Sistemi Dinamici Discreti

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Chapter 1

Topological dynamics

1.1 Basic definitions and first examples

To study the dynamics of a self-map $f\colon X \to X$ means to study the qualitative behavior of the sequences $\{f^k(x)\}$ as k goes to infinity when x varies in X, where f^k denotes the composition of f with itself k times.

Definition 1.1.1: A *(discrete) dynamical system* is a pair (X, f), where X is a topological space and $f\colon X \to X$ a continuous* self-map of X. For $k \in \mathbb{N}$ we shall always denote by f^k the *k-th iterate* of f, inductively defined by $f^0 = \mathrm{id}_X$ and $f^k = f \circ f^{k-1}$. When f is invertible, we denote by f^{-1} its inverse, and set $f^{-k} = (f^{-1})^k$. If $x \in X$, the points $f^k(x)$ for $k \in \mathbb{N}$ are called *(forward) iterates* of x, and the set $\{f^k(x)\}_{k\in\mathbb{N}}$ is the *(forward) orbit* of x. When f is invertible, the points $f^{-k}(x)$ for $k \in \mathbb{N}$ are the *backward iterates* of x, while $\{f^{-k}(x)\}_{k\in\mathbb{N}}$ is the *backward orbit*, and $\{f^k(x)\}_{k\in\mathbb{Z}}$ the *full orbit*.

Definition 1.1.2: Let (X, f) be a dynamical system. A point $x \in X$ is *fixed* if $f(x) = x$; we denote by $\mathrm{Fix}(f)$ the set of all fixed points of f. If $f^m(x) = x$ for some $m \geq 1$, we say that x is *periodic* of *period* m; the minimal such m is called the *exact period* of x. The set of all periodic points of f is denoted by $\mathrm{Per}\,(f)$. The orbit of a periodic point will be called a *cycle*.

Definition 1.1.3: Let (X, f) be a dynamical system. A subset $Y \subset X$ is *f-invariant* if $f(Y) \subseteq Y$; *completely f-invariant* if $f(Y) \cup f^{-1}(Y) \subseteq Y$.

Exercise 1.1.1. Show that a subset $Y \subseteq X$ is completely f-invariant if and only if $f^{-1}(Y) = Y$.

To warm us up, let us begin discussing a few elementary examples.

EXAMPLE 1.1.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax$, with $a \in \mathbb{R}$. Then clearly 0 is the only fixed point, $f^k(x) = a^k x$ for any $k \geq 1$, and so we can easily describe the behavior of the orbits as follows:
- if $|a| < 1$ then $f^n(x) \to 0$ for every $x \in \mathbb{R}$;
- if $|a| > 1$ then $|f^n(x)| \to +\infty$ for every $x \in \mathbb{R}$, $x \neq 0$;
- if $a = 1$ then $f = \mathrm{id}_{\mathbb{R}}$;
- if $a = -1$ then $f^2 = \mathrm{id}_{\mathbb{R}}$, and all points are periodic of period (at most) two.

We can describe in a quantitative way how the orbits in the previous example converge (or diverge):

Definition 1.1.4: Let (X, d) be a metric space. We say that a sequence $\{x_k\} \subset X$ *is exponentially convergent* to $p \in X$ if there are $C > 0$ and $0 \leq \lambda < 1$ such that

$$\forall k \in \mathbb{N} \qquad d(x_k, p) \leq C\,\lambda^k.$$

We say that the sequence *is exponentially diverging to infinity* if there are $c > 0$ and $\mu > 1$ such that

$$\forall k \in \mathbb{N} \qquad d(x_k, q) \geq c\,\mu^k$$

for some $q \in X$; the definition is obviously independent of q.

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* In a later chapter, when we shall discuss ergodic theory, X will be a measure space and f a measurable self-map.

So the orbits in Example 1.1.1 are exponentially converging to the origin if $|a| < 1$, and exponentially diverging to infinity if $|a| > 1$.

EXAMPLE 1.1.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + c$, with $c \in \mathbb{R}^*$. Then there are no fixed (or periodic) points, and $f^k(x) = x + ck$. Therefore we have $|f^k(x)| \to +\infty$ for all $x \in \mathbb{R}$ and $c \in \mathbb{R}^*$.

This time the orbits are not exponentially diverging to infinity, but they are diverging nonetheless:

Definition 1.1.5: Let (X, d) be a metric space. We say that a sequence $\{x_k\} \subset X$ *is polynomially diverging to infinity* if there are $0 < c_1 < c_2$ and m_1, $m_2 \in \mathbb{N}^*$ such that

$$\forall k \in \mathbb{N} \qquad\qquad c_1 k^{m_1} \le d(x_k, q) \le c_2 k^{m_2}$$

for some $q \in X$; the definition is obviously independent of q.

So the orbits of the Example 1.1.2 are polynomially diverging to infinity, with $m_1 = m_2 = 1$.

Exercise 1.1.2. Study the dynamics of the maps $f_{a,b}\colon \mathbb{R} \to \mathbb{R}$ given by $f_{a,b}(x) = ax + b$ with a, $b \in \mathbb{R}^*$.

EXAMPLE 1.1.3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^x$; we claim that all orbits are diverging to infinity. The first observation is that since $e^x > x$ then every orbit $\{f^k(x)\}$ is strictly increasing, and thus it is either converging or diverging to $+\infty$. To prove that the latter holds always, it would suffice to show that every orbit is unbounded; but there is a cleverer way. Indeed, we have that $e^x \ge x + 1$; therefore by induction it follows that $f^k(x) \ge x + k$, and we are done.

Exercise 1.1.3. Prove that the orbits of $f(x) = e^x$ are superexponentially diverging to infinity, that is that for every $x \in \mathbb{R}$ and $\mu > 1$ there exists a $C_\mu > 0$ such that $|f^k(x)| \ge C_\mu \mu^k$ for all $k \in \mathbb{N}$.

Exercise 1.1.4. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be two continuous functions such that g increasing, $f \ge g$, and $g^k(x) \to +\infty$ for all $x \in \mathbb{R}$. Prove that $f^k(x) \to +\infty$ for all $x \in \mathbb{R}$.

Exercise 1.1.5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a homeomorphism. Prove that:
(i) f is strctly monotone;
(ii) if f has a periodic point of exact period at least 2 then f is decreasing;
(iii) f cannot have periodic points of odd period;
(iv) f cannot have periodic points of exact period greater than 2;
(v) there is an homeomorphism of $\mathbb{R}$ with periodic points of exact period 2.

EXAMPLE 1.1.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$. The function f has an unique (exercise) fixed point, the origin. Furthermore, the image of f is the interval $I = [-1, 1]$, and so to study the dynamics of f it suffices to see what happens on I. If $x \in (0, 1]$ we have $0 < \sin x < x$; therefore the orbit $\{f^k(x)\}$ is strctly decreasing, and thus it converges to a point $x_\infty \in [0, 1]$. Now,

$$f(x_\infty) = f\bigl(\lim_{k\to\infty} f^k(x)\bigr) = \lim_{k\to\infty} f^{k+1}(x) = x_\infty;$$

therefore x_∞ should be a fixed point, and thus $x_\infty = 0$. In the same way one proves that $f^k(x) \to 0$ if $x \in [-1, 0)$; therefore all orbits of f converge to the origin.

Remark 1.1.1. The argument used in the previous example shows that if an orbit converges to a point, this point must necessarily be fixed.

Exercise 1.1.6. Do the orbits of $f(x) = \sin x$ converge exponentially to the origin?

EXAMPLE 1.1.5. Let $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \cos x$. Again we have $f(\mathbb{R}) \subseteq [-1, 1]$, and again the function f has a (again exercise) unique fixed point p, belonging to $(0, 1)$. We claim that all orbits of f coverge exponentially to p. First of all, notice that we actually have $f^2(\mathbb{R}) \subseteq I = [\cos(1), 1] \subset (0, 1]$ and $f(I) \subseteq I$, and so it suffices to study the orbits starting (and thus staying) in I. Now, the average value theorem says that for every $x \in I$ there exists a $y \in I$ such that

$$\cos x - p = \cos x - \cos p = -(\sin y)(x - p);$$

therefore we get

$$|\cos x - p| \leq \lambda |x - p|,$$

where $\lambda = \sup_{y \in I} |\sin y| < 1$. By induction we then get

$$|f^k(x) - p| \leq \lambda^k |x - p|,$$

and thus every orbit converges exponentially to the fixed point p.

In the next section we shall generalize the argument used in this example, showing that the orbits of a contractions always converge to a (unique) fixed point.

Exercise 1.1.7. Let $f\colon [a,b] \to [a,b]$ be continuous. Show that f has at least one fixed point.

1.2 Contractions

This section is devoted to the study of one of the easiest example of dynamical system, a contracting map on a complete metric space.

Definition 1.2.1: A self-map $f\colon X \to X$ of a metric space X is a *contraction* if there is $\lambda < 1$ such that

$$\forall x, y \in X \qquad d\big(f(x), f(y)\big) \leq \lambda\, d(x,y). \tag{1.2.1}$$

The constant $\lambda = \sup_{x \neq y} \big\{ d\big(f(x), f(y)\big)/d(x,y) \big\}$ is the *contraction costant* of f.

We now prove the main result of this section, the *contraction principle,* which is both simple and extremely useful.

Theorem 1.2.1: (Contraction principle) *Let X be a complete metric space, and let $f\colon X \to X$ be a contraction. Then f has a unique fixed point $p \in X$, and for every $x \in X$ the orbit of x is esponentially convergent to p.*

Proof: Applying (1.2.1) several times we get

$$\forall x, y \in X\ \forall n \in \mathbb{N} \qquad d\big(f^n(x), f^n(y)\big) \leq \lambda^n d(x,y). \tag{1.2.2}$$

As a consequence, for every $x \in X$ the sequence $\{f^n(x)\}$ is a Cauchy sequence. Indeed, for every $m > n$ we get

$$\begin{aligned} d\big(f^m(x), f^n(x)\big) &\leq \sum_{k=0}^{m-n-1} d\big(f^{n+k+1}(x), f^{n+k}(x)\big) \\ &\leq \sum_{k=0}^{m-n-1} \lambda^{n+k} d\big(f(x), x\big) < \frac{\lambda^n}{1-\lambda} d\big(f(x), x\big) \longrightarrow 0. \end{aligned} \tag{1.2.3}$$

Thus for every $x \in X$ the sequence $\{f^n(x)\}$ is converging, and (1.2.2) says that the limit does not depend on x. Let p be this limit; then $f(p) = p$, that is p is a fixed point of f. Indeed,

$$f(p) = \lim_{n \to \infty} f\big(f^n(p)\big) = \lim_{n \to \infty} f^{n+1}(p) = p.$$

Being f a contraction, p is the unique fixed point of f. Finally, letting m go to infinity in (1.2.3) we find

$$d\big(p, f^n(x)\big) \leq \frac{\lambda^n}{1-\lambda} d\big(f(x), x\big),$$

and so the sequence $\{f^n(x)\}$ is esponentially convergent to $p \in X$. □

For differentiable maps between Riemannian manifolds there is an easy sufficient condition for a map to be a contraction. We first recall a standard definition:

Definition 1.2.2: Let $L: V \to W$ be a linear map between finite-dimensionale normed vector spaces. Then the *norm* of L is given by

$$\|L\| = \sup_{\|v\|_V=1} \|L(v)\|_W = \sup_{v\in V\setminus\{O\}} \frac{\|L(v)\|_W}{\|v\|_V},$$

where $\|\cdot\|_V$ (respectively, $\|\cdot\|_W$) is the norm in V (respectively, in W). It is easy to check (exercise) that this yields a norm on the space $\mathrm{Hom}(V,W)$ of linear maps from V into W.

Proposition 1.2.2: *Let $F: M \to N$ be a C^1-map between (connected) Riemannian manifolds, and assume that*

$$K = \sup_{x\in M} \|dF_x\| < +\infty.$$

Then we have

$$\forall x, y \in M \qquad d_N\bigl(F(x), F(y)\bigr) \le K\, d_M(x,y),$$

where d_M (respectively, d_N) is the Riemannian distance in M (respectively, N). In particular, if $K < 1$ then F is a contraction.

Proof: Let $\sigma: [0,r] \to M$ be a piecewise regular curve and such that $\sigma(0) = x$ and $\sigma(r) = y$. Then $F \circ \sigma$ is a piecewise regular curve in N from $F(x)$ to $F(y)$, and hence

$$\begin{aligned} d_N\bigl(F(x), F(y)\bigr) &\le \mathrm{Length}(F\circ\sigma) \\ &= \int_0^r \|(F\circ\sigma)'(t)\|_{F\circ\sigma(t)}\, dt = \int_0^r \bigl\|dF_{\sigma(t)}\bigl(\sigma'(t)\bigr)\bigr\|_{F\circ\sigma(t)}\, dt \\ &\le K \int_0^r \|\sigma'(t)\|_{\sigma(t)}\, dt = K\,\mathrm{Length}(\sigma). \end{aligned}$$

Taking the infimum of the right-hand side as σ runs over all piecewise regular curves connecting x and y we get the assertion. □

Corollary 1.2.3: *Let $F: U \to \mathbb{R}^m$ be a C^1-map, where U is a convex open subset of $\mathbb{R}^n$, and assume that*

$$K = \sup_{x\in U} \|dF_x\| < +\infty.$$

Then we have

$$\forall x, y \in U \qquad \|F(x) - F(y)\| \le K\,\|x - y\|.$$

In particular, if $K < 1$ then F is a contraction.

Proof: It suffices to apply the previous Proposition to the euclidean metrics. □

The fixed point of a contraction depends continuously on the contraction and on the contraction constant λ. To precisely state this, we need the following

Definition 1.2.3: If Y is a metric space and f, $g: X \to Y$ are continuous maps into Y, we set

$$d_0(f,g) = \sup_{x\in X} d_Y\bigl(f(x), g(x)\bigr) \in [0, +\infty].$$

If X is compact, then d_0 is a distance on the space $C^0(X,Y)$ of continuous functions from X into Y.

Proposition 1.2.4: *Let $f: X \to X$ be a contraction of a complete metric space X, with contraction constant λ and fixed point x_0. Then for every $\varepsilon > 0$ there is $0 < \delta < 1-\lambda$ so that for every contraction $g: X \to X$ with contraction constant at most $\lambda + \delta$ and such that $d_0(f,g) < \delta$ we have $d(x_0, y_0) < \varepsilon$, where y_0 is the fixed point of g.*

Proof: Put $\delta = \varepsilon(1-\lambda)/(1+\varepsilon)$. Since $g^n(x_0) \to y_0$ we have

$$d(x_0, y_0) \le \sum_{n=0}^{\infty} d\bigl(g^n(x_0), g^{n+1}(x_0)\bigr) \le d\bigl(x_0, g(x_0)\bigr) \sum_{n=0}^{\infty} (\lambda+\delta)^n < \frac{\delta}{1-\lambda-\delta} = \varepsilon.$$

□

Exercise 1.2.1. Find a complete metric space X and a map $f\colon X \to X$ such that $d\big(f(x), f(y)\big) < d(x,y)$ for every $x \neq y \in X$, f is fixed point free and $d\big(f^n(x), f^n(y)\big)$ does not converge to zero for some x, $y \in X$.

