Discrete local holomorphic dynamics

Informal notes

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1. Introduction

Let M be a complex manifold, and $p \in M$. In this survey, a *(discrete) holomorphic local dynamical system* at p will be a holomorphic map $f: U \to M$ such that $f(p) = p$, where $U \subseteq M$ is an open neighbourhood of p; we shall also assume that $f \not\equiv \mathrm{id}_U$. We shall denote by $\mathrm{End}(M, p)$ the set of holomorphic local dynamical systems at p.

Remark 1.1: Since we are mainly concerned with the behavior of f nearby p, we shall sometimes replace f by its restriction to some suitable open neighbourhood of p. It is possible to formalize this fact by using germs of maps and germs of sets at p, but for our purposes it will be enough to use a somewhat less formal approach.

Remark 1.2: In this survey we shall never have the occasion of discussing continuous holomorphic dynamical systems (i.e., holomorphic foliations). So from now on all dynamical systems in this paper will be discrete, except where explicitely noted otherwise.

To talk about the dynamics of an $f \in \mathrm{End}(M, p)$ we need to define the iterates of f. If f is defined on the set U, then the second iterate $f^2 = f \circ f$ is defined on $U \cap f^{-1}(U)$ only, which still is an open neighbourhood of p. More generally, the k-th iterate $f^k = f \circ f^{k-1}$ is defined on $U \cap f^{-1}(U) \cap \cdots \cap f^{-(k-1)}(U)$. Thus it is natural to introduce the *stable set* K_f of f by setting

$$K_f = \bigcap_{k=0}^{\infty} f^{-k}(U).$$

Clearly, $p \in K_f$, and so the stable set is never empty (but it can happen that $K_f = \{p\}$; see the next section for an example). The stable set of f is the set of all points $z \in U$ such that the *orbit* $\{f^k(z) \mid k \in \mathbb{N}\}$ is well-defined. If $z \in U \setminus K_f$, we shall say that z (or its orbit) *escapes* from U.

Thus the first natural question in local holomorphic dynamics is:

(Q1) *What is the topological structure of K_f?*

For instance, when does K_f have non-empty interior? As we shall see in section 4, holomorphic local dynamical systems such that p belongs to the interior of the stable set enjoy special properties; we shall then say that p is *stable* for $f \in \mathrm{End}(M, p)$ if it belongs to the interior of K_f.

Remark 1.3: Both the definition of stable set and Question 1 (as well as several other definitions or questions we shall meet later on) are topological in character; we might state them for local dynamical systems which are continuous only. As we shall see, however, the *answers* will strongly depend on the holomorphicity of the dynamical system.

Clearly, the stable set K_f is *completely f-invariant*, that is $f^{-1}(K_f) = K_f$ (this implies, in particular, that $f(K_f) \subseteq K_f$). Therefore the pair (K_f, f) is a discrete dynamical system in the usual sense, and so the second natural question in local holomorphic dynamics is

(Q2) *What is the dynamical structure of (K_f, f)?*

For instance, what is the asymptotic behavior of the orbits? Do they converge to p, or have they a chaotic behavior? Is there a dense orbit? Do there exist proper *f-invariant* subsets, that is sets $L \subset K_f$ such that $f(L) \subseteq L$? If they do exist, what is the dynamics on them?

To answer all these questions, the most efficient way is to replace f by a "dynamically equivalent" but simpler (e.g., linear) map g. In our context, "dynamically equivalent" means "locally conjugated"; and we have at least three kinds of conjugacy to consider.

