guarantees that $\phi_s(t,s)$ exists for $0 \le s < t < T$; in addition, for each $\eta > 0$ and

$0 \le s < t < T$,

$$\|\phi_s(t,s)\|_{\mathcal{L}(H)} \le c_{T-s}(T-t)^{2\alpha-1}(t-s)^{-1}, \tag{3.37}$$

$$\left\|\phi_s(t,s)A(s)^{-1}\right\|_{\mathcal{L}(H)} \le c_{T-s}(T-t)^{2\alpha-1}(t-s)^{\alpha-1}. \tag{3.38}$$

(ii) Using (3.5) we deduce that $P_T^{1/2}\phi_s(t,s)$ exists even when $t = T$, and for each $s \in [0, T[$ we obtain

$$P_T^{1/2}\phi_s(T,s) = P_T^{1/2}\phi\left(T, \frac{T+s}{2}\right)\phi_s\left(\frac{T+s}{2}, s\right), \tag{3.39}$$

$$\left\|P_T^{1/2}\phi_s(T,s)\right\|_{\mathcal{L}(H)} \le c_{T-s}. \tag{3.40}$$

4. Uniqueness of the Riccati Operator

In [AT5] we proved that the operator $P(t)$, defined in (0.5), is a classical solution of the differential Riccati equation (0.4); moreover, it is self-adjoint and nonnegative, and satisfies

$$\begin{aligned} &P \in L^\infty(0,T;\Sigma(H)) \cap C([0,T[,\Sigma(H)),\\ &\qquad [-A(\cdot)^*]^{1-\alpha}P(\cdot) \in B_{1-\alpha}([0,T[,\mathcal{L}(H)), \end{aligned} \tag{4.1}$$

$$\|P(t)x - P_Tx\|_H \to 0 \qquad \text{as} \quad t \to T^-. \tag{4.2}$$

In particular, by the general results of [AT1] on nonautonomous parabolic equations, for each $\varepsilon \in]0,T[$, $P(\cdot)$ solves the Riccati equation in mild form in the interval $[0, T-\varepsilon]$, i.e.,

$$\begin{aligned} P(t) = U(T-\varepsilon,t)^*P(T-\varepsilon)U(T-\varepsilon,t) + \int_t^{T-\varepsilon} U(r,t)^*\\ \times\left[M(r) - P(r)A(r)G(r)N(r)^{-1}G(r)^*A(r)^*P(r)\right]U(r,t)\,dr. \end{aligned} \tag{4.3}$$

We are unable to prove uniqueness of the Riccati operator $P(t)$ in its full generality, i.e., within the class

$$\begin{aligned} &\left\{Q \in L^\infty(0,T,\Sigma(H)) \cap C([0,T[,\Sigma(H)) \colon [-A(\cdot)^*]^{1-\alpha}Q(\cdot) \in C_{1-\alpha}([0,T[,\mathcal{L}(H)),\right.\\ &\left. Q(t) \ge 0, \forall t \in [0,T[, Q(t)x \to P_Tx \text{ in } H \text{ as } t \to T^-\right\}. \end{aligned}$$

We need some regularity of the final datum P_T, which allows us to get a better behavior of the Riccati operator near the point T. In fact, following Theorem 6.4 of [LT3], we can prove:

Theorem 4.1. *Under Hypotheses* 1.1–1.7, *assume in addition that there exists* $\beta \in]1-2\alpha, 1-\alpha]$ *such that* $P_T \in \mathcal{L}(H, D([-A(T)^*]^\beta))$. *Then the solution of the differential Riccati equation* (0.4) *is unique within the class*

$$\begin{aligned} &\left\{Q \in L^\infty(0,T,\Sigma(H)) \cap C([0,T[,\Sigma(H)) \colon [-A(\cdot)^*]^{1-\alpha}Q(\cdot) \in C_{1-\alpha-\beta}([0,T[,\mathcal{L}(H)),\right.\\ &\left. P(t) \ge 0, \forall t \in [0,T[, P(t)x \to P_Tx \textit{ in } H \textit{ as } t \to T^-\right\}. \end{aligned} \tag{4.4}$$

Proof. To start with, if $P(t)$ is the operator (0.5), the assumption on P_T along with Hypothesis 1.3 yields

$$\left\| [-A(t)^*]^{1-\alpha} P(t) \right\|_{\mathcal{L}(H)} \le c(T-t)^{\alpha+\beta-1}, \qquad \forall t \in [0, T[; \tag{4.5}$$

this follows by Theorem 3.13 of [AFT], but can be proved directly by adapting to the nonautonomous situation the argument of Section 6 of [LT3]. Now assume that $P_1(t)$, $P_2(t)$ solve the integral equation (4.3) for each $\varepsilon \in]0, T[$ and both belong to the class (4.4). The integrand in (4.3) is bounded by $c(T-t)^{2(\alpha+\beta-1)}$ in view of (4.5) and Hypotheses 1.5 and 1.6, and we can let $\varepsilon \to 0$ since $2(\alpha+\beta-1) > -1$. Hence the operator $Q(t) := P_1(t) - P_2(t)$ solves

$$\begin{aligned} Q(t) = &- \int_t^T U(r,t)^* Q(r) A(r) G(r) N(r)^{-1} G(r)^* A(r)^* P_1(r) U(r,t)\, dr \\ &- \int_t^T U(r,t)^* P_2(r) A(r) G(r) N(r)^{-1} G(r)^* A(r)^* Q(r) U(r,t)\, dr. \end{aligned} \tag{4.6}$$

Hence applying $[-A(t)^*]^{1-\alpha}$ to both members we find for each $t \in [0, T[$, due to (4.4) and Hypotheses 1.3, 1.5, and 1.6,

$$\begin{aligned} &\left\| [-A(t)^*]^{1-\alpha} Q(t) \right\|_{\mathcal{L}(H)} \\ &\quad \le c \int_t^T (r-t)^{\alpha-1} (T-r)^{2(\alpha+\beta-1)}\, dr \, \| [-A(\cdot)^*]^{1-\alpha} Q(\cdot) \|_{C_{1-\alpha-\beta}([0,T[,\mathcal{L}(H))}; \end{aligned}$$

thus, setting

$$K_t := \left\| [-A(\cdot)^*]^{1-\alpha} Q(\cdot) \right\|_{C_{1-\alpha-\beta}([t,T[,\mathcal{L}(H))},$$

we obtain

$$K_t \le c(T-t)^{2\alpha+\beta-1} K_t, \qquad \forall t \in [0, T[.$$

Finally, if $t \in [T-\delta, T[$, with a sufficiently small $\delta > 0$, we deduce that $K_{T-\delta} = 0$, which in turn yields $P_1 \equiv P_2$ in $[T-\delta, T]$. This argument can be iterated, starting again from (4.6), which now holds with T replaced by $T-\delta$; in a finite number of steps we get $P_1 \equiv P_2$ in $[0, T]$. □

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