Exercise 1.2.2. Let X be a compact metric space, and $f\colon X \to X$ a map such that $d\big(f(x), f(y)\big) < d(x,y)$ for every $x \neq y \in X$. Prove that the orbit of every $x \in X$ converges to the unique fixed point of f, and find an example where the convergence is not exponential.

As first example of application of this principle we shall show that, assuming a non-degeneracy condition on the differential, the existence of a periodic point of period m is stable under small perturbations. But first we have to define a topology on the space of C^r maps.

Definition 1.2.4: Let $f\colon U \to \mathbb{R}^m$ be a C^r map ($0 \le r < \infty$) defined on an open subset of $\mathbb{R}^n$, and $K \subset U$ a compact subset. Let $\|f\|_{r,K}$ be the maximum over K of the norm of f and of all its partial derivatives up to order r included. In this way we get a set of semi-norms that can be used to define a topology on $C^r(U, \mathbb{R}^m)$ taking as basis for the open sets the finite intersections of sets of the form

$$\mathcal{U}_{r,K}(f, \varepsilon) = \{g \mid \|f - g\|_{r,K} < \varepsilon\},$$

where $\varepsilon > 0$ and K is a compact subset of U. If M is a smooth manifold, the same trick yields a topology on $C^r(M, \mathbb{R}^m)$ by using only compact subsets contained inside local charts, and computing partial derivatives with respect to the local coordinates. Finally, if N is another smooth manifold, we can consider N imbedded in some $\mathbb{R}^m$ and give to $C^r(M, N)$ the topology induced by the topology of $C^r(M, \mathbb{R}^m)$. By the way, it is not difficult to prove (because a manifold is a countable union of compact subsets) that this topology is induced by a complete metric; furthermore, $C^r(M, N)$ is a Baire space and has a countable basis of open sets. Roughly speaking, $f_n \to g$ in the C^r topology if all the derivatives (up to order r) of f_n converge to the derivatives of g. Using the seminorms $\|\cdot\|_{r,K}$ for all finite r one defines analogously a topology on $C^\infty(M, N)$; in this case $f_n \to g$ in the C^∞ topology if all the derivatives of f_n converge to the derivatives of g. Finally, if M and N are complex manifolds, we put on $\mathrm{Hol}(M, N)$ the usual compact-open topology (which is just the C^0 topology); thanks to the wonderful properties of holomorphic maps, convergence in the C^0 topology implies convergence of all derivatives, that is in $\mathrm{Hol}(M, N)$ the C^0 topology and the C^∞ topology agree.

Exercise 1.2.3. Prove that

$$\|A\| \le \sqrt{mn}\, \|A\|_\infty$$

for all $A \in M_{m,n}(\mathbb{R})$, where $\|A\|_\infty = \max\{|a_{ij}|\}$, and we have endowed $\mathbb{R}^n$ and $\mathbb{R}^n$ with the euclidean metric. In particular, if $f \in C^1(U, V)$ is a C^1 map between open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, we have $\|df_x\| \le \sqrt{mn}\, \|f\|_{1,U}$ for all $x \in U$.

Proposition 1.2.5: *Let M be a smooth manifold, and $f\colon M \to M$ of class C^1. Let $p \in M$ be a periodic point of period m such that 1 is not an eigenvalue of df_p^m. Then every map $g\colon M \to M$ sufficiently C^1-close to f has a unique periodic point of period m close to p. More precisely, there exists a compact neighbourhood $B \subset M$ of p and an $\varepsilon > 0$ so that $\|g - f\|_{1, B \cup \cdots \cup f^{m-1}(B)} < \varepsilon$ implies that g has a unique periodic point of period m in B.*

Proof: Since if $\|g - f\|_{1, K \cup \cdots \cup f^{m-1}(K)}$ is small then $\|g^m - f^m\|_{1,K}$ is small, we can assume $m = 1$. Being a local problem, we can also assume $M = \mathbb{R}^n$ and $p = O$. Since 1 is not an eigenvalue of df_p, the Inverse Function Theorem implies that $F = f - \mathrm{id}$ is invertible in a compact neighbourhood B of the origin, that we may take to be a closed ball of radius $R > 0$. We look for an $\varepsilon > 0$ such that if $\|g - f\|_{1,B} < \varepsilon$ then g has a unique fixed point in B.

Set $H = f - g$. Then $x \in B$ is a fixed point of g if and only if

$$x = g(x) = (f - H)(x) = (F + \mathrm{id} - H)(x),$$

that is if and only if $(F - H)(x) = O$, that is if and only if

$$x = F^{-1} \circ H(x).$$

So it suffices to prove that for ε small enough the map $F^{-1} \circ H$ is a contraction of B into itself.

First of all, we take $\varepsilon \le R$, so that $H(B) \subseteq B$. Furthermore, if $L = \max_{x \in B} \|dF_x^{-1}\|$ then

$$\|F^{-1} \circ H(O)\| \le L\|H(O)\| \le L\,\varepsilon,$$

thanks to Corollary 1.2.3, because $F^{-1}(O) = O$. Then

$$\|F^{-1} \circ H(x)\| \le \|F^{-1} \circ H(x) - F^{-1} \circ H(O)\| + \|F^{-1} \circ H(O)\| \le \varepsilon\,L(m\|x\| + 1) \le \varepsilon L(nR + 1),$$

where we used Exercise 1.2.3. So to have $F^{-1} \circ H(B) \subseteq B$ it suffices to require $\varepsilon \le R/L(nR + 1)$.

Finally, if x, $y \in B$ we have

$$\|F^{-1} \circ H(x) - F^{-1} \circ H(y)\| \le n\varepsilon\,L\|x - y\|,$$

and so if we also have $\varepsilon < 1/nL$ we are done. □

The assumption that 1 is not an eigenvalue of the differential is essential, as shown in the following example:

Example 1.2.1. Let $f_\lambda\colon \mathbb{R} \to \mathbb{R}$ be given by $f_\lambda(x) = \lambda x + x^2$. Then $\mathrm{Fix}(f_1) = \{0\}$, while for $\lambda \ne 1$ the function f_λ has two distinct fixed points: $\mathrm{Fix}(f_\lambda) = \{0, 1-\lambda\}$. We have $f'_\lambda(0) = \lambda$, and in particular $f'_1(0) = 1$. Now, it is easy to see that

$$\|f_1 - f_\lambda\|_{1,[-r,r]} = |1 - \lambda| r$$

for all $r > 0$, and so for every compact neighbourhood B of the origin and every $\varepsilon > 0$ there is a $\lambda > 0$ such that $\|f_1 - f_\lambda\|_{1,B} < \varepsilon$ but f_λ has two distinct fixed points in B.

Often, discrete dynamical systems appear in families depending on one (or more) parameters, as in the previous example. A variation of the dynamical behavior when the parameter crosses a particular value is called a *bifurcation*. To describe examples of bifurcations, let us introduce the following terminology:

Definition 1.2.5: Let $f\colon M \to M$ a C^1 self-map of a manifold M. A periodic point $p \in M$ of exact period m is said *attracting* if all the eigenvalues of $d(f^m)_p$ have absolute value less than 1; *repelling* if all the eigenvalues of $d(f^m)_p$ have absolute value greater than 1; *hyperbolic* if all the eigenvalues of $d(f^m)_p$ have absolute value different from 1.

Remark 1.2.1. $d(f^m)_p = df_{f^{m-1}(p)} \circ df_{f^{m-2}(p)} \circ \cdots \circ df_p$.

Definition 1.2.6: Let (X, f) be a discrete dynamical system on a metric space X. Given $p \in X$, we shall say that $x \in X$ is (*forward*) *asymptotic* to p if $d\big(f^k(x), f^k(p)\big) \to 0$ as $k \to +\infty$. The set of points asymptotic to p is the *stable set* $W^s(p)$ of p. Finally, if f is invertible we shall say that x is *backward asymptotic* to p if it is asymptotic to p with respect to f^{-1}, and the set of points backward asymptotic to p is the *unstable set* $W^u(p)$ of p.

Exercise 1.2.4. Prove that the stable sets give a partition of the set X.

If $p \in X$ is fixed, then its stable set is the set of points whose orbit is converging to p. More generally, if $p \in X$ is periodic of exact period m, then $x \in W^s(p)$ if and only if $f^{mk}(x) \to p$ (exercise).

Example 1.2.2. If $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \sin x$, then $W^s(0) = \mathbb{R}$.

Example 1.2.3. If $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^3$ then $W^s(0) = (-1, 1)$, $W^s(1) = \{1\}$, $W^s(-1) = \{-1\}$, $W^u(0) = \{0\}$, $W^u(1) = \mathbb{R}^+ \setminus \{0\}$, and $W^u(-1) = \mathbb{R}^- \setminus \{0\}$.

We shall later see (Corollary 1.3.4) that the stable set of an attracting point p contains an open neighbourhood of p, justifying the adjective.

Example 1.2.4. In the Example 1.2.1, the origin is an attractive fixed point for $|\lambda| < 1$, and it is repelling for $|\lambda| > 1$. The second fixed point is repelling for $\lambda < 1$ (and $\lambda > 3$), while it is attracting for $1 < \lambda < 3$. Therefore we have a bifurcation for $\lambda = 1$: roughly speaking, the two fixed points for $0 < \lambda < 1$ merge at $\lambda = 1$ and exchange their roles for $1 < \lambda < 3$. (We shall discuss below what happens for $\lambda = 3$).

The previous bifurcation is sort of exceptional, because it happens at parameter speed 0: indeed,

$$\frac{d}{d\lambda} f_\lambda(0)\Big|_{\lambda=1} = 0.$$

We end this section with two examples of more typical bifurcations:

EXAMPLE 1.2.5. Let $f_c\colon \mathbb{R} \to \mathbb{R}$ given by $f_c(x) = x^2 + c$. We have $\mathrm{Fix}(f_c) = \varnothing$ if $c > 1/4$, $\mathrm{Fix}(f_{1/4}) = \{1/2\}$, and $\mathrm{Fix}(f_c)$ contains two points if $c < 1/4$. The appearance of two fixed points when the parameter crosses a specific value is called a *saddle-node bifurcation.*

Exercise 1.2.5. Discuss the type (attracting, etc.) of the fixed points in the previous example.

EXAMPLE 1.2.6. Let $F_\mu\colon \mathbb{R} \to \mathbb{R}$ be given by $F_\mu(x) = \mu x(1 - x)$. We have two fixed points, the origin and $p_\mu = 1 - 1/\mu$. The origin is attracting for $|\mu| < 1$, and repelling for $|\mu| > 1$, while p_μ is repelling for $\mu < 1$ and $\mu > 3$, and attracting for $1 < \mu < 3$. The bifurcation at $\mu = 1$ is exactly the same as the one discussed in Example 1.2.1. Much more interesting is the bifurcation at $\mu = 3$. Indeed, if for $1 < \mu \le 3$ we have one repelling and one attracting fixed point and no periodic points of exact period 2, when μ crosses 3 we see that the attracting fixed point becomes repelling but it is born an attracting cycle of period 2 (check that this is true). This is called a *period-doubling bifurcation.*

1.3 Linear maps

This section is devoted to the study of the dynamics of linear maps of a complex or real finite-dimensional vector space.

Definition 1.3.1: Let $T\colon V \to V$ be a linear self-map of a finite-dimensional vector space V on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The set of all (complex) eigenvalues of T is the *spectrum* $\mathrm{sp}(T) \subset \mathbb{C}$ of T. The *spectral radius* is $r(T) = \max\{|\lambda| \mid \lambda \in \mathrm{sp}(T)\}$.

It is easy to see (exercise) that $\|T\| \ge r(T)$ always. Conversely we have:

Lemma 1.3.1: *Let $T\colon V \to V$ be a linear self-map of a finite-dimensional vector space V on $\mathbb{R}$ or $\mathbb{C}$. Then for every $\delta > 0$ there exists a scalar (respectively, hermitian) product whose norm $\|\cdot\|$ is such that*

$$\|T\| \le r(T) + \delta.$$

Proof: Let $\mathcal{B}$ be a (real or complex) Jordan basis of V, so that T is represented by a (real or complex) canonical Jordan matrix

$$A = \begin{vmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{vmatrix},$$

where each block is of the form

$$\begin{vmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{vmatrix}$$

with $\lambda \in \mathrm{sp}(T)$, or, in the real case only, of the form

$$\begin{vmatrix} \rho R_\varphi & I_2 & & & \\ & \rho R_\varphi & I_2 & & \\ & & \ddots & \ddots & \\ & & & \rho R_\varphi & I_2 \\ & & & & \rho R_\varphi \end{vmatrix}$$

corresponding to two complex conjugates eigenvalues $\lambda=\rho e^{i\varphi}$ e $\overline{\lambda}=\rho e^{-i\varphi}$, where $R_\varphi=\begin{vmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{vmatrix}$ and I_2 is the identity matrix of order 2.

For $t\in\mathbb{R}$ let us denote by $A(t)$ the matrix with the same block structure as A but where the off-diagonal 1's are replaced by t. We then put on V the scalar (hermitian) product for which the basis $\mathcal{B}$ is orthonormal, and denote by $|||\cdot|||$ the corresponding norm. Now, $t\mapsto |||A(t)|||$ is a continuous function, and (exercise) $|||A(0)|||=r(A)$. Therefore given $\delta>0$ we can find $t_\delta>0$ such that $|||A(t_\delta)|||<r(A)+\delta$. But $A(t_\delta)=\Lambda A\Lambda^{-1}$, where Λ is the diagonal matrix composed by blocks of the form

$$\begin{vmatrix}1 & & & \\ & t_\delta^{-1} & & \\ & & \ddots & \\ & & & t_\delta^{-m+1}\end{vmatrix}$$

for each block of order m of the first kind, and by blocks of the form

$$\begin{vmatrix}I_2 & & & \\ & t_\delta^{-1}I_2 & & \\ & & \ddots & \\ & & & t_\delta^{-m+1}I_2\end{vmatrix}$$

for each block of order $2m$ of the second kind. So setting $\|v\|=|||\Lambda v|||$ we get (exercise)

$$\|A\|=|||A(t_\delta)|||<r(A)+\delta,$$

and we are done. □

Corollary 1.3.2: *Let $T\colon V\to V$ be a linear self-map of a finite-dimensional vector space V on $\mathbb{R}$ or $\mathbb{C}$. Then for every norm $\|\cdot\|$ on V and every $\delta>0$ there is a constant C_δ such that for every $v\in V$ and every $k\in\mathbb{N}$ we have*

$$\|T^k(v)\|\le C_\delta(r(T)+\delta)^k\|v\|. \tag{1.3.1}$$

Proof: Given $\delta>0$, let $|||\cdot|||$ be norm given by the previous Lemma. Since V is finite-dimensional, there are c_1, $c_2>0$ such that $c_1\|v\|\le|||v|||\le c_2\|v\|$ for all $v\in V$ (exercise). Then

$$\|T^k(v)\|\le c_2|||T^k(v)|||\le c_2|||T^k|||\,|||v|||\le\frac{c_2}{c_1}|||T|||^k\|v\|\le\frac{c_2}{c_1}\bigl(r(t)+\delta\bigr)^k\|v\|.$$

□

Corollary 1.3.3: *Let $T\colon V\to V$ be a linear self-map of a finite-dimensional vector space V on the field $\mathbb{K}$ with all eigenvalues of absolute value less than 1 (that is such that $r(T)<1$). Then the orbit of every point is exponentially convergent to the origin. If furthermore T is invertible, that is if $0\notin\mathrm{sp}(T)$, then every backward orbit is exponentially divergent to infinity.*

Proof: Choose $\delta>0$ so that $r(T)+\delta<1$; then (1.3.1) shows that every orbit is exponentially convergent to the origin. If T is invertible, then (1.3.1) yields

$$\|T^{-k}(v)\|\ge\frac{\|v\|}{C_\delta}\left(\frac{1}{r(T)+\delta}\right)^k,$$

and we are done. □

We can now fulfill a previous promise:

Corollary 1.3.4: *Let $p \in M$ be a fixed point of a C^1 self-map $f: M \to M$ of a manifold M. Then:*

(i) *if p is attracting then there exists a neighbourhood U of p attracted by p, that is such that $f^k(x) \to p$ for all $x \in U$;*

(ii) *if p is repelling then there exists a neighbourhood U of p repelled by p, that is so that for every $x \in U \setminus \{p\}$ there exists a $k > 0$ such that $f^k(x) \notin U$.*

Proof: Since it is a local statement, we can assume that $M = \mathbb{R}^n$ and $p = O$. If p is attracting, we have $r(df_O) < 1$; therefore we can find a norm on $\mathbb{R}^n$ such that $\|df_O\| < 1$. Since f is C^1, this means that we can find a closed ball U centered at the origin and $0 < \lambda < 1$ such that $\|df_x\| \le \lambda$ for all $x \in U$. The assertion then follows from Corollary 1.2.3 and Theorem 1.2.1.