Let $f_1\colon U_1 \to M_1$ and $f_2\colon U_2 \to M_2$ be two holomorphic local dynamical systems at $p_1 \in M_1$ and $p_2 \in M_2$ respectively. We shall say that f_1 and f_2 are *holomorphically* (respectively, *topologically*) *locally conjugated* if there are open neighbourhoods $W_1 \subseteq U_1$ of p_1, $W_2 \subseteq U_2$ of p_2, and a biholomorphism (respectively, a homeomorphism) $\varphi\colon W_1 \to W_2$ with $\varphi(p_1) = p_2$ such that

$$f_1 = \varphi^{-1} \circ f_2 \circ \varphi \qquad \text{on} \qquad \varphi^{-1}\bigl(W_2 \cap f_2^{-1}(W_2)\bigr) = W_1 \cap f_1^{-1}(W_1).$$

In particular we have

$$\forall k \in \mathbb{N} \qquad f_1^k = \varphi^{-1} \circ f_2^k \circ \varphi \quad \text{on} \quad \varphi^{-1}\bigl(W_2 \cap \cdots \cap f_2^{-(k-1)}(W_2)\bigr) = W_1 \cap \cdots \cap f_1^{-(k-1)}(W_1),$$

and thus $K_{f_2|_{W_2}} = \varphi(K_{f_1|_{W_1}})$. So the local dynamics of f_1 about p_1 is to all purposes equivalent to the local dynamics of f_2 about p_2.

Remark 1.4: Using local coordinates centered at $p \in M$ it is easy to show that any holomorphic local dynamical system at p is holomorphically locally conjugated to a holomorphic local dynamical system at $O \in \mathbb{C}^n$, where $n = \dim M$.

Whenever we have an equivalence relation in a class of objects, there are classification problems. So the third natural question in local holomorphic dynamics is

(Q3) *Find a (possibly small) class $\mathcal{F}$ of holomorphic local dynamical systems at $O \in \mathbb{C}^n$ such that every holomorphic local dynamical system f at a point in an n-dimensional complex manifold is holomorphically (respectively, topologically) locally conjugated to a (possibly) unique element of $\mathcal{F}$, called the* holomorphic *(respectively,* topological*)* normal form *of f.*

Unfortunately, the holomorphic classification is often too complicated to be practical; the family $\mathcal{F}$ of normal forms might be uncountable. A possible replacement is looking for invariants instead of normal forms:

(Q4) *Find a way to associate a (possibly small) class of (possibly computable) objects, called* invariants*, to any holomorphic local dynamical system f at $O \in \mathbb{C}^n$ so that two holomorphic local dynamical systems at O can be holomorphically conjugated only if they have the same invariants. The class of invariants is furthermore said* complete *if two holomorphic local dynamical systems at O are holomorphically conjugated if and only if they have the same invariants.*

As remarked before, up to now all the questions we asked make sense for topological local dynamical systems; the next one instead makes sense only for holomorphic local dynamical systems.

A holomorphic local dynamical system at $O \in \mathbb{C}^n$ is clearly given by an element of $\mathbb{C}_0\{z_1, \ldots, z_n\}^n$, the space of n-uples of converging power series in $z_1, \ldots, z_n$ without constant terms. The space $\mathbb{C}_0\{z_1, \ldots, z_n\}^n$ is a subspace of the space $\mathbb{C}_0[[z_1, \ldots, z_n]]^n$ of n-uples of formal power series without constant terms. An element $\Phi \in \mathbb{C}_0[[z_1, \ldots, z_n]]^n$ has an inverse (with respect to composition) still belonging to $\mathbb{C}_0[[z_1, \ldots, z_n]]^n$ if and only if its linear part is a linear automorphism of $\mathbb{C}^n$. We shall say that two holomorphic local dynamical systems f_1, $f_2 \in \mathbb{C}_0\{z_1, \ldots, z_n\}^n$ are *formally conjugated* if there exists an invertible $\Phi \in \mathbb{C}_0[[z_1, \ldots, z_n]]^n$ such that $f_1 = \Phi^{-1} \circ f_2 \circ \Phi$ in $\mathbb{C}_0[[z_1, \ldots, z_n]]^n$.

It is clear that two holomorphically locally conjugated holomorphic local dynamical systems are both formally and topologically locally conjugated too. On the other hand, we shall see examples of holomorphic local dynamical systems that are topologically locally conjugated without being neither formally nor holomorphically locally conjugated, and examples of holomorphic local dynamical systems that are formally conjugated without being neither holomorphically nor topologically locally conjugated. So the last natural question in local holomorphic dynamics we shall deal with is

(Q5) *Find normal forms and invariants with respect to the relation of formal conjugacy for holomorphic local dynamical systems at $O \in \mathbb{C}^n$.*

In this survey we shall present some of the main results known on these questions, starting from the one-dimensional situation.

2. One complex variable: the hyperbolic case

Let us then start by discussing holomorphic local dynamical systems at $0 \in \mathbb{C}$. As remarked in the previous section, such a system is given by a converging power series f without constant term:

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbb{C}_0\{z\}.$$

The number $a_1 = f'(0)$ is the *multiplier* of f.

Since $a_1 z$ is the best linear approximation of f, it is sensible to expect that the local dynamics of f will be strongly influenced by the value of a_1. For this reason we introduce the following definitions:

- if $|a_1| < 1$ we say that the fixed point 0 is *attracting;*
- if $a_1 = 0$ we say that the fixed point 0 is *superattracting;*
- if $|a_1| > 1$ we say that the fixed point 0 is *repelling;*
- if $|a_1| \neq 0, 1$ we say that the fixed point 0 is *hyperbolic;*
- if $a_1 \in S^1$ is a root of unity, we say that the fixed point 0 is *parabolic* (or *rationally indifferent*);
- if $a_1 \in S^1$ is not a root of unity, we say that the fixed point 0 is *elliptic* (or *irrationally indifferent*).

As we shall see in a minute, the dynamics of one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point is pretty elementary; so we start with this case. Notice that if 0 is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C}, 0)$ with non-zero multiplier (we shall discuss the superattracting case momentarily), then it is a repelling fixed point for the inverse map $f^{-1} \in \mathrm{End}(\mathbb{C}, 0)$.

Assume first that 0 is attracting for the holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}, 0)$. Then we can write $f(z) = a_1 z + O(z^2)$, with $0 < |a_1| < 1$; hence we can find a large constant $M > 0$, a small constant $\varepsilon > 0$ and $0 < \delta < 1$ such that if $|z| < \varepsilon$ then

$$|f(z)| \leq (|a_1| + M\varepsilon)|z| \leq \delta|z|. \tag{2.1}$$

In particular, if Δ_ε denotes the disk of center 0 and radius ε, we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$ for $\varepsilon > 0$ small enough, and the stable set of $f|_{\Delta_\varepsilon}$ is Δ_ε itself (in particular, an one-dimensional attracting fixed point is always stable). Furthermore,

$$|f^k(z)| \leq \delta^k |z| \to 0$$

as $k \to +\infty$, and thus every orbit starting in Δ_ε is attracted by the origin, which is the reason of the name "attracting" for such a fixed point.

If instead 0 is a repelling fixed point, a similar argument (or the observation that 0 is attracting for f^{-1}) shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ reduces to the origin only: all (non-trivial) orbits escape.

It is also not difficult to find holomorphic and topological normal forms for one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point, as shown in the following result, which has marked the beginning of the theory of holomorphic dynamical systems:

Theorem 2.1: (Kœnigs, 1884 [Kœ]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be an one-dimensional holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let $a_1 \in \mathbb{C}^*$ be its multiplier. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to its linear part $g(z) = a_1 z$. The conjugation φ is uniquely determined by the condition $\varphi'(0) = 1$.*

(ii) *Two such holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same multiplier.*

(iii) *f is topologically locally conjugated to the map $g_<(z) = z/2$ if $|a_1| < 1$, and to the map $g_>(z) = 2z$ if $|a_1| > 1$.*

Proof: Let us assume $0 < |a_1| < 1$; if $|a_1| > 1$ it will suffice to apply the same argument to f^{-1}.