Conversely, if p is repelling then f^{-1} is defined in a neighbourhood of the origin, and p is attracting for f^{-1}. In particular, we find a closed ball U centered at the origin and a $0 < \lambda < 1$ so that $\|f^{-1}(x)\| \le \lambda \|x\|$ for all $x \in U$.

Now take $x \in U$, and assume that $f^k(x) \in U$ for all $k \in \mathbb{N}$. Since $f^k(x) = f^{-1}\bigl(f^{k+1}(x)\bigr)$, it easily follows by induction that

$$\|f^k(x)\| \ge \frac{1}{\lambda^k}\|x\|;$$

therefore the orbit of x is unbounded, again the hypothesis that it was contained in U, unless $x = O$, and we are done. □

Exercise 1.3.1. Let $f: \mathbb{R} \to \mathbb{R}$ be a diffeomorphism of class C^1. Show that all hyperbolic periodic points of f are isolated.

Exercise 1.3.2. Let $f: \mathbb{R} \to \mathbb{R}$ be a diffeomorphism of class C^1 such that $f(x) = x + x^3 \sin \frac{1}{x}$ for $x \in (0, 1/\pi)$. Show that f has a non-isolated (non-hyperbolic) fixed point.

Coming back to a linear self-map $T: V \to V$, if all eigenvalues of T have absolute value greater than 1, we can apply Corollary 1.3.3 to T^{-1}, showing that every orbit is exponentially divergent to infinity, and that every backward orbit is exponentially convergent to the origin.

To study the dynamics of more general linear maps we need a few more definitions.

Definition 1.3.2: Let $T: V \to V$ be a linear self-map of a finite-dimensional vector space V on the field $\mathbb{K}$, and take $\lambda \in \mathrm{sp}(T)$. If $\lambda \in \mathbb{K}$, we denote by V_λ the eigenspace relative to λ, and by E_λ the *root space* (or *generalized eigenspace*) relative to λ, given by all $v \in V$ such that $(T - \lambda\,\mathrm{id})^k(v) = O$ for some $k \ge 1$. If $\mathbb{K} = \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we denote by $V_\lambda \subseteq V^{\mathbb{C}}$ the eigenspace relative to λ for the action of T on the complexified space $V^{\mathbb{C}} = V \oplus iV$, and by $E_\lambda \subseteq V^{\mathbb{C}}$ the corresponding root space. In this case we then set $V_{\lambda,\overline{\lambda}} = (V_\lambda \oplus V_{\overline{\lambda}}) \cap V$ and $E_{\lambda,\overline{\lambda}} = (E_\lambda \oplus E_{\overline{\lambda}}) \cap V$. Using the root spaces we now define a number of dynamically relevant subspaces. The *stable subspace* of T is

$$E^s = E^s(T) = \bigoplus_{|\lambda|<1, \lambda \in \mathbb{K}} E_\lambda \oplus \bigoplus_{|\lambda|<1, \lambda \in \mathbb{C}\setminus\mathbb{K}} E_{\lambda,\overline{\lambda}};$$

the *invertible stable subspace* is

$$E^{si} = E^{si}(T) = \bigoplus_{0<|\lambda|<1, \lambda \in \mathbb{K}} E_\lambda \oplus \bigoplus_{|\lambda|<1, \lambda \in \mathbb{C}\setminus\mathbb{K}} E_{\lambda,\overline{\lambda}},$$

so that $E^s = E_0 \oplus E^{si}$; the *unstable subspace* is

$$E^u = E^u(T) = \bigoplus_{|\lambda|>1, \lambda \in \mathbb{K}} E_\lambda \oplus \bigoplus_{|\lambda|>1, \lambda \in \mathbb{C}\setminus\mathbb{K}} E_{\lambda,\overline{\lambda}};$$

the *central subspace* is

$$E^c = E^c(T) = \bigoplus_{|\lambda|=1, \lambda \in \mathbb{K}} E_\lambda \oplus \bigoplus_{|\lambda|=1, \lambda \in \mathbb{C}\setminus\mathbb{K}} E_{\lambda,\overline{\lambda}};$$

the *core subspace* is

$$C = C(T) = \bigoplus_{|\lambda|=1,\lambda\in\mathbb{K}} V_\lambda \oplus \bigoplus_{|\lambda|=1,\lambda\in\mathbb{C}\setminus\mathbb{K}} V_{\lambda,\overline{\lambda}};$$

and the *invertible subspace* is

$$E^i = E^{si} \oplus E^c \oplus E^u.$$

All these subspaces are T-invariant; furthermore, $T|_{E^i}$ is invertible. Finally, $V = E^s \oplus E^c \oplus E^u = E_0 \oplus E^i$.

Theorem 1.3.5: *Let $T\colon V \to V$ be a linear self-map of a finite-dimensional vector space V on the field $\mathbb{K}$. Then:*

(i) *There exists a scalar (or hermitian) product on V such that $T|_{E^s}$ and $(T|_{E^u})^{-1}$ are contractions, and $T|_C$ an isometry, with respect to the induced norm $\|\cdot\|$.*
(ii) *For every $v \in V$ we have $v \in E^s$ if and only if the orbit of v is exponentially convergent to the origin; furthermore, if $v \in E^{si}$ then the backward (with respect to $T|_{E^i}$) orbit of v is exponentially divergent to infinity.*
(iii) *For every $v \in V$ we have $v \in E^u$ if and only if $v \in E^i$ and the backward orbit of v is exponentially convergent to the origin; furthermore, if $v \in E^u$ then the orbit of v is exponentially divergent to infinity.*
(iv) *If $v \in E^c \setminus C$ then both the orbit and the backward orbit of v are polynomially divergent to infinity.*
(v) *More generally, given $v \in V$ write $v = v^s + v^c + v^u$ with $v^s \in E^s$, $v^c \in E^c$ and $v^u \in E^u$. Then:*
   - *if $v^u \neq O$ then the orbit of v is exponentially divergent to infinity;*
   - *if $v^u = O$ and $v^c \in E^c \setminus C$ then the orbit of v is polynomially divergent to infinity;*
   - *if $v^u = O$ and $v^c \in C \setminus \{O\}$ then the orbit of v is bounded and bounded away from the origin;*
   - *if $v^u = v^c = O$ (that is, if $v \in E^s$) then the orbit of v is exponentially convergent to the origin.*

Proof: (i) Applying Lemma 1.3.1 to $T|_{E^s}$ and to $(T|_{E^u})^{-1}$ we get on E^s and E^u norms induced by a scalar (or hermitian) product such that $T|_{E^s}$ and $(T|_{E^u})^{-1}$ are contractions. Then we choose a Jordan basis for T restricted to E^c, and we put on E^c the scalar (hermitian) product making this basis orthonormal; clearly $T|_C$ turns out to be an isometry. Finally, we combine these three scalar (hermitian) products to get a scalar (or hermitian) product on V by declaring the subspaces E^u, E^c and E^s to be orthogonal.

(ii) The behavior of the orbits follows from Corollary 1.3.3 applied to $T|_{E^s}$; the characterization of the elements of E^s will follow from (v).

(iii) The behavior of the orbits follows from the same Corollary applied to $(T|_{E^u})^{-1}$; the characterization follows from (v) applied to $(T|_{E^i})^{-1}$.

(iv) Using the norm described in (i) it is clear that it suffices to prove the assertion when $T|_{E^c}$is represented by a single Jordan block A relative to the eigenvalue $\lambda \in S^1$. It is then an exercise in linear algebra; see Exercise 1.3.1.

(v) If $v^u \neq O$ we get

$$\begin{aligned}\|T^k(v)\| &= \|T^k(v^u) + T^k(v^c) + T^k(v^s)\| \\ &\geq \|T^k(v^u)\| - \|T^k(v^c)\| - \|T^k(v^s)\| \\ &\geq c_1\mu^k - c_2k^m - c_3\lambda^k \geq c_4\mu^k\end{aligned}$$

for suitable c_1, c_2, c_3, $c_4 > 0$, $\mu > 1 > \lambda$ and $m \geq 1$, all independent of k. Similar arguments yield the remaining assertions. □

Exercise 1.3.3. Let $A \in GL(\ell,\mathbb{C})$ be a Jordan block of order $\ell \geq 2$ relative to an eigenvalue $\lambda \in S^1$.

(i) Prove that there are c_1, $c_2 > 0$ and $1 \leq m_1 \leq m_2 \leq \ell - 1$ such that

$$c_1k^{m_1} \leq |a^k_{ij}| \leq c_2k^{m_2}$$

for all $1 \leq i < j \leq \ell$ and all $k \in \mathbb{N}$, where a^k_{ij} is the (i,j)-element of the matrix A^k.
(ii) Prove that if $v \in \mathbb{C}^\ell$ is not an eigenvector of A then there exist c_3, $c_4 > 0$ so that $c_3k^{m_1} \leq \|A^kv\| \leq c_4k^{m_2}$ for all $k \in \mathbb{N}$.
(iii) Prove that if $A \in GL(2\ell,\mathbb{R})$ is a pseudo-Jordan block of order 2ℓ relative to an eigenvalue $\lambda \in S^1 \setminus \{\pm1\}$, and $v \in \mathbb{R}^{2\ell} \setminus (\mathbb{R}^{2\ell})_{\lambda,\overline{\lambda}}$, there are c_5, $c_6 > 0$ and $1 \leq m_3 \leq m_4 \leq 2\ell - 1$ so that $c_5k^{m_3} \leq \|A^kv\| \leq c_6k^{m_4}$ for all $k \in \mathbb{N}$.

Definition 1.3.3: A linear self-map $T: V \to V$ of a finite-dimensional vector space V on the field $\mathbb{K}$ is *hyperbolic* if T has no eigenvalues of modulus one, that is if $\mathrm{sp}(T) \cap S^1 = \varnothing$.

The behavior described in the previous theorem is particularly simple for hyperbolic linear maps: the space V decomposes in a T-invariant direct sum $V = E^s \oplus E^u$ so that $T|_{E^s}$ and $(T_{E^u})^{-1}$ are contractions, and all orbits are exponentially divergent to infinity except for orbits inside E_s which are exponentially convergent to the origin. Furthermore, the orbits of points in $V \setminus (E^s \cup E^u)$ are forward asymptotic to E^u and backward asymptotic to E^s, and thus have a sort of "hyperbolic" shape.

Most linear maps are hyperbolic, as shown in the next exercise.

Exercise 1.3.4. Let V be a finite-dimensional vector space on the field $\mathbb{K}$. Prove that the set of hyperbolic linear self-maps of V is an open dense subset of the space of all linear self-maps of V.

Exercise 1.3.5. Prove that the eigenvalues of a linear map $T: V \to V$ depend continuously on T, that is prove that if $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of T with multiplicity respectively $m_1, \ldots, m_k$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|T - S\| < \delta$ then in the open disk of center λ_j and radius ε the linear map S has exactly m_j eigenvalues (counted according to their multiplicities) for $j = 1, \ldots, k$.

Theorem 1.3.5 describes he dynamics of a linear self-map everywhere except on the core subspace, or, more precisely, everywhere but on the subspaces $V_{\lambda,\overline{\lambda}}$ (or V_λ if $\mathbb{K} = \mathbb{C}$) with $|\lambda| = 1$ but $\lambda \notin \mathbb{R}$. Such a subspace decomposes in a sum of real 2-dimensional (or complex 1-dimensional) T-invariant subspaces, and the action of T on any such subspace is a rotation of angle $2\pi\alpha$, where $\lambda = e^{2\pi i\alpha}$. If $\alpha \in \mathbb{Q}$ then a suitable iterate of T is the identity, and there is nothing else to say. But if $\alpha \notin \mathbb{Q}$ then new phenomena appear, as we shall see in the next section.

1.4 Translations of the torus and topological transitivity

Let us start with the rotations of $S^1 = \{e^{2\pi i\varphi} \mid \varphi \in \mathbb{R}\} = \mathbb{R}/\mathbb{Z}$. Using additive notations (that is thinking of S^1 as $\mathbb{R}/\mathbb{Z}$), the rotation R_α of angle $2\pi\alpha$ is represented by

$$R_\alpha(x) = x + \alpha \pmod 1,$$

so that $R_\alpha^k(x) = x + k\alpha \pmod 1$. In particular, if $\alpha = p/q \in \mathbb{Q}$ we have $R_{p/q}^q = \mathrm{id}_{S^1}$, that is the rotation of angle $2\pi p/q$ is globally periodic of period q.

The situation is much more interesting when $\alpha \notin \mathbb{Q}$. In all the examples we have seen so far, the orbits were either converging or diverging to infinity, and coming back on themselves only if periodic. In other words, we have not yet seen non-trivial *recurrence* phenomena: orbits coming arbitrarily close to the starting point without being periodic.

The two main cases of recurrent behavior are topological transitivity and minimality.

Definition 1.4.1: A dynamical system (X, f) is *topologically transitive* if there is $x \in X$ whose orbit is dense in X. It is *minimal* if every orbit is dense.

Remark 1.4.1. This is not the standard definition of topological transitivity. The standard definition (which is more useful but less intuitive than the one we just gave) is described in Proposition 1.4.3, where we show that in "good" topological spaces it is equivalent to ours.

Exercise 1.4.1. Prove that the closure of a f-invariant subset is still f-invariant.

Exercise 1.4.2. Prove that a dynamical system (X, f) is minimal if and only if there are no proper closed f-invariant subsets of X.

Proposition 1.4.1: *If $\alpha \in \mathbb{R}$ is irrational then the rotation $R_\alpha: S^1 \to S^1$ is minimal.*

Proof: Let $Y \subseteq S^1$ be the closure of an orbit. If the orbit is not dense, the complement $S^1 \setminus Y$ of Y is open and not empty, and thus it is an (at most countable) union of open disjoint intervals, its connected components; furthermore, since $R_\alpha(Y) \subseteq Y$, we have $R_\alpha^{-1}(S^1 \setminus Y) \subseteq S^1 \setminus Y$.