(i) Choose $0 < \delta < 1$ such that $\delta^2 < |a_1| < \delta$. Writing $f(z) = a_1 z + z^2 r(z)$ for a suitable holomorphic germ r, we can clearly find $\varepsilon > 0$ such that $|a_1| + M\varepsilon < \delta$, where $M = \max_{z \in \overline{\Delta_\varepsilon}} |r(z)|$. So we have

$$|f(z) - a_1 z| \leq M|z|^2$$

and

$$|f^k(z)| \le \delta^k |z|$$

for all $z \in \overline{\Delta_\varepsilon}$ and $k \in \mathbb{N}$.

Put $\varphi_k = f^k/a_1^k$; we claim that the sequence $\{\varphi_k\}$ converges to a holomorphic map $\varphi\colon \Delta_\varepsilon \to \mathbb{C}$. Indeed we have

$$|\varphi_{k+1}(z) - \varphi_k(z)| = \frac{1}{|a_1|^{k+1}} \bigl| f\bigl(f^k(z)\bigr) - a_1 f^k(z) \bigr| \le \frac{M}{|a_1|^{k+1}} |f^k(z)|^2 \le \frac{M}{|a_1|} \left(\frac{\delta^2}{|a_1|} \right)^k |z|^2$$

for all $z \in \overline{\Delta_\varepsilon}$, and so the telescopic series $\sum_k (\varphi_{k+1} - \varphi_k)$ is uniformly convergent in Δ_ε to $\varphi - \varphi_0$.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrink ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi\bigl(f(z)\bigr) = \lim_{k\to+\infty} \frac{f^k\bigl(f(z)\bigr)}{a_1^k} = a_1 \lim_{k\to+\infty} \frac{f^{k+1}(z)}{a_1^{k+1}} = a_1 \varphi(z),$$

that is $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another local holomorphic function such that $\psi'(0) = 1$ and $\psi^{-1} \circ g \circ \psi = f$, it follows that $\psi \circ \varphi^{-1}(\lambda z) = \lambda \psi \circ \varphi^{-1}(z)$; comparing the expansion in power series of both sides we find $\psi \circ \varphi^{-1} \equiv \mathrm{id}$, that is $\psi \equiv \varphi$, as claimed.

(ii) Since $f_1 = \varphi^{-1} \circ f_2 \circ \varphi$ implies $f_1'(0) = f_2'(0)$, the multiplier is invariant under holomorphic local conjugation, and so two one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point are holomorphically locally conjugated if and only if they have the same multiplier.

(iii) Since $|a_1| < 1$ it is easy to build a topological conjugacy between g and $g_<$ on Δ_ε. First choose a homeomorphism χ between the annulus $\{|a_1|\varepsilon \le |z| \le \varepsilon\}$ and the annulus $\{\varepsilon/2 \le |z| \le \varepsilon\}$ which is the identity on the outer circle and given by $\chi(z) = z/2|a_1|$ on the inner circle. Now extend χ by induction to a homeomorphism between the annuli $\{|a_1|^k \varepsilon \le |z| \le |a_1|^{k-1}\varepsilon\}$ and $\{\varepsilon/2^k \le |z| \le \varepsilon/2^{k-1}\}$ by prescribing

$$\chi(|a_1| z) = \tfrac{1}{2}\chi(z).$$

Putting finally $\chi(0) = 0$ we then get a homeomorphism χ of Δ_ε with itself such that $g = \chi^{-1} \circ g_< \circ \chi$, as required. □

Notice that $g_<(z) = \frac{1}{2} z$ and $g_>(z) = 2z$ cannot be topologically conjugated, because (for instance) the origin is stable for $g_<$ and it is not stable for $g_>$.

Thus the dynamics in the one-dimensional hyperbolic case is completely clear. The superattracting case can be treated similarly. If 0 is a superattracting point for an $f \in \mathrm{End}(\mathbb{C}, 0)$, we can write

$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \cdots$$

with $a_r \ne 0$; the number $r \ge 2$ is the *order* of the superattracting point. An argument similar to the one described above shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ still is all of Δ_ε, and the orbits converge (faster than in the attracting case) to the origin. Furthermore, we can prove the following

Theorem 2.2: (Böttcher, 1904 [B]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be an one-dimensional holomorphic local dynamical system with a superattracting fixed point at the origin, and let $r \ge 2$ be its order. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to the map $g(z) = z^r$.*
(ii) *two such holomorphic local dynamical systems are holomorphically (or topologically) conjugated if and only if they have the same order.*

Proof: First of all, up to a linear conjugation $z \mapsto \mu z$ with $\mu^{r-1} = a_r$ we can assume $a_r = 1$.

Now write $f(z) = z^r h_1(z)$ for a suitable holomorphic germ h_1 with $h_1(0) = 1$. By induction, it is easy to see that we can write $f^k(z) = z^{r^k} h_k(z)$ for a suitable holomorphic germ h_k with $h_k(0) = 1$. Furthermore, the equalities $f \circ f^{k-1} = f^k = f^{k-1} \circ f$ yields

$$h_{k-1}(z)^r h_1\bigl(f^{k-1}(z)\bigr) = h_k(z) = z^{-r} h_1(z)^{r^{k-1}} h_{k-1}\bigl(f(z)\bigr). \tag{2.2}$$

Choose $0<\delta<1$. Then we can clearly find $1>\varepsilon>0$ such that $M\varepsilon<\delta$, where $M=\max_{z\in\overline{\Delta_\varepsilon}}|h_1(z)|$; we can also assume that $h_1(z)\ne 0$ for all $z\in\overline{\Delta_\varepsilon}$. Since

$$\forall z\in\overline{\Delta_\varepsilon}\qquad |f(z)|\le M|z|^r<\delta|z|^{r-1},$$

we have $f(\Delta_\varepsilon)\subset\Delta_\varepsilon$, as anticipated before.

We also remark that (2.2) implies that each h_k is well-defined and never vanishing on $\overline{\Delta_\varepsilon}$. So for every $k\ge 1$ we can choose a unique ψ_k holomorphic in Δ_ε such that $\psi_k(z)^{r^k}=h_k(z)$ on Δ_ε and with $\psi_k(0)=1$.

Set $\varphi_k(z)=z\psi_k(z)$, so that $\varphi_k'(0)=1$ and $\varphi_k(z)^{r^k}=f_k(z)$ on Δ_ε. We claim that the sequence $\{\varphi_k\}$ converges to a holomorphic function φ on Δ_ε. Indeed, we have

$$\begin{aligned}\left|\frac{\varphi_{k+1}(z)}{\varphi_k(z)}\right|&=\left|\frac{\psi_{k+1}(z)^{r^{k+1}}}{\psi_k(z)^{r^{k+1}}}\right|^{1/r^{k+1}}=\left|\frac{h_{k+1}(z)}{h_k(z)^r}\right|^{1/r^{k+1}}=\left|h_1\bigl(f^k(z)\bigr)\right|^{1/r^{k+1}}\\&=\left|1+O\bigl(|f^k(z)|\bigr)\right|^{1/r^{k+1}}=1+\frac{1}{r^{k+1}}O\bigl(|f^k(z)|\bigr)=1+O\left(\frac{1}{r^{k+1}}\right),\end{aligned}$$

and so the telescopic product $\prod_k(\varphi_{k+1}/\varphi_k)$ converges to φ/φ_1 uniformly in Δ_ε.

Since $\varphi_k'(0)=1$ for all $k\in\mathbb{N}$, we have $\varphi'(0)=1$ and so, up to possibly shrink ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi_k\bigl(f(z)\bigr)^{r^k}=f(z)^{r^k}\psi_k\bigl(f(z)\bigr)^{r^k}=z^{r^k}h_1(z)^{r^k}h_k\bigl(f(z)\bigr)=z^{r^{k+1}}h_{k+1}(z)=\bigl[\varphi_{k+1}(z)^r\bigr]^{r^k},$$

and thus $\varphi_k\circ f=[\varphi_{k+1}]^r$. Passing to the limit we get $f=\varphi^{-1}\circ g\circ\varphi$, as claimed.