Let I be the longest connected component of $S^1 \setminus Y$ (or one of the longest if there are more than one; in any case, notice that only a finite number of connected components can be longer than any given $\varepsilon > 0$, because they are all disjoint and $S^1 \setminus Y$ has finite length). If $R_\alpha^{-k}(I) = I$ for some $k \geq 0$, an end x of I would be periodic, that is $x + k\alpha = x \pmod 1$, and so α would be rational. On the other hand, also $R_\alpha^{-k}(I) \cap I \neq \varnothing$ is impossible, because otherwise $S^1 \setminus Y$ would contain an interval longer than I. Therefore $\{R_\alpha^{-k}(I)\}_{k\in\mathbb{N}}$ is a disjoint collection of intervals of S^1 all of the same positive length, and this is clearly impossible. $\square$

Since the plane decomposes in a disjoint union of circles R_α-invariant (including the origin as a degenerate circle of radius zero), this result completes the description of the dynamics of linear maps.

Exercise 1.4.3. Prove that the decimal expansion of the number 2^k may begin with any finite string of digits.

In the proof of the previous proposition it did not matter which orbit we chose. This is not accidental:

Lemma 1.4.2: *Let G be a topological group. If $g_0 \in G$ is such that the left (or right) translation L_{g_0} is topologically transitive, then L_{g_0} is minimal.*

Proof: Let $x_0 \in G$ be such that its orbit $\{g_0^k x_0\}_{k\in\mathbb{N}}$ under L_{g_0} is dense in G, and let $x \in G$ be any other point. Now, $g_0^k x = (g_0^k x_0)(x_0^{-1}x)$; therefore the orbit of x is obtained applying the right translation $R_{x_0^{-1}x}$ to the orbit of x_0. Since right translations are homeomorphisms, the assertion follows. $\square$

Exercise 1.4.4. Let G be a metrizable topological group. Prove that if for some $g_0 \in G$ the left (respectively, right) translation L_{g_0} (respectively, R_{g_0}) is topologically transitive then G is abelian.

Now we proceed to the study of more interesting examples. We start with the direct generalization of rotations on S^1, that is with translations on the torus $\mathbb{T}^n = (S^1)^n = (\mathbb{R}/\mathbb{Z})^n$.

Definition 1.4.2: If $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{T}^n$, the associated *translation* $T_\gamma\colon \mathbb{T}^n \to \mathbb{T}^n$ is given by

$$T_\gamma(x_1, \ldots, x_n) = (x_1 + \gamma_1, \ldots, x_n + \gamma_n) \pmod 1.$$

Again, if all coordinates of γ are rational then T_γ is globally periodic. However, contrarily to the case of S^1, it is not anymore true that the only alternative is minimality. For instance, if $n = 2$ and $\gamma = (\alpha, 0)$ with α irrational, then the torus $\mathbb{T}^2$ is the disjoint union of the circles $x_2 =$ const., which are completely T_γ-invariant and minimal.

To find a characterization of minimal translations, we need some general criteria of topological transitivity and minimality.

Proposition 1.4.3: *Let X be a locally compact Hausdorff space with a countable basis of open sets and no isolated points, and $f\colon X \to X$ a continuous self-map. Then f is topologically transitive if and only if for every pair of not empty open sets U, $V \subset X$ there is $N = N(U,V) \in \mathbb{N}$ such that $f^N(U) \cap V \neq \varnothing$.*

Proof: Let f be topologically transitive, and $x \in X$ with a dense orbit. Since X is Hausdorff and has no isolated points, the orbit of x must intersect every open subset of X infinitely many times. In particular, given the open sets U and V there are h, $k \in \mathbb{N}$ such that $f^h(x) \in U$ and $f^k(x) \in V$, and we can assume $k \geq h$. But then $f^k(x) \in f^{k-h}(U) \cap V$, as required.

Conversely, let us assume that the condition on the intersections holds. Let $U_1, U_2, \ldots$ be a countable basis of open sets of X; by local compactness, we can also assume that $\overline{U_1}$ is compact. To prove the topological transitivity of f it suffices to find an orbit intersecting every U_n. By assumption, there exists $N_1 \in \mathbb{N}$ such that $f^{N_1}(U_1) \cap U_2$ is not empty. Let V_1 be a not empty open set such that $\overline{V_1} \subset U_1 \cap f^{-N_1}(U_2)$; in particular, $\overline{V_1}$ is compact. Now, there exists a natural number N_2 such that $f^{N_2}(V_1) \cap U_3$ is not empty; let V_2 be a not empty open set such that $\overline{V_2} \subset V_1 \cap f^{-N_2}(U_3)$. Proceeding by induction we build a decreasing sequence of open sets V_n such that $\overline{V_{n+1}} \subset V_n \cap f^{-N_{n+1}}(U_{n+2})$. Then if x belongs to the intersection $\cap_{n=1}^\infty \overline{V_n}$, which is not empty by compactness, we have $f^{N_{n-1}}(x) \in U_n$ for every $n \in \mathbb{N}$. $\square$

Remark 1.4.2. The previous proof shows that if for every pair of not empty open sets U, $V \subseteq X$ yjere is $N \in \mathbb{N}$ such that $f^N(U) \cap V \neq \varnothing$ then there is a dense orbit even when X has isolated points. The converse is *not* true if X has isolated points. For instance, let $X = S^1 \cup \{p_0\}$, where p_0 is an isolated point, and take $f\colon X \to X$ given by $f(p_0) = 1$ and $f|_{S^1} = R - \alpha$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then the orbit of p_0 is dense, but $f^N(S^1) \cap \{p_0\} = \varnothing$ for all $N \geq 0$, even though both S^1 and $\{p_0\}$ are open in X.

Remark 1.4.3. A locally compact Haudorff space is always regular; a regular Hausdorff space with a countable basis of open sets is always metrizable; so every locally compact Hausdorff space with a countable basis of open sets is always metrizable. For this reason, in the sequel we shall often work with metrizable spaces instead of merely Hausdorff spaces.

When $f\colon X \to X$ is a homeomorphism, we can check the topological transitivity using full orbits:

Proposition 1.4.4: *Let X be a locally compact Hausdorff space with a countable basis of open sets and no isolated points, and $f\colon X \to X$ a homeomorphism. Then the following assertions are equivalent:*
(i) *f is topologically transitive;*
(ii) *there exists $x \in X$ with a dense full orbit;*
(iii) *for every pair of not empty open sets $U, V \subset X$ there exists $N = N(U,V) \in \mathbb{N}$ such that $f^N(U) \cap V \neq \varnothing$;*
(iv) *for every pair of not empty open sets $U, V \subset X$ there exists $N = N(U,V) \in \mathbb{Z}$ such that $f^N(U) \cap V \neq \varnothing$.*

Remark 1.4.4. The equivalence of (ii) and (iv) holds even when there are isolated points.

Proof: We have already seen that (i) is equivalent to (iii); the same argument (even when there are isolated points) shows that (ii) is equivalent to (iv). Since (i) clearly implies (ii), it suffices to prove that (ii) and (iv) together imply (iii).

We first show that for every $M \in \mathbb{N}$ and every open subset $W \subset X$ there exists $n > M$ such that $f^{-n}(W) \cap W \neq \varnothing$. Indeed, since X has no isolated points condition (ii) implies that there is $x \in X$ and an infinite subset $H \subseteq \mathbb{Z}$ such that $f^h(x) \in W$ for every $h \in H$. Then taking h_0, $h_1 \in H$ such that $h_0 - h_1 > M$ (this can be done because H is infinite) we get $f^{h_1-h_0}(W) \cap W \neq \varnothing$, as desired.

Let now U and V be not empty open sets. Condition (iv) yields $N \in \mathbb{Z}$ such that $W = f^N(U) \cap V \neq \varnothing$. If $N \geq 0$ we are done; if instead $N < 0$, let $n > |N|$ be such that $f^{-n}(W) \cap W \neq \varnothing$. But then

$$\varnothing \neq f^n\bigl(f^{-n}(W) \cap W\bigr) = W \cap f^n(W) = V \cap f^N(U) \cap f^n(V) \cap f^{n+N}(U) \subseteq f^{n+N}(U) \cap V,$$

and (iii) is verified, because $n + N > 0$ by the choice of n. □

Corollary 1.4.5: *Let X be a locally compact Hausdorff space with a countable basis of open sets and no isolated points, and $f\colon X \to X$ a homeomorphism. Then f is topologically transitive if and only if there do not exist two not empty disjoint completely f-invariant open subsets of X.*

Proof: An invariant subset contains any orbit starting in it; therefore if f is topologically transitive then two not empty invariant open sets always have a not empty intersection. Conversely, let U and V be two not empty open subsets of X. Then the sets $\tilde{U} = \bigcup_{k\in\mathbb{Z}} f^k(U)$ and $\tilde{V} = \bigcup_{k\in\mathbb{Z}} f^k(V)$ are completely f-invariant open subsets, and thus their intersection is not empty. This means that there are h, $k \in \mathbb{Z}$ such that $f^h(U) \cap f^k(V) \neq \varnothing$, so that $f^{h-k}(U) \cap V \neq \varnothing$, and Proposition 1.4.4 implies that f is topologically transitive. □

Remark 1.4.5. This proof shows that if $f\colon X \to X$ is merely continuous and topological transitive, then there do not exist two non empty disjoint f-invariant open subsets of X. If f is not an homeomorphism, the converse is not true. Take $X = S^1 \times \mathbb{Z}_2$, and let $f\colon X \to X$ be given by $f(x,0) = (x,1)$ and $f(x,1) = \bigl(R_\alpha(x), 1\bigr)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Since R_α is minimal, the only not empty open completely R_α-invariant subset of S^1 is S^1 itself (why?); therefor the only not empty open completely f-invariant subset of X is X itself. But no orbit of f is dense, and $f^N(S^1 \times \{1\}) \cap S^1 \times \{0\} = \varnothing$ for all $N \in \mathbb{N}$.

A necessary condition for topological transitivity can be stated using functions.

Definition 1.4.3: Let (X,f) be a dynamical system. A map $\varphi\colon X \to Y$ is called *f-invariant* if $\varphi\bigl(f(x)\bigr) = \varphi(x)$ for every $x \in X$.

Lemma 1.4.6: *Let (X,f) be a topologically transitive dynamical system. Then every continuous f-invariant function $\varphi\colon X \to Y$ is constant.*

Proof: Let $x \in X$ be a point with dense orbit. The f-invariance of φ yields

$$\varphi\bigl(f^k(x)\bigr) = \varphi(x)$$

for all $k \in \mathbb{N}$; therefore φ is constant on the orbit of x — and thus everywhere. □

This is what we need to characterize the minimal translations of the torus.

Definition 1.4.4: The numbers $\gamma_0, \ldots, \gamma_n \in \mathbb{R}$ are *rationally dependent* if there are integers $k_0, \ldots, k_n \in \mathbb{Z}$, with at least one k_j different from zero, such that $k_0\gamma_0 + \cdots + k_n\gamma_n = 0$; they are *rationally independent* if they are not rationally dependent.

Proposition 1.4.7: *A translation $T_\gamma: \mathbb{T}^n \to \mathbb{T}^n$ is minimal if and only if the numbers $1, \gamma_1, \ldots, \gamma_n$ are rationally independent.*

Proof: Let us assume that $1, \gamma_1, \ldots, \gamma_n$ are not rationally independent, e let $k_0, \ldots, k_n \in \mathbb{Z}$ be such that

$$k_1\gamma_1 + \cdots + k_n\gamma_n = k_0,$$

with $k_j \neq 0$ for at least one $1 \leq j \leq n$. Define $\varphi: \mathbb{T}^n \to \mathbb{R}$ by

$$\varphi(x) = \sin 2\pi(k_1x_1 + \cdots + k_nx_n).$$

It is a well-defined continuous function on the torus, it is not constant because there is at least one k_j different from zero, and it is clearly T_γ-invariant; therefore, by Lemma 1.4.6, T_γ cannot be topologically transitive.

Conversely, suppose that T_γ is not minimal (and thus not even topologically transitive, by Lemma 1.4.2). Corollary 1.4.5 then yields two disjoint not empty open sets U and V completely T_γ-invariant. Let χ be the characteristic function of U; being U completely invariant, χ is T_γ-invariant. Let

$$\chi(x_1, \ldots, x_n) = \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} \chi_{k_1,\ldots,k_n} \exp\bigl(2\pi i(k_1x_1 + \cdots + k_nx_n)\bigr)$$

be the Fourier expansion of χ. Since

$$\begin{aligned}\chi\bigl(T_\gamma(x)\bigr) = \chi(x_1+\gamma_1, \ldots, x_n+\gamma_n) &= \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} \chi_{k_1,\ldots,k_n} \exp\bigl[2\pi i\bigl(k_1(x_1+\gamma_1) + \cdots + k_n(x_n+\gamma_n)\bigr)\bigr] \\ &= \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} \chi_{k_1,\ldots,k_n} \exp\bigl(2\pi i(k_1\gamma_1 + \cdots + k_n\gamma_n)\bigr) \exp\bigl(2\pi i(k_1x_1 + \cdots + k_nx_n)\bigr),\end{aligned}$$

the T_γ-invariance of χ and the uniquess of the Fourier expansion imply that for every $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ we have

$$\chi_{k_1,\ldots,k_n}\left[1 - \exp\bigl(2\pi i(k_1x_1 + \cdots + k_nx_n)\bigr)\right] = 0.$$

But this may happen if and only if $\chi_{k_1,\ldots,k_n} = 0$ or $k_1\gamma_1 + \cdots + k_n\gamma_n \in \mathbb{Z}$ for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. Since both U and its complement contain not empty open subsets, and thus have positive Lebesgue measure, χ is not almost everywhere constant. Hence there must exists $(k_1, \ldots, k_n) \neq O$ such that $\chi_{k_1,\ldots,k_n} \neq 0$, and so $k_1\gamma_1 + \cdots + k_n\gamma_n \in \mathbb{Z}$, that is $1, \gamma_1, \ldots, \gamma_n$ are not rationally independent. □

Exercise 1.4.5. Prove that for every translation $T_\gamma: \mathbb{T}^n \to \mathbb{T}^n$ and every $x \in \mathbb{T}^n$ the closure $C(x)$ of the full orbit $\{T_\gamma^k(x)\}_{k\in\mathbb{Z}}$ of x is a finite union of tori of dimension $0 \leq m \leq n$, such that the restriction of T_γ to $C(x)$ is minimal.

Exercise 1.4.6. Prove that the map $A_\alpha: \mathbb{T}^2 \to \mathbb{T}^2$ given by $A_\alpha(x, y) = (x+\alpha, y+x) \pmod 1$ is topologically transitive if and only if α is irrational.

1.5 Toral automorphisms and chaotic dynamical systems

In this section we study another kind of toral automorphism, anticipating several characteristics of general hyperbolic dynamical systems.

Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix

$$L = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}.$$

Since L belongs to $SL(2, \mathbb{Z})$, it sends $\mathbb{Z}^2$ into itself, and thus defines a map $F_L: \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$F_L(x, y) = (2x + y, x + y) \pmod 1.$$

The map F_L is continuous, invertibile (because $\det L = 1$ implies $L^{-1} \in SL(2, \mathbb{Z})$ too) and it is an automorphism of $\mathbb{T}^2$ as topological group.

As linear map, L is hyperbolic: its eigenvalues are

$$\lambda_1 = \frac{3 + \sqrt{5}}{2} > 1, \qquad \lambda_1^{-1} = \lambda_2 = \frac{3 - \sqrt{5}}{2} < 1.$$

Being L symmetric, the eigenvectors are orthogonal. The eigenspace relative to λ_1 has equation $y = \omega x$, where $\omega = \frac{\sqrt{5}-1}{2}$; the family of lines parallel to it is sent into itself by L. Moreover, L expands uniformly the distances on these lines, by a factor λ_1. More precisely, if we denote by $\ell_c \subset \mathbb{R}^2$ the line of equation $y = \omega x + c$, with $c \in \mathbb{R}$, then $L(\ell_c) = \ell_{(1-\omega)c}$, and

$$\|L(v_1) - L(v_0)\| = \lambda_1 \|v_1 - v_0\|$$

for all v_0, $v_1 \in \ell_c$.

Analogously, the lines ℓ'_c parallel to the eigenspace relative to λ_2 have equation $y = (\omega - \sqrt{5})x + c$, and L sends ℓ'_c into $\ell'_{(1-\omega+\sqrt{5})c}$ contracting distances by a factor λ_2.

To describe the dynamical properties of F_L we shall need the following well-known and interesting

Theorem 1.5.1: (Pick) *Let $P \subset \mathbb{R}^2$ be a simple (i.e., whose sides intersect at the vertices only) polygon, and assume that the vertices of P belong to $\mathbb{Z}^2$. Then*

$$\mathrm{Area}(P) = i + \frac{e}{2} - 1,$$

where i is the number of points of $\mathbb{Z}^2$ belonging to the interior of P, and e is the number of points of $\mathbb{Z}^2$ belonging to the boundary of P.

Proposition 1.5.2: *The map $F_L: \mathbb{T}^2 \to \mathbb{T}^2$ is topologically transitive, and its periodic points are dense in $\mathbb{T}^2$. Furthermore, the number of periodic points of (not necessarily exact) period $k \in \mathbb{N}$ is given by $\lambda_1^k + \lambda_1^{-k} - 2$.*

Proof: Let us begin by proving that all points with rational coordinates are periodic points of F_L; this in particular implies that periodic points are dense. Choose $x = r/q$ and $y = s/q$, with r, s, $q \in \mathbb{Z}$ not necessarily coprime. Then

$$F_L(r/q, s/q) = \big((2r + s)/q, (r + s)/q\big),$$

that is $F_L(x, y)$ is a rational point whose coordinates still have denominator q. But there are only q^2 points in $\mathbb{T}^2$ whose coordinates are represented by rational numbers with q as denominator, and the whole orbit of $(r/q, s/q)$ is contained in this finite set; therefore there exist $h \geq k$ such that $F_L^h(r/q, s/q) = F_L^k(r/q, s/q)$. Being F_L invertible, this implies that $F_L^{h-k}(r/q, s/q) = (r/q, s/q)$, that is $(r/q, s/q)$ is a periodic point for F_L, as claimed.

We now prove that these are the only periodic points. Suppose that $F_L^k(x, y) = (x, y)$. But

$$F_L^k(x, y) = (ax + by, cx + dy) \pmod 1$$

for suitable integers a, b, c, $d \in \mathbb{N}$; therefore

$$\begin{cases} ax + by = x + r, \\ cx + dy = y + s, \end{cases} \tag{1.5.1}$$

for suitable integers r, $s \in \mathbb{Z}$. Since 1 is not an eigenvalue of L^k we can solve (1.5.1) obtaining

$$x = \frac{(d-1)r - bs}{(a-1)(d-1) - cb}, \qquad y = \frac{(a-1)s - cr}{(a-1)(d-1) - cb},$$

and hence x and y are rational.

Let us now show that F_L is topologically transitive (but it is not minimal: a periodic orbit cannot be dense). Let U and V be two not empty open subsets of $\mathbb{T}^2$; since periodic points are dense, we can find two periodic points $p \in U$, $q \in V$ of common (not necessarily exact) period $k \in \mathbb{N}$. Let $c_0 \in [0,1]$ be such that the projection $\pi(\ell_{c_0})$ of ℓ_{c_0} on the torus contains p, and $d_0 \in [0,1]$ such that the projection $\pi(\ell'_{d_0})$ contains q, where $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ is the canonical projection. Since $F_L^k(p) = p$, and L sends the family of lines ℓ_c into itself, we necessarily have $F_L^k\bigl(\pi(\ell_{c_0})\bigr) = \pi(\ell_{c_0})$. Analogously, $F_L^k\bigl(\pi(\ell'_{d_0})\bigr) = \pi(\ell'_{d_0})$.

Let $r \in \mathbb{T}^2$ be in the intersection of these two curves. Since L^k acts on the lines ℓ'_d contracting the distances by a factor λ_2^k, we necessarily have $F_L^{mk}(r) \to q$ for $m \to +\infty$ (why?). Analogously, we find $F_L^{-mk}(r) \to p$ for $m \to +\infty$. Hence if $m > 0$ is large enough we have $F_L^{-mk}(r) \in U$ and $F_L^{mk}(r) \in V$; therefore $F_L^{2mk}(U) \cap V \neq \varnothing$, and F_L is topologically transitive thanks to Proposition 1.4.3.

We are left with counting the periodic points. As before, we have $F_L^k(x,y) = (x,y)$ if and only if $(a-1)x + by$ and $cx + (d-1)y$ are integers. Thus the number of periodic points of period k is exactly equal to the cardinality of the intersection of $\mathbb{Z}^2$ with the set $P = (L^k - \mathrm{id})\bigl([0,1) \times [0,1)\bigr)$. Now, the boundary of P is the parallelogram with vertices the origin, $(a-1, c)$, $(b, d-1)$ and $(a+b-1, c+d-1)$; therefore, using the symbols introduced in the statement of Theorem 1.5.1, the number of periodic points of period k is given by

$$i + \frac{e}{2} - 1 = \mathrm{Area}(P) = |\det(L^k - \mathrm{id})| = |(\lambda_1^k - 1)(\lambda_1^{-k} - 1)| = \lambda_1^k + \lambda_1^{-k} - 2.$$

□

Exercise 1.5.1. Prove that $L^k = \begin{vmatrix} F_{2k} & F_{2k-1} \\ F_{2k-1} & F_{2k-2} \end{vmatrix}$, where F_k is the k-th Fibonacci number.

We remark a basic difference between the automorphism F_L and the translations T_γ: while in the latter case every orbit is dense, in the former case arbitrarily close to dense orbits there are periodic orbits, that is orbits with a completely different behavior. In other words, the dynamical behavior is very sensitive to initial conditions. This is a typical phenomenon for *chaotic* dynamical systems.

Definition 1.5.1: A dynamical system (X, f) is *chaotic* if it is topologically transitive and the periodic points are dense.

Example 1.5.1. The dynamical system $(\mathbb{T}^2, F_L)$ is chaotic.

In good topological spaces we have an interesting characterization of chaotic dynamical systems:

Proposition 1.5.3: *Let X be a Hausdorff locally compact topological space with a countable basis of open sets and no isolated points. Then a dynamical system (X, f) is chaotic if and only if every pair of not empty open sets share a periodic orbit, that is if for every open pair of not empty subsets U, $V \subset X$ there is a periodic point $x \in U$ such that $O^+(x) \cap V \neq \varnothing$, or, equivalently, there exists $N = N(U,V) \in \mathbb{N}$ such that $f^N(U) \cap V \cap \mathrm{Per}\,(f) \neq \varnothing$.*

Proof: Assume that every pair of not empty open subsets share a periodic orbit. Then, by Proposition 1.4.3, f is topologically transitive. Furthermore, every open set must contain a periodic point, and so periodic points are dense.

Conversely, assume that f is chaotic. Again by Proposition 1.4.3, we know that for every pair of open sets U, $V \subset X$ there are $y \in U$ and $k \in \mathbb{N}$ such that $f^k(y) \in V$. Let $W = f^{-k}(V) \cap U$. Clearly, W is open and not empty, because $y \in W$; therefore it must contain a periodic point x. But then $f^k(x) \in V$, and $O^+(x) \cap V \neq \varnothing$, as required. □

Remark 1.5.1. If we assume as definition of topological transitivity the characterization given in Proposition 1.4.3, both the previous result and Proposition 1.5.4 hold without topological assumptions.

Exercise 1.5.2. Let X be a Hausdorff locally compact topological space with a countable basis of open sets and no isolated points. Prove that if $f\colon X \to X$ is a chaotic dynamical system then for every finite collection of non-empty open subsets of X there are infinitely many periodic orbits intersecting all the sets in the collection.

There is a property of chaotic dynamical systems expressing very well the intuitive notion of chaos.

Definition 1.5.2: A dynamical system (X, f) on a metric space X *has sensitive dependence on initial conditions* if there is $\delta > 0$ such that for every $x \in X$ and every neighbourhood $U \subset X$ of x there exist $y \in U$ and $k \geq 0$ such that $d\big(f^k(x), f^k(y)\big) > \delta$.

Proposition 1.5.4: *Let X be a locally compact metric space with a countable basis of open sets and without isolated points. Then every chaotic dynamical system on X has sensitive dependence on initial conditions.*

Proof: Let $f\colon X \to X$ be topologically transitive and having a dense set of periodic points; we shall prove that it has sensitive dependence on initial conditions.

First of all, there is a $\delta_0 > 0$ such that for all $x \in X$ there is a periodic point $q \in X$ whose orbit is at distance at least $\delta_0/2$ from x. Indeed, choose two arbitrary periodic points q_1 and q_2 with disjoint orbits, and let δ_0 be the distance between these two orbits. Then every $x \in X$ is at least $\delta_0/2$ away from one of these two orbits.

Put $\delta = \delta_0/8$. Let $x \in X$ be arbitrary, and U a neighbourhood of x. Since periodic points are dense, there is a periodic point $p \in U \cap B(x, \delta)$, where $B(x, \delta)$ is the open ball of center x and radius δ. Let $k_0 \in \mathbb{N}$ be the period of p. By the previous observation, there is a periodic point $q \in X$ whose orbit is at least 4δ away from x. Set

$$V = \bigcap_{j=0}^{k_0} f^{-j}\big(B\big(f^j(q), \delta\big)\big);$$

this is open and not empty because $q \in V$. By Proposition 1.4.3, there are $y \in U$ and $k \in \mathbb{N}$ such that $f^k(y) \in V$.

Now let l be the integer part of $(k/k_0) + 1$, so that $1 \leq k_0 l - k \leq k_0$. Then

$$f^{k_0 l}(y) = f^{k_0 l - k}\big(f^k(y)\big) \in f^{k_0 l - k}(V) \subseteq B\big(f^{k_0 l - k}(q), \delta\big).$$

Now $f^{k_0 l}(p) = p$, and so

$$d\big(f^{k_0 l}(p), f^{k_0 l}(y)\big) \geq d\big(x, f^{k_0 l - k}(q)\big) - d\big(f^{k_0 l - k}(q), f^{k_0 l}(y)\big) - d(p, x) > 4\delta - \delta - \delta = 2\delta.$$

Therefore at least one of $d\big(f^{k_0 l}(x), f^{k_0 l}(y)\big)$ and $d\big(f^{k_0 l}(x), f^{k_0 l}(p)\big)$ must be greater than δ, and we are done. □

Remark 1.5.2. Devaney defined a chaotic system as a dynamical system topologically transitive, having sensitive dependence to initial conditions and having a dense set of periodic points. Propositions 1.5.3 and 1.5.4 show that, in locally compact metric spaces with a countable basis of open sets and no isolated points, Devaney's definition is equivalent to ours.

Even the type of topological transitivity of T_γ and F_L is different.

Definition 1.5.3: A dynamical system (X, f) is *topologically mixing* if for every pair of not empty open sets U, $V \subset X$ there is $N = N(U, V) \in \mathbb{N}$ so that for every $k \geq N$ the intersection $f^k(U) \cap V$ is not empty.

Thanks to Proposition 1.4.3, every topologically mixing dynamical system on a sensible topological space is topologically transitive; the converse is false.

Lemma 1.5.5: *Let (X, f) be a dynamical system. If there is a f-invariant distance d generating the topology of X, then f cannot be topologically mixing. In particular, the translations T_γ are not topologically mixing.*

Proof: Choose three distinct points x, y_1, $y_2 \in X$, and let δ be equal to one-fifth of the minimal distance between them. Let U, V_1, V_2 be the balls of radius δ and center x, y_1, y_2 respectively. Since f preserves the diameter of any subset of X, the diameter of $f^k(U)$ cannot exceed 2δ, while the distance between two points $p \in V_1$ and $q \in V_2$ is at least 3δ. Hence for every $k > 0$ at least one of the intersections $f^k(U) \cap V_1$ and $f^k(U) \cap V_2$ is empty.

Finally, T_γ preserves the distance induced by the euclidean distance of $\mathbb{R}^n$, and thus it is not topologically mixing. □

On the other hand

Proposition 1.5.6: *The dynamical system $(\mathbb{T}^2, F_L)$ is topologically mixing.*

Proof: The projection $\hat{\ell}_c = \pi(\ell_c)$ of a line ℓ_c is invariant under the translation $T_{(1,\omega)}$, which is minimal by Proposition 1.4.7; in particular, every such $\hat{\ell}_c$ is dense in $\mathbb{T}^2$, and thus every open set U contains a segment J of any $\hat{\ell}_c$.

Now fix $\varepsilon > 0$ and a $c \in [0, 1]$. We claim that there is $T = T(\varepsilon, c) > 0$ such that every segment of $\hat{\ell}_c$ of length T intersects every ε-ball on the torus. Since every segment of $\hat{\ell}_c$ of length T is obtained by a given one applying a translation of the form $T_{(t,t\omega)}$ for a suitable $t \in \mathbb{R}$, it suffices to prove that there is one segment of $\hat{\ell}_c$ of finite length intersecting every ε-ball. Suppose this is not the case; then we can find an increasing sequence $I_n \subset I_{n+1}$ of segments of $\hat{\ell}_c$ of length at least n and a sequence $\{x_n\} \subset \mathbb{T}^2$ such that $I_n \cap B(x_n, \varepsilon) = \varnothing$. Up to a subsequence we can assume that $x_n \to y \in \mathbb{T}^2$. Since $\hat{\ell}_c$ is dense, we have $\hat{\ell}_c \cap B(y, \varepsilon/2) \neq \varnothing$; being $\hat{\ell}_c$ the union of the I_n, this means that there is $n_0 \in \mathbb{N}$ such that $I_n \cap B(y, \varepsilon/2) \neq \varnothing$ for every $n \geq n_0$. But then if we choose $n \geq n_0$ so that $d(x_n, y) < \varepsilon/2$ we get $I_n \cap B(x_n, \varepsilon) \neq \varnothing$, contradiction.

Now we claim that we can find a $T(\varepsilon)$ independent of c. Suppose not: then for every $n > 0$ we can find a $c_n \in [0, 1]$, a segment $I_n \subset \hat{\ell}_{c_n}$ of length at least n, and a $x_n \in X$ such that $I_n \cap B(x_n, \varepsilon) = \varnothing$. Again, we can assume that $x_n \to y$ and $c_n \to d$. By the previous statement, for every segment $I \subset \hat{\ell}_d$ of length $T = T(\varepsilon/3, d)$ we have $I \cap B(y, \varepsilon/3) \neq \varnothing$. Then it is clear that for n large enough we have $I_n \cap B(x_n, \varepsilon) \neq \varnothing$, contradiction.