Finally, (ii) follows because z^r and z^s are locally topologically conjugated if and only if $r=s$. □

Therefore the one-dimensional local dynamics about a hyperbolic or superattracting fixed point is completely clear; let us now discuss what happens about a parabolic fixed point.

3. One complex variable: the parabolic case

Let $f\in\mathrm{End}(\mathbb{C},0)$ be a (non-linear) holomorphic local dynamical system with a parabolic fixed point at the origin. Then we can write

$$f(z)=e^{2i\pi p/q}z+a_{r+1}z^{r+1}+a_{r+2}z^{r+2}+\cdots,\tag{3.1}$$

with $a_{r+1}\ne 0$, where $p/q\in\mathbb{Q}\cap[0,1)$ is the *rotation number* of f, and the number $r+1\ge 2$ is the *multiplicity* of f at the fixed point.

The first observation is that such a dynamical system is never locally conjugated to its linear part, not even topologically, unless it is of finite order. Indeed, if we had $\varphi^{-1}\circ f\circ\varphi(z)=e^{2\pi ip/q}z$ we would have $\varphi^{-1}\circ f^q\circ\varphi=\mathrm{id}$, that is $f^q=\mathrm{id}$.

In particular, if the rotation number is 0 (that is the multiplier is 1, and we shall say that f is *tangent to the identity*), then f *cannot* be locally conjugated to the identity (unless it was the identity to begin with, which is not a very interesting case dynamically speaking). More precisely, the stable set of such an f is never a neighbourhood of the origin. To understand why, let us first consider a map of the form

$$f(z)=z(1+az^r)$$

for some $a\ne 0$. Let $v\in S^1\subset\mathbb{C}$ be such that av^r is real and positive. Then for any $c>0$ we have

$$f(cv)=c(1+c^rav^r)v\in\mathbb{R}^+v;$$

moreover, $|f(cv)|>|cv|$. In other words, the half-line $\mathbb{R}^+v$ is f-invariant and repelled from the origin, that is $K_f\cap\mathbb{R}^+v=\varnothing$. Conversely, if av^r is real and negative then the segment $[0,|a|^{-1/r}]v$ is f-invariant and attracted by the origin. So K_f neither is a neighbourhood of the origin nor reduces to $\{0\}$.

This example suggests the following definition. Let $f\in\mathrm{End}(\mathbb{C},0)$ be of the form (3.1) and tangent to the identity. Then a unit vector $v\in S^1$ is an *attracting* (respectively, *repelling*) *direction* for f at the origin if $a_{r+1}v^r$ is real and negative (respectively, positive). Clearly, there are r equally spaced attracting directions, separated by r equally spaced repelling directions: if $a_{r+1} = |a_{r+1}|e^{i\alpha}$, then $v = e^{i\theta}$ is attracting (respectively, repelling) if and only if

$$\theta = \frac{2k+1}{r}\pi - \frac{\alpha}{r} \qquad \left(\text{respectively, } \theta = \frac{2k}{r}\pi - \frac{\alpha}{r}\right).$$

Furthermore, a repelling (attracting) direction for f is attracting (repelling) for f^{-1}, which is defined in a neighbourhood of the origin.

It turns out that to every attracting direction is associated a connected component of $K_f \setminus \{0\}$. Let $v \in S^1$ be an attracting direction for an f tangent to the identity. The *basin* centered at v is the set of points $z \in K_f \setminus \{0\}$ such that $f^k(z) \to 0$ and $f^k(z)/|f^k(z)| \to v$ (notice that, up to shrinking the domain of f, we can assume that $f(z) \neq 0$ for all $z \in K_f \setminus \{0\}$). If z belongs to the basin centered at v, we shall say that the orbit of z *tends to* 0 *tangent to* v.