Finally, let V be any not empty open set, and $\varepsilon > 0$ such that V contains one ball of radius ε. Let $J \subset U$ be a segment of a $\hat{\ell}_c$. Then there is a $N = N(J, \varepsilon)$ such that $F_L^k(J)$ is a segment of some $\hat{\ell}_d$ of length at least $T(\varepsilon)$ for any $k \geq N$; therefore $F_L^k(J) \cap V \neq \varnothing$, and we are done. □

The automorphism F_L is just one example of a family of toral automorphisms.

Definition 1.5.4: A *hyperbolic toral endomorphism* is a map fo the form $F_A : \mathbb{T}^n \to \mathbb{T}^n$ where $A \in GL(n, \mathbb{Z})$ is a hyperbolic matrix with integer entries. If moreover $\det A = \pm 1$, then F_A is invertible, and it is a *hyperbolic toral automorphism.*

We end this ection with a couple of exercises describing some properties of hyperbolic toral automorphisms.

Exercise 1.5.3. Let $A \in GL(n, \mathbb{Z})$ be any matrix with integer entries and determinant ± 1. Prove that the induced map $L_A : \mathbb{T}^n \to \mathbb{T}^n$ has a finite number of periodic points of period k for all $k \in \mathbb{N}$ if and only if A is hyperbolic.

Exercise 1.5.4. Prove that the periodic points of a hyperbolic endomorphism of $\mathbb{T}^n$ are dense.

Exercise 1.5.5. Let $f : S^1 \to S^1$ be given by $f(x) = 2x \pmod 1$. Prove that f is chaotic.

1.6 The quadratic family

In this section we begin the study of the *quadratic family* $F_\mu : \mathbb{R} \to \mathbb{R}$ defined by

$$F_\mu(x) = \mu x(1 - x)$$

for $\mu \in \mathbb{R}^*$. It is easy to check that

$$\mathrm{Fix}(F_\mu) = \left\{0, 1 - \frac{1}{\mu}\right\},\ F_\mu(1) = 0,\ F_\mu\left(\frac{1}{\mu}\right) = 1 - \frac{1}{\mu},\ F'_\mu(x) = \mu(1-2x);\ F'_\mu(0) = \mu;\ F'_\mu\left(1 - \frac{1}{\mu}\right) = 2 - \mu.$$

In particular, the derivative of F_μ vanishes only at $x = 1/2$, where $F_\mu(1/2) = \mu/4$, which is an absolute maximum if $\mu > 0$, and an absolute minimum if $\mu < 0$.

The dynamics of F_μ for $0 < |\mu| \le 1$ is easily described:

Exercise 1.6.1. Assume that $0 < |\mu| \le 1$. Prove that:

(i) if $0 < \mu \le 1$ and $x \in (-\infty, 1 - 1/\mu) \cup (1/\mu, +\infty)$ then $F_\mu^k(x) \to -\infty$ as $k \to +\infty$;
(ii) if $0 < \mu \le 1$ and $x \in (1 - 1/\mu, 1/\mu)$ then $F_\mu^k(x) \to 0$ as $k \to +\infty$;
(iii) if $-1 \le \mu < 0$ and $x \in (-\infty, 1/\mu) \cup (1 - 1/\mu, +\infty)$ then $F_\mu^k(x) \to +\infty$ as $k \to +\infty$;
(iv) if $-1 \le \mu < 0$ and $x \in (1/\mu, 1 - 1/\mu)$ then $F_\mu^k(x) \to 0$ as $k \to +\infty$.

If $|\mu| > 1$ then the origin becomes a repelling fixed point, and the dynamics becomes more interesting. In this section we shall concentrate our attention on the case $\mu > 1$. For simplicity, we put

$$p_\mu = 1 - \frac{1}{\mu};$$

since $\mu > 1$ we have $p_\mu \in (0, 1)$.

Lemma 1.6.1: *If $\mu > 1$ and $x \in (-\infty, 0) \cup (1, +\infty)$ then $F_\mu^k(x) \to -\infty$.*

Proof: If $x < 0$ we clearly have $\mu x < x$ and hence $F_\mu(x) < x$. So the orbit $\{F_\mu^k(x)\}$ is strictly decreasing, and hence converge to a point $x_\infty \in [-\infty, 0)$. But x_∞ cannot be finite, because otherwise, by Remark 1.1.1, it would be a negative fixed point of F_μ, and F_μ has no negative fixed points. Therefore $F_\mu^k(x) \to -\infty$ for all $x \in \mathbb{R}^-$. Since $F_\mu\bigl(1, +\infty)\bigr) = (-\infty, 0)$, we are done. □

So the interesting dynamics (if any) should live in the interval $I = [0, 1]$; in particular, we know that as soon as an orbit leaves the interval I then it necessarily escapes to infinity.

The case $1 < \mu < 3$ is not difficult to study:

Proposition 1.6.2: *If $1 < \mu < 3$ and $x \in (0, 1)$ then $F_\mu^k(x) \to p_\mu$.*

Proof: When $1 < \mu < 3$ we know that p_μ is an attractive fixed point, and hence there is a neighbourhood of points whose orbit converges to p_μ; our aim is to show that this neighbourhood is the whole interval $(0, 1)$.

Let us first assume $1 < \mu \le 2$. Then $p_\mu \le 1/2$, and F_μ is increasing in the interval $[0, p_\mu]$; in particular, since 0 and p_μ are fixed, we have $F_\mu([0, p_\mu]) = [0, p_\mu]$. Furthermore, if $x \in (0, p_\mu)$ we have

$$0 < x < F_\mu(x) < p_\mu;$$

hence the orbit $\{F_\mu^k(x)\}$ is strictly increasing and bounded, and thus converges (by Remark 1.1.1) to p_μ.

If $x \in (p_\mu, 1/2]$ we instead have $p_\mu < F_\mu(x) < x$; therefore we again obtain $F_\mu^k(x) \to p_\mu$. Finally, since $F_\mu\bigl((1/2, 1)\bigr) \subset (0, 1/2)$ because $F_\mu(1/2) = \mu/4 \le 1/2$, we are done in this case.

Let us now assume $2 < \mu < 3$, so that $p_\mu > 1/2 > 1/\mu$. Then $F_\mu\bigl((1/\mu, p_\mu)\bigr) = (p_\mu, \mu/4]$ and

$$F_\mu^2\bigl((1/\mu, p_\mu)\bigr) = \left[\frac{\mu^2}{4}\left(1 - \frac{\mu}{4}\right), p_\mu\right) \subset \left(\frac{1}{2}, p_\mu\right) \subset \left(\frac{1}{\mu}, p_\mu\right).$$

In particular, $F_\mu^2(1/\mu) > 1/\mu$ and $F_\mu^2(p_\mu) = p_\mu$. Since we have already seen (Example 1.2.6) that F_μ has no periodic points of exact period 2 when $2 < \mu < 3$, it follows that $F_\mu^2 - \mathrm{id}$ does not vanish in $(1/\mu, p_\mu)$, and so $p_\mu > F_\mu^2(x) > x > 1/\mu$ for all $x \in (1/\mu, p_\mu)$. The usual argument based on Remark 1.1.1 then shows that $F_\mu^{2k}(x) \to p_\mu$ for all $x \in (1/\mu, p_\mu)$. But then

$$F_\mu^{2k+1}(x) = F_\mu\bigl(F_\mu^{2k}(x)\bigr) \to F_\mu(p_\mu) = p_\mu,$$

and hence we have $F_\mu^k(x) \to p_\mu$ for all $x \in (1/\mu, p_\mu)$.

Take now $x \in (0, 1/\mu)$, so that $F_\mu(x) > x$. If the orbit of x were contained in $(0, 1/\mu)$, then it would be strictly increasing and bounded, and thus converging to a fixed point of F_μ in $(0, 1/\mu]$, impossible. Therefore there is a $k_0 > 0$ such that $F_\mu^{k_0}(x) \in [1/\mu, p_\mu)$, and the previous argument yields $F_\mu^k(x) \to p_\mu$.

Finally, $F_\mu\bigl((p_\mu, 1)\bigr) = (0, p_\mu)$, and we are done. □

Exercise 1.6.2. Describe the dynamics of F_3.

As noticed in Example 1.2.6, when μ crosses 3 two things happen: p_μ becomes repelling, and it appears an attracting cycle of period 2. This is the beginning of an interesting story that we shall (I hope!) describe later. Now instead we discuss what happen when $\mu > 4$, where a completely new phenomenon appears. Indeed, when $\mu > 4$ the maximum of F_μ is greater than 1; as a consequence, there are orbits starting in $[0,1]$ that escapes to infinity. On the other hand, there are orbits never leaving I, e.g., periodic orbits. This means that the study of the dynamics of F_μ for $\mu > 4$ boils down to answering two main questions:

(A) What is the (topological) structure of the set $\Lambda \subset I$ of points with bounded orbit?
(B) What is the dynamics of F_μ restricted to Λ?

We shall answer question (A) in this section, deferring the answer to question (B) to the next section.

The set of points $x \in I$ whose image via F_μ is outside I is an open interval $A_0 \subset I$ centered at $1/2$; the complement $I \setminus A_0$ is the union of two closed intervals I_0 and I_1, where I_0 contains 0 and I_1 contains 1. It is easy to see that $F_\mu(I_0) = F_\mu(I_1) = I$ and that F_μ is increasing in I_0 and decreasing in I_1.

Now let

$$A_1 = F_\mu^{-1}(A_0) = \{x \in I \mid F_\mu(x) \in I, F_\mu^2(x) \notin I\};$$

this is the union of two open intervals, one contained in I_0 and the second in I_1. So $I \setminus (A_0 \cup A_1)$ is the union of four closed intervals that we shall call (going from left to right) I_{00}, I_{01}, I_{11} and I_{10}. The names are chosen so that

$$I_{s_0 s_1} \subset I_{s_0} \quad \text{and} \quad F_\mu(I_{s_0 s_1}) = I_{s_1}$$

for all s_0, $s_1 = 0, 1$. In particular, $F_\mu^2(I_{s_0 s_1}) = I$, and F_μ^2 is increasing on I_{00} and I_{11}, while it is decreasing on I_{01} and I_{10}.

Let us then now define by induction

$$A_k = F_\mu^{-1}(A_{k-1}) = \{x \in I \mid F_\mu(x), \ldots, F_\mu^k(x) \in I, F_\mu^{k+1}(x) \notin I\}.$$

Then it is easy to check (do it!) that $I \setminus (A_0 \cup \cdots \cup A_k)$ is the union of 2^{k+1} disjoint closed intervals that we can label $I_{0\ldots 0}, \ldots, I_{1\ldots 1}$ in such a way that

$$I_{s_0 \ldots s_k} \subset I_{s_0} \quad \text{and} \quad F_\mu(I_{s_0 \ldots s_k}) = I_{s_1 \ldots s_k}$$

for all $s_0, \ldots, s_k = 0, 1$. In particular, it is easy to see by induction that $I_{s_0 \ldots s_k} \subset I_{s_0 \ldots s_{k-1}}$; indeed, this is true for $k = 1$, and

$$I_{s_0 \ldots s_k} = I_{s_0} \cap F_\mu^{-1}(I_{s_1 \ldots s_k}) \subset I_0 \cap F_\mu^{-1}(I_{s_1 \ldots s_{k-1}}) = I_{s_0 \ldots s_{k-1}}. \tag{1.6.1}$$

Now let $x \in I$ be such that its orbit is not contained in I. This means that x must belong to some A_k, and then its orbit diverges to infinity. Hence the set Λ of points with bounded orbits coincides with the complementary of the A_k, that is

$$\Lambda = I \setminus \bigcup_{k=0}^{\infty} A_k = \bigcap_{k=0}^{\infty} I \setminus (A_0 \cup \cdots \cup A_k).$$

We are now able to describe the topological structure of Λ:

Theorem 1.6.3: *Let $\mu > 4$. Then Λ is compact, totally disconnected and without isolated points.*

Proof: Since Λ is closed in I, it is clearly compact. Let us prove that it has no isolated points. First of all, notice that the end points of each interval in an A_k belongs to Λ, because their orbit ends in 0 after $k+2$ iterations. If $p \in \Lambda$ is isolated, there must exist a neighbourhood U of p in I such that $U \setminus \{p\} \subseteq \bigcup_k A_k$. Since the end points of the intervals in each A_k belong to Λ, they cannot accumulate in p; therefore the only possibility is that p is the right end point of an interval in some A_{k_1}, and the left end point of an interval in some A_{k_2}. But then, up to shrink U, we can find $k_0 > 0$ such that $F_\mu^{k_0}(p) = 0$ and $F_\mu^{k_0}(U \setminus \{p\}) \subset \mathbb{R}^-$.

So $F_\mu^{k_0}$ has a local maximum in p, which implies $(F_\mu^{k_0})'(p) = 0$. Hence $F'_\mu\bigl(F_\mu^j(p)\bigr) = 0$ for some $0 \le j < k_0$, which forces $F_\mu^j(p) = 1/2$. Therefore $F_\mu^{j+1}(p) \notin I$ and $p \notin \Lambda$, contradiction.

To prove that Λ is totally disconnected, we assume that $\mu > 2 + \sqrt{5}$ (see Remark 1.6.1 for the general case). Under this assumption, we have (check!) $|F'_\mu(x)| > 1$ for all $x \in I_0 \cup I_1$. Therefore we can find $c > 1$ such that $|F'_\mu(x)| \ge c$ for all $x \in \Lambda$, and hence

$$\forall x \in \Lambda, \forall k \in \mathbb{N} \qquad |(F_\mu^k)'(x)| \ge c^k. \tag{1.6.2}$$

Let us assume, by contradiction, that Λ contains an interval $[x, y]$. Choose $k \in \mathbb{N}$ so that $c^k|x - y| > 1$. But then Lagrange's theorem yields

$$|F_\mu^k(x) - F_\mu^k(y)| = |(F_\mu^k)'(\xi)||x - y| \ge c^k|x - y| > 1$$

for a suitable $\xi \in [x, y] \subset \Lambda$. But this means that either $F_\mu^k(x)$ or $F_\mu^k(y)$ does not belong to I, contradiction. □

Remark 1.6.1. In the previous proof, we needed to assume $\mu > 2 + \sqrt{5}$ only to prove (1.6.2). Clearly, the same proof would also work under the slightly weaker assumption that there exist $K > 0$ and $c > 1$ so that

$$\forall x \in \Lambda, \forall k \in \mathbb{N} \qquad |(F_\mu^k)'(x)| \ge Kc^k. \tag{1.6.3}$$

In Section 1.8 we shall describe how to prove this for all $\mu > 4$, thus completing the proof of Theorem 1.6.3.

Definition 1.6.1: A *Cantor set* is a compact, totally disconnected topological space without isolated points.

Remark 1.6.2. It can be proved that all Cantor sets are homeomorphic (and metrizable, locally compact and with a countable base of open sets). Furthermore, their cardinality is always uncountable. Finally, the classical one-third Cantor set is a Cantor set according to this definition.

So Theorem 1.6.3 can be expressed saying that *the set* Λ *is a Cantor set;* and this answers completely question (A). Notices that in particular Λ is uncountable. This means that besides the periodic orbits (that are countable) and the pre-periodic orbits (that are still countable) it contains infinitely many non-trivial bounded orbits.

To fully understand the dynamics on Λ, answering question (B), we need a new tool.

1.7 Symbolic dynamics

To describe the behavior of a dynamical system one needs suitable models. A good source of models is provided by sequence spaces and shift mappings.

Definition 1.7.1: Given $N \ge 2$, set $\mathbb{Z}_N = \{0, \ldots, N-1\}$. The *sequence space* Ω_N^+ on N symbols is

$$\Omega_N^+ = (\mathbb{Z}_N)^{\mathbb{N}} = \{\mathbf{s} = (s_0 s_1 \ldots) \mid s_j \in \mathbb{Z}_N \text{ for all } j \in \mathbb{N}\}.$$

Analogously, the *two-sided sequence space* Ω_N on N symbols is given by

$$\Omega_N = (\mathbb{Z}_N)^{\mathbb{Z}} = \{\mathbf{s} = (\ldots s_{-1} s_0 s_1 \ldots) \mid s_j \in \mathbb{Z}_N \text{ for all } j \in \mathbb{Z}\}.$$

Remark 1.7.1. To simplify the exposition, in the sequel we shall sometimes think of Ω_N^+ as a subset of Ω_N, identifying a one-sided sequence $(s_0 s_1 \ldots)$ with the two-sided sequence $(\ldots 00 s_0 s_1 \ldots)$.

It is not difficult to define a distance on Ω_N^+ and Ω_N:

Lemma 1.7.1: *Let* $d\colon \Omega_N \times \Omega_N \to \mathbb{R}^+$ *be given by*

$$d(\mathbf{s}, \mathbf{t}) = \sum_{j=-\infty}^{+\infty} \frac{|s_j - t_j|}{N^{|j|}}.$$

Then d *is a distance on* Ω_N *such that* $\operatorname{diam}(\Omega_N) \le N + 1$ *and* $\operatorname{diam}(\Omega_N^+) \le N$.

Proof: The fact that d is a distance is an easy exercise. Furthermore,

$$d(\mathbf{s}, \mathbf{t}) \le (N-1) \sum_{j=-\infty}^{+\infty} \frac{1}{N^{|j|}} = (N-1)\left(2\frac{1}{1 - \frac{1}{N}} - 1\right) = N + 1$$

for all $\mathbf{s}$, $\mathbf{t} \in \Omega_N$. A similar estimate yields the assertion for Ω_N^+. □

Remark 1.7.2. In some cases it might be useful to consider the distance

$$d_\lambda(\mathbf{s},\mathbf{t}) = \sum_{j=-\infty}^{+\infty} \frac{|s_j - t_j|}{\lambda^{|j|}},$$

where $\lambda > 1$.

Exercise 1.7.1. Prove the analogous of Lemma 1.7.1 for d_λ, and prove that two such distances always induce the same topology.

It is also easy to see when two sequence are close with respect to this distance:

Lemma 1.7.2: (i) *Let $\mathbf{s}, \mathbf{t} \in \Omega_N^+$. If $s_j = t_j$ for $0 \le j \le k$ then $d(\mathbf{s},\mathbf{t}) \le 1/N^k$. Conversely, if $d(\mathbf{s},\mathbf{t}) < 1/N^k$ then $s_j = t_j$ for $0 \le j \le k$.*
(ii) *Let $\mathbf{s}, \mathbf{t} \in \Omega_N$. If $s_j = t_j$ for $0 \le |j| \le k$ then $d(\mathbf{s},\mathbf{t}) \le 2/N^k$. Conversely, if $d(\mathbf{s},\mathbf{t}) < 1/N^k$ then $s_j = t_j$ for $0 \le |j| \le k$.*

Proof: (i) If $s_j = t_j$ for $0 \le j \le k$ we have

$$d(\mathbf{s},\mathbf{t}) = \sum_{j=k+1}^{\infty} \frac{|s_j - t_j|}{N^j} = \frac{1}{N^{k+1}} \sum_{j=0}^{\infty} \frac{|s_{j+k+1} - t_{j+k+1}|}{N^j} \le \frac{1}{N^k}.$$

Conversely, if $s_j \ne t_j$ for some $0 \le j \le k$ we clearly have $d(\mathbf{s},\mathbf{t}) \ge 1/N^j \ge 1/N^k$.
(ii) If $s_j = t_j$ for $0 \le |j| \le k$ we have

$$\begin{aligned} d(\mathbf{s},\mathbf{t}) &= \sum_{j=k+1}^{\infty} \frac{|s_j - t_j|}{N^j} + \sum_{j=-\infty}^{-k-1} \frac{|s_j - t_j|}{N^{|j|}} = \frac{1}{N^{k+1}} \left[\sum_{j=0}^{\infty} \frac{|s_{j+k+1} - t_{j+k+1}|}{N^j} + \sum_{j=-\infty}^{0} \frac{|s_{j-k-1} - t_{j-k-1}|}{N^{|j|}} \right] \\ &\le \frac{2}{N^k}. \end{aligned}$$

Conversely, if $s_j \ne t_j$ for some $0 \le |j| \le k$ we clearly have $d(\mathbf{s},\mathbf{t}) \ge 1/N^{|j|} \ge 1/N^k$. □

As shown by the latter lemma, it is not easy to explicitely describe the balls for this distance. Luckily, we can use another basis for the topology:

Definition 1.7.2: A *cylinder* (of *rank* 1) in Ω_N^+ (or Ω_N) is a set of the form

$$C_a^m = \{\mathbf{s} \in \Omega_N^+ \mid s_m = a\},$$

where $m \in \mathbb{N}$ (or $j \in \mathbb{Z}$) and $a \in \mathbb{Z}_N$. More generally, a *cylinder of rank* $r \ge 1$ is the intersection of r cylinders of rank 1:

$$C_{a_1 \ldots a_r}^{m_1 \ldots m_r} = C_{a_1}^{m_1} \cap \cdots \cap C_{a_r}^{m_r} = \{\mathbf{s} \in \Omega_N^+ \mid s_{m_h} = a_h \text{ for } h = 1, \ldots, r\}.$$

Lemma 1.7.3: *The cylinders are open and closed subsets of Ω_N^+ and Ω_N. In particular, the topology induced by the distance d coincides with the product tpology on $(\mathbb{Z}_N)^{\mathbb{N}}$ and $(\mathbb{Z}_N)^{\mathbb{Z}}$, where we have endowed $\mathbb{Z}_N$ with the discrete topology.*

Proof: Take $\mathbf{s} \in C_a^m$. If $\mathbf{t} \in \Omega_N^+$ is such that $d(\mathbf{s},\mathbf{t}) < 1/N^m$ we necessarily have $t_m = s_m = a$, and so $\mathbf{t} \in C_a^m$. In other words, we have proved that $B(\mathbf{s}, 1/N^m) \subset C_a^m$, and hence C_a^m is open. It is also closed, because

$$\Omega_N^+ \setminus C_a^m = \bigcup_{\substack{b \in \mathbb{Z}_N \\ b \ne a}} C_b^m$$

is open. As a consequence, all the cylinders are open and closed, being finite intersections of open and closed sets.

Now, the cylinders are exactly a basis of the product topology; so to prove that the product topology coincides with the distance topology we are left to proving that every ball for the distance can be written as union of cylinders. But indeed, given $\mathbf{t} \in B(\mathbf{s}, \varepsilon)$, choose $r > 0$ so that $1/N^r < \varepsilon - d(\mathbf{s}, \mathbf{t})$. Then if $\mathbf{r} \in C^{0\ldots r}_{t_0\ldots t_r}$ we have

$$d(\mathbf{s}, \mathbf{r}) \le d(\mathbf{s}, \mathbf{t}) + d(\mathbf{t}, \mathbf{r}) \le d(\mathbf{s}, \mathbf{t}) + \frac{1}{N^r} < \varepsilon;$$

therefore $\mathbf{t} \in C^{0\ldots r}_{t_0\ldots t_r} \subset B(\mathbf{s}, \varepsilon)$,as desired.

A completely analogous argument works for Ω_N, and we are done. □

We are now able to explicitely describe the topology on the sequence spaces:

Proposition 1.7.4: *Ω_N^+ and Ω_N are Cantor sets.*

Proof: Since it is easy to prove (exercise) that Ω_N and Ω_N^+ are homeomorphic, we shall work with the latter.

To prove that Ω_N^+ is compact, instead of quoting Tychonoff's theorem on the product of compact spaces we shall directly prove that Ω_N^+ is sequentially compact. Let $\{\mathbf{s}^{(n)}\}$ be a sequence in Ω_N^+. Since $s_0^{(n)}$ runs in a finite set, we can extract a subsequence $\{\mathbf{s}^{(0,n)}\}$ such that $s_0^{(0,n)}$ is constant, say equal to $s_0^\infty \in \mathbb{Z}_N$. For the same reason, we can extract now a sub-subsequence $\{\mathbf{s}^{(1,n)}\}$ such that $s_0^{(1,n)} \equiv s_0^\infty$ and $s_1^{(1,n)} \equiv s_1^\infty$ for a suitable $s_1^\infty \in \mathbb{Z}_N$. Proceeding in this way, by induction we build a sequence $\{\mathbf{s}^{(k,n)}\}$ of nested subsequences such that $s_0^{(k,n)} \equiv s_0^\infty, \ldots, s_k^{(k,n)} \equiv s_k^\infty$. But then it is easy to check that the diagonal subsequence $\{\mathbf{s}^{(n,n)}\}$ converges to the sequence $\mathbf{s}^\infty = (s_0^\infty s_1^\infty \ldots)$.

If $\mathbf{s} \in \Omega_N^+$ then the sequence $\{\mathbf{s}^{(n)}\}$ defined by

$$s_j^{(n)} = \begin{cases} s_j & \text{if } j \neq n, \\ s_j + 1 \pmod N & \text{if } j = n, \end{cases}$$

is composed by distinct elements and converges to $\mathbf{s}$; therefore Ω_N^+ has no isolated points.

Finally, Ω_N^+ is totally disconnected: if $\mathbf{s} \neq \mathbf{t}$ take $m \in \mathbb{N}$ so that $s_m \neq t_m$. Then $U = C^m_{s_m}$ and $V = \cup_{a \neq s_m} C^m_a$ are two disjoint open sets such that $\mathbf{s} \in U$, $\mathbf{t} \in V$ and $\Omega_N^+ = U \cup V$. □

Now we we know our space, we can define our model dynamical system.

Definition 1.7.3: The *full left shift* $\sigma_N \colon \Omega_N \to \Omega_N$ is defined by $\sigma_N(\mathbf{s}) = \mathbf{s}'$, where $s'_n = s_{n+1}$. It sends cylinders onto cylinders, it is invertible and thus it is a homeomorphism. We analogously define the *left shift* $\sigma_N \colon \Omega_N^+ \to \Omega_N^+$ by setting

$$\sigma_N(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots).$$

It is continuous, open and surjective, but it is not invertible: every point has exactly N preimages.

Proposition 1.7.5: *The dynamical systems (Ω_N, σ_N) and (Ω_N^+, σ_N) are both chaotic. Furthermore, σ_N is topologically mixing on both spaces, and has N^k periodic points of period k for all $k \ge 1$.*

Proof: We shall discuss the case of Ω_N^+; similar arguments work in Ω_N too.

A periodic point of period k for σ_N is a periodic sequence of period k; since such a sequence is uniquely determined by its beginning segment $(s_0 \ldots s_{k-1})$, we have exactly N^k periodic points of period k.

To prove that σ_N is chaotic, since every open set contains a cylinder, it suffices to show that every pair of cylinders shares a periodic orbit, by Proposition 1.5.3. Since every cylinder contains a cylinder of the form $C^{0\ldots r}_{a_0\ldots a_r}$, it suffices to prove that two cylinders $C^{0\ldots r}_{a_0\ldots a_r}$ and $C^{0\ldots r'}_{b_0\ldots b_{r'}}$ share a periodic orbit. Define $\mathbf{s} \in \Omega_N^+$ by setting

$$s_j = \begin{cases} a_j & \text{for } 0 \le j \le r, \\ b_{j-r-1} & \text{for } j = r+1, \ldots, r+r'+1, \end{cases}$$

and then repeating the sequence. Then $\mathbf{s}$ is periodic, it belongs to $C^{0\ldots r}_{a_0\ldots a_r}$ and $\sigma_N^{r+1}(\mathbf{s}) \in C^{0\ldots r'}_{b_0\ldots b_{r'}}$.

We can prove analogously that σ_N is topologically mixing. Take $p > 0$, and consider any sequence $\mathbf{s}$ such that

$$s_j = \begin{cases} a_j & \text{for } 0 \le j \le r, \\ b_{j-r-1-p} & \text{for } j = r+p+1, \ldots, r+r'+p+1. \end{cases}$$

Then $\mathbf{s} \in C^{0\ldots r}_{a_0\ldots a_r}$ and $\sigma_N^{r+1+p}(\mathbf{s}) \in C^{0\ldots r'}_{b_0\ldots b_{r'}}$. Therefore $\sigma_N^k(C^{0\ldots r}_{a_0\ldots a_r}) \cap C^{0\ldots r'}_{b_0\ldots b_{r'}} \neq \varnothing$ for any $k > r+1$. □

Remark 1.7.3. A point $\mathbf{s} \in \Omega_N^+$ with a dense orbit is obtaining starting with the sequence $0 \ldots N-1$, the adding all the possible sequences of two symbols, and then all the possible sequences of three symbols, and so on. It is also easy to see directly that the periodic points are dense. Given $\mathbf{s} \in \Omega_N^+$ and $n \geq 0$, let $\mathbf{s}^{(n)}$ be defined repeating the initial segment $(s_0 \ldots s_n)$ of $\mathbf{s}$. Then each $\mathbf{s}^{(n)}$ is periodic and $\mathbf{s}^{(n)} \to \mathbf{s}$ as $n \to +\infty$.

The idea is that the dynamical system (Ω_2^+, σ_2) is, in a suitable sense, equivalent to the dynamical system $(\Lambda, F_\mu|_\Lambda)$, where F_μ is the quadratic map (with $\mu > 4$) studied in the previous section.

As usual, as soon as a new class of objects is introduced one needs a notion of equivalence stating when two such objects are to be considered identical. In dynamics we have several notions of equivalence, mostly depending on the differentiability of the maps under consideration.

Definition 1.7.4: Two dynamical systems (X, f) and (Y, g) on topological spaces are (*topologically*) *conjugated* (or *equivalent*) if there is a homeomorphism $\varphi: X \to Y$, the *conjugacy*, such that

$$\varphi \circ f = g \circ \varphi, \tag{1.7.1}$$

so that $f = \varphi^{-1} \circ g \circ \varphi$. More generally, if M and N are manifolds, two C^r maps $f: M \to M$ and $g: N \to N$ are C^m *equivalent* or C^m *conjugate* (with $0 \leq m \leq r \leq \infty$) if there is a C^m diffeomorphism $\varphi: M \to N$, the *conjugacy*, such that $\varphi \circ f = g \circ \varphi$. If M and N are complex manifolds, there are similar definitions in the holomorphic category.

The whole point of (1.7.1) is that it implies

$$\varphi \circ f^k = g^k \circ \varphi \tag{1.7.2}$$

for all $k \in \mathbb{N}$, and hence

$$\forall k \in \mathbb{N} \qquad f^k = \varphi^{-1} \circ g^k \circ \varphi;$$

this means that all the dynamical properties of f can be read in g, via the map φ.