A slightly more specialized (but more useful) object is the following: an *attracting petal* centered at an attracting direction v is an open simply connected f-invariant set $P \subseteq K_f \setminus \{0\}$ such that a point $z \in K_f \setminus \{0\}$ belongs to the basin centered at v if and only if its orbit intersects P. In other words, the orbit of a point tends to 0 tangent to v if and only if it is eventually contained in P. A *repelling petal* (centered at a repelling direction) is an attracting petal for the inverse of f.

It turns out that the basins centered at the attracting directions are exactly the connected components of $K_f \setminus \{0\}$, as shown in the *Leau-Fatou flower theorem:*

Theorem 3.1: (Leau, 1897 [L]; Fatou, 1919-20 [F1–3]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r + 1 \geq 2$ at the fixed point. Let $v_1, v_3, \ldots, v_{2r-1} \in S^1$ be the r attracting directions of f at the origin, and $v_2, v_4, \ldots, v_{2r} \in S^1$ the r repelling directions. Then*

(i) *There exists for each attracting (repelling) direction v_{2j-1} (v_{2j}) an attracting (repelling) petal P_{2j-1} (P_{2j}), so that the union of these $2r$ petals together with the origin forms a neighbourhood of the origin. Furthermore, the $2r$ petals are arranged ciclically so that two petals intersect if and only if the angle between their central directions is π/r.*

(ii) *$K_f \setminus \{0\}$ is the (disjoint) union of the basins centered at the r attracting directions.*

(iii) *If B is a basin centered at one of the attracting directions, there is a function $\varphi: B \to \mathbb{C}$ such that $\varphi \circ f(z) = \varphi(z) + 1$ for all $z \in B$. Furthermore, if P is the corresponding petal constructed in part* (i)*, then $\varphi|_P$ is a biholomorphism with an open subset of the complex plane containing a right half-plane — and so $f|_P$ is holomorphically conjugated to the translation $z \mapsto z + 1$.*

Proof: Up to a linear conjugation, we can assume that $a_{r+1} = -1$, so that the attracting directions are the r-th roots of unity. For any $\delta > 0$, the set $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$ has exactly r connected components, each one symmetric with respect to a different r-th root of unity; it will turns out that, for δ small enough, these connected components are attracting petals of f, even though to get a pointed neighbourhood of the origin we shall need larger petals.

For $j = 1, 3, \ldots, 2r - 1$ let $\Sigma_j \subset \mathbb{C}^*$ denote the sector centered about the attractive direction v_j and bounded by two consecutive repelling directions, that is

$$\Sigma_j = \left\{ z \in \mathbb{C}^* \;\middle|\; \frac{2j-3}{r}\,\pi < \arg z < \frac{2j-1}{r}\,\pi \right\}.$$

Notice that each Σ_j contains a unique connected component $P_{j,\delta}$ of $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$; moreover, $P_{j,\delta}$ is tangent at the origin to the sector centered about v_j of amplitude π/r.

The main technical trick in this proof consists in transfering the setting to a neighbourhood of infinity in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Let $\psi: \mathbb{C}^* \to \mathbb{C}^*$ be given by

$$\psi(z) = \frac{1}{rz^r};$$

it is a biholomorphism between Σ_j and $\mathbb{C}^* \setminus \mathbb{R}^-$, with inverse $\psi^{-1}(w) = (rw)^{-1/r}$, after choosing suitably the r-th root. Furthermore, $\psi(P_{j,\delta})$ is the right half-plane $H_\delta = \{w \in \mathbb{C} \mid \mathrm{Re}\, w > 1/(2r\delta)\}$.

When $|w|$ is so large that $\psi^{-1}(w)$ belongs to the domain of definition of f, the composition $F = \psi\circ f\circ\psi^{-1}$ makes sense, and we have

$$F(w) = w + 1 + O(w^{-1/r}). \tag{3.2}$$

Thus to study the dynamics of f in a neighbourhood of the origin in Σ_j it suffices to study the dynamics of F in a neighbourhood of infinity.