Actually, to get (1.7.2) from (1.7.1) it is not necessary that φ be invertible, and thus it is sometimes useful the following

Definition 1.7.5: Given two dynamical systems (X, f) and (Y, g), if there is a *surjective* map $\varphi: X \to Y$ such that $\varphi \circ f = g \circ \varphi$, we shall say that g is *semiconjugate* to (or a *factor* of) f, and that φ is a *semiconjugation*.

Exercise 1.7.2. Let $\varphi: X \to Y$ be a semiconjugation between (X, f) and (Y, g). Prove that:

(i) $\varphi\bigl(\mathrm{Per}(f)\bigr) \subseteq \mathrm{Per}(g)$;
(ii) if the periodic points of f are dense in X then the periodic points of g are dense in Y;
(iii) if f is topologically transitive then also g is;
(iv) if f is chaotic then also g is.

Exercise 1.7.3. Let $f: S^1 \to S^1$ be given by $f(x) = 2x \pmod 1$, and $g: [-1, 1] \to [-1, 1]$ given by $g(x) = 2x^2 - 1$. Prove that g is semiconjugated to f, and deduce that g is chaotic on $[-1.1]$.

We can now state in a rigourous way the equivalence of $(\Lambda, F_\mu|\Lambda)$ and (Ω_2^+, σ_2):

Theorem 1.7.6: *Let $\mu > 4$. Then $(\Lambda, F_\mu|\Lambda)$ and (Ω_2^+, σ_2) are topologically conjugated.*

Proof: We must define a homeomorphism $S: \Lambda \to \Omega_2^+$ such that $S \circ F_\mu = \sigma_2 \circ S$. We shall use all the notations introduced in the previous section.

For any $x \in \Lambda$, we necessarily have $F_\mu^j(x) \in I_0 \cup I_1$ for all $j \in \mathbb{N}$; we then define $S(x)$ by setting $S(x)_j = s$ if and only if $F_\mu^j(x) \in I_s$.

Let us first show that S is bijective. Given $\mathbf{s} \in \Omega_2^+$, put

$$I_{\mathbf{s}} = \bigcap_{n=0}^{\infty} I_{s_0 \ldots s_n}.$$

Thanks to (1.6.1), $I_{\mathbf{s}}$ is a decreasing intersection of closed intervals; hence it is a not empty closed interval. Now, by construction we have

$$I_{s_0\ldots s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap \cdots \cap F_\mu^{-n}(I_{s_n});$$

therefore $S(x) = \mathbf{s}$ if and only if $x \in I_{\mathbf{s}}$, and in particular $I_{\mathbf{s}} \subset \Lambda$. But Λ is totally disconnected; hence $I_{\mathbf{s}}$ must be a single point, and hence S is bijective.

Now we show that S is continuous. Choose $x \in \Lambda$ and $\varepsilon > 0$, set $S(x) = \mathbf{s}$, and choose $n \geq 0$ so that $1/2^n < \varepsilon$. Now, Λ is contained in the finite union of intervals of the form $I_{t_0\ldots t_n}$; let

$$\delta = \operatorname{dist}\left(I_{s_0\ldots s_n}, \bigcup_{(t_0\ldots t_n)\neq(s_0\ldots s_n)} I_{t_0\ldots t_n} \right) > 0.$$

Then $y \in \Lambda$ and $|x-y| < \delta$ implies $y \in I_{s_0\ldots s_n}$, and hence $d\big(S(x), S(y)\big) \leq 1/2^n < \varepsilon$, and S is continuous. Since Λ is compact and Ω_2^+ is Hausdorff, it follows that S is a homeomorphism.

Finally, let us show that S conjugates F_μ with the shift. But indeed $S(x) = \mathbf{s}$ yields

$$F_\mu^{j-1}\big(F_\mu(x)\big) = F_\mu^j(x) \in I_{s_j},$$

and hence $S\big(F_\mu(x)\big)_j = s_{j+1}$, which means exactly that $S \circ F_\mu = \sigma_2 \circ S$. □

Remark 1.7.4. Notice that to build the conjugation S we have partitioned Λ in two sets and then tracked the *itinerary* of an orbit in the two sets. This is the standard way for building conjugations (or, at least, semiconjugatinos) between a complicated dynamical system and a symbolic dynamical system.

Corollary 1.7.7: *Let $\mu > 4$. Then $(\Lambda, F_\mu|\Lambda)$ is chaotic, topologically mixing, and it has 2^k periodic points of period $k \in \mathbb{N}$.*

Proof: It follows from the previous Theorem and from Proposition 1.7.5. □

Exercise 1.7.4. Let $\Omega = \Omega_2^+/\sim$, where $\sim$ is the equivalence relation given by

$$(s_0 \ldots s_{n-1}0100\ldots) \sim (s_0 \ldots s_{n-1}1100\ldots);$$

since $\mathbf{s} \sim \mathbf{t}$ implies $\sigma_2(\mathbf{s}) \sim \sigma_2(\mathbf{t})$, the shift σ_2 induces a well-defined continuous map $\sigma\colon \Omega \to \Omega$.

(i) Prove that (Ω, σ) is chaotic.

(ii) Let $T\colon [0,1] \to [0,1]$ be the *tent map* given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2-2x & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Prove that T is topologically conjugated to σ.

(iii) Prove that F_4 is topologically conjugated to T, and hence it is chaotic on $[0,1]$.

We end this section with a few more definitions and exercises.

Definition 1.7.6: The restriction of the shift σ_N to a closed shift-invariant subset of Ω_N^+ (or Ω_N) is called a *symbolic dynamical system.*

Definition 1.7.7: A *binary matrix* is a matrix $A \in M_{N,N}(\mathbb{Z}_2)$, with row and column indeces running from 0 to $N-1$. Given a binary matrix A, we set

$$\Omega_A = \{\mathbf{s} \in \Omega_N \mid a_{s_j, s_{j+1}} = 1 \text{ for all } j \in \mathbb{Z}\}.$$

Clearly Ω_A is closed and shift-invariant; the corresponding symbolic dynamical system is called a *topological Markov chain.* Sometimes, $\sigma_A = \sigma_N|_{\Omega_A}$ is said a *subshift of finite type.* Analogous definitions hold in Ω_N^+.

Exercise 1.7.5. Let $A = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$. Prove that (Ω_A^+, σ_2) is chaotic. Furthermore, shows that if p_k denotes the number of periodic points of period k then $p_1 = 1$, $p_2 = 3$ and $p_k = p_{k-1} + p_{k-2}$ for $k \geq 3$.

Definition 1.7.8: Let σ_A be a topological Markov chain. The *associated graph* G_A has $\{0, 1, \ldots, N-1\}$ as sets of vertices, and a pair (i, j) is a directed edge if and only if $a_{ij} = 1$. A sequence of vertices of G_A is *admissible* if any two consecutive vertices in the sequence are connected by a directed edge.

The following exercises contain further informations on topological Markov chains.

Exercise 1.7.6. Let A be a binary matrix. Prove that for every $i,\ j \in \{0, \ldots, N-1\}$ the number of admissible sequences in G_A of length $m+1$ beginning at i and ending at j is equal to the entry a_{ij}^m of the matrix A^m.

Exercise 1.7.7. Let A be a binary matrix. Prove that the number of periodic points of period k for σ_A is given by $\mathrm{tr}(A^k)$.

Definition 1.7.9: A binary matrix A and the corresponding topological Markov chain σ_A are called *transitive* if for some $m > 0$ all the entries of A^m are positive.

Exercise 1.7.8. Let A be a binary matrix. Prove that if for some $n > 0$ all the entries of A^n are positive then this is true for A^m for any $m \geq n$.

Exercise 1.7.9. Let A be a transitive binary matrix. Prove that if $\alpha = (\alpha_{-r}, \ldots, \alpha_r)$ is an admissible sequence then the intersection $\Omega_A \cap C^{-r \ldots r}_{\alpha_{-r} \ldots \alpha_r}$ contains a periodic point.

Exercise 1.7.10. Prove that any transitive topological Markov chain is chaotic and topologically mixing.

In the following exercises A is a binary matrix with at least one 1 in every row and column.

Exercise 1.7.11. Prove that for all $j \in \{0, \ldots, N-1\}$ the set $\Omega_{A,j} = \{\mathbf{s} \in \Omega_A \mid s_0 = j\}$ is not empty.

Exercise 1.7.12. Prove that if there is $\mathbf{s} \in \Omega_A$ containing the symbol j at least twice then there is a periodic element $\mathbf{s}' \in \Omega_A$ such that $s'_0 = j$.

Exercise 1.7.13. Let us call *essential* the symbols j satisfying the condition of the previous exercise. Prove that if $\mathbf{s} \in \Omega_A$ is in the closure of a forward orbit of σ_A then $\mathbf{s}$ contains only essential symbols.

Exercise 1.7.14. We shall say that two essential symbols i and j are *equivalent* if there are $\mathbf{s}$, $\mathbf{s}' \in \Omega_A$, $k_1 < k_2$ and $l_1 < l_2$ such that $s_{k_1} = s'_{l_2} = i$ and $s_{k_2} = s'_{l_1} = j$. Prove that this is an equivalence relation on the set of essential symbols.

Exercise 1.7.15. Prove that σ_A is topologically transitive if and only if all symbols are essential and equivalent.

Exercise 1.7.16. Assuming that σ_A is topologically transitive, prove that there exist a positive integer M and a partition of Ω_A into closed disjoint subsets $\Lambda_1, \ldots, \Lambda_M = \Lambda_0$ so that $\sigma_A(\Lambda_j) = \Lambda_{j+1}$ for $j = 0, \ldots, M-1$ and the restriction of $(\sigma_A)^M$ to each Λ_j is topologically mixing. Furthermore this corresponds to a decomposition of the set $\{0, \ldots, N-1\}$ into M equal groups such that every $\omega \in \Omega_A$ has only symbols from one group in positions equal modulo M.

1.8 The Schwarzian derivative

In this section we shall introduce a tool very useful in one-dimensional dynamics, and we shall describe how to use it to complete the proof of Theorem 1.6.3.

Definition 1.8.1: Let $f: I \to \mathbb{R}$ be of class C^3, where $I \subseteq \mathbb{R}$ is an interval, and denote by $\mathrm{Crit}(f)$ the set of critical points of f. The *Schwarzian derivative* of f is the function $Sf: I \setminus \mathrm{Crit}(f) \to \mathbb{R}$ given by

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

If $x_0 \in \mathrm{Crit}(f)$ we shall also put

$$Sf(x_0) = \lim_{x \to x_0} Sf(x) \in \mathbb{R} \cup \{\pm\infty\}$$

when the limit exists (finite or infinite).

A way to understand the meaning of the Schwarzian derivative is to see which functions have vanishing Schwarzian derivative:

Proposition 1.8.1: *Let $f: I \to \mathbb{R}$ be of class C^3. Then $Sf \equiv 0$ if and only if there are a, b, c, $d \in \mathbb{R}$ not all vanishing such that*

$$f(x) = \frac{ax+b}{cx+d}.$$

The Schwarzian derivative behaves very well under composition:

Proposition 1.8.2: *Let $f: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$ two functions of class C^3 with $g(J) \subseteq I$. Then*

$$S(f \circ g) = \big((Sf) \circ g\big)(g')^2 + Sg.$$

In particular, if Sf, $Sg < 0$ then $S(f \circ g) < 0$.

Example 1.8.1. It is easy to check that

$$SF_\mu(x) = -\frac{6}{(1-2x)^2},$$

and hence $S(F_\mu)^k < 0$ for all $\mu \in \mathbb{R}^*$ and $k \geq 1$. Notice that $SF_\mu(1/2) = -\infty$.

Exercise 1.8.1. Prove that if $P \in \mathbb{R}[x]$ is a polynomial such that all the roots of P' are real and distinct then $SP < 0$.

The first main result on the dynamics of functions with negative Schwarzian derivative is:

Theorem 1.8.3: *Let $f: I \to I$ be a function of class C^3 with n critical points, and assume that $Sf < 0$ (Sf may assume the value $-\infty$ in some critical point). Then f has at most $n+2$ non-repelling periodic points. Furthermore, every periodic cycle (except at most two) must attract a critical point.*

The proof depends on the following lemmata:

Lemma 1.8.4: *Let $f: I \to I$ be a function of class C^3 with $Sf < 0$. Then f' cannot have either a positive local minimum or a negative local maximum.*

Lemma 1.8.5: *Let $f: I \to I$ be a function of class C^3 with $Sf < 0$. Then between two isolated critical points of f' there always is a critical point of f.*

Lemma 1.8.6: *Let $f: I \to I$ be a function of class C^3 with a finite numebr of critical points. Then Crit(f^k) is a finite set for all $k \in \mathbb{N}$.*

Lemma 1.8.7: *Let $f: I \to I$ be a function of class C^3 with a finite number of critical points, and such that $Sf < 0$. Then the number of periodic points of period k is finite for any $k \geq 1$.*

Theorem 1.8.3 applies to the quadratic family, of course. But checking the proof one gets something slightly better:

Corollary 1.8.8: *For any $\mu \in \mathbb{R}^*$ the function F_μ has at most one non-repelling periodic cycle, and it attracts the orbit of $1/2$.*

Another very important property of functions with negative Schwarzian derivative is the following

Proposition 1.8.9: (Minimum principle) *Let $f: [a,b] \to \mathbb{R}$ be a function of class C^3 without critical points and such that $Sf < 0$. Then*

$$\forall x \in (a,b) \qquad |f'(x)| > \min\{|f'(a)|, |f'(b)|\}.$$

We have almost all we need to complete the proof of Theorem 1.6.3.

Definition 1.8.2: Let $f: I \to I$ be of class C^1, where $I \subseteq \mathbb{R}$ is an interval. A subset $\Lambda \subseteq I$ is a *hyperbolic repeller* if it is compact, f-invariant, and there are $K > 0$ and $c > 1$ such that

$$\forall x \in \Lambda, \forall k \in \mathbb{N} \qquad |(f^k)'(x)| \geq Kc^k.$$

So to conclude the proof of Theorem 1.6.3 it suffices to show that the set Λ is a hyperbolic repeller for any $\mu > 4$. A necessary and sufficient condition for a subset to be a hyperbolic repeller is the following:

Proposition 1.8.10: *Let $f: I \to I$ be of class C^1. An f-invariant compact subset $\Lambda \subseteq I$ is a hyperbolic repeller if and only if for every $x \in \Lambda$ there is $k = k(x) \in \mathbb{N}$ such that $|(f^k)'(x)| > 1$.*

The final ingredient is the following deep theorem:

Theorem 1.8.11: (Misiurewicz) *Let $f: I \to I$ be of class C^3 with a finite number of critical points and such that $Sf < 0$, where $I \subset \mathbb{R}$ is a closed interval. Then a compact f-invariant set $\Lambda \subseteq I$ is a hyperbolic repeller if it does not contain either critical points or non-repelling periodic points.*

Corollary 1.8.12: *Let $\mu > 4$. Then the set Λ of points with bounded orbit is a hyperbolic repeller for F_μ.*

Proof: Indeed, Corollary 1.8.8 shows that F_μ has no non-repelling periodic points, and we know that the unique critical point $1/2$ does not belong to Λ. Therefore we can apply Misiurewicz's Theorem 1.8.11. □

Hence the proof of Theorem 1.6.3 holds for all $\mu > 4$, as claimed.