The first observation is that when $\operatorname{Re} w$ is large enough then

$$\operatorname{Re} F(w) > \operatorname{Re} w + \frac{1}{2};$$

this implies that for δ small enough H_δ is F-invariant (and thus $P_{j,\delta}$ is f-invariant). Furthermore, by induction one has

$$\forall w \in H_\delta \qquad \operatorname{Re} F^k(w) > \operatorname{Re} w + \frac{k}{2}, \tag{3.3}$$

which implies that $F^k(w) \to \infty$ in H_δ (and $f^k(z) \to 0$ in $P_{j,\delta}$) as $k \to \infty$.

Now we claim that the argument of $w_k = F^k(w)$ tends to zero. Indeed, (3.2) and (3.3) yield

$$\frac{w_k}{k} = \frac{w}{k} + 1 + \frac{1}{k}\sum_{l=0}^{k-1} O(w_l^{-1/r});$$

so Cesaro's theorem on the averages of a converging sequence implies

$$\frac{w_k}{k} \to 1, \tag{3.4}$$

and thus $\arg w_k \to 0$ as $k \to \infty$. Going back to $P_{j,\delta}$, this implies that $f^k(z)/|f^k(z)| \to v_j$ for every $z \in P_{j,\delta}$. Since furthermore $P_{j,\delta}$ is centered about v_j, every orbit converging to 0 tangent to v_j must intersect $P_{j,\delta}$, and thus we have proved that $P_{j,\delta}$ is an attracting petal.

Arguing in the same way with f^{-1} we get repelling petals; unfortunately, these petals are too small to obtain a full pointed neighbourhood of the origin. In fact, as remarked before each $P_{j,\delta}$ is contained in a sector centered about v_j of amplitude π/r; therefore the repelling and attracting petals obtained in this way do not intersect but are tangent to each other. We need larger petals.

So our aim is to find an f-invariant subset $\tilde{P}_j$ of Σ_j containing $P_{j,\delta}$ and which is tangent at the origin to a sector centered about v_j of amplitude strictly greater than π/r. To do so, first of all remark that there are R, $C > 0$ such that

$$|F(w) - w - 1| \le \frac{C}{|w|^{1/r}} \tag{3.5}$$

as soon as $|w| > R$. Choose $\varepsilon \in (0,1)$ and select $\delta > 0$ so that $|w| > 1/(r\delta)$ implies

$$|F(w) - w - 1| \le \varepsilon/2.$$

Set $M_\varepsilon = (1+\varepsilon)/(2r\delta)$ and let

$$\tilde{H}_\varepsilon = \{w \in \mathbb{C} \mid |\operatorname{Im} w| > -\varepsilon \operatorname{Re} w + M_\varepsilon\} \cup H_\delta.$$

If $w \in \tilde{H}_\varepsilon$ we have

$$\operatorname{Re} F(w) > \operatorname{Re} w + 1 - \varepsilon/2 \qquad \text{and} \qquad |\operatorname{Im} F(w) - \operatorname{Im} w| < \varepsilon/2;$$

it is then easy to check that $F(\tilde{H}_\varepsilon) \subset \tilde{H}_\varepsilon$ and that every orbit starting in $\tilde{H}_\varepsilon$ must eventually enter H_δ. Thus $\tilde{P}_j = \psi^{-1}(\tilde{H}_\varepsilon)$ is as required, and we have proved (i).

To prove (ii) we need a further property of $\tilde H_\varepsilon$. Since $f^{-1}(z)=z+z^{r+1}+O(z^{r+2})$, we have

$$F^{-1}(w)=w-1+O(w^{-1/r});$$

up to decreasing δ we can thus assume that $|F^{-1}(w)-w+1|<\varepsilon/2$ on $\tilde H_\varepsilon$. But then if $w\in\tilde H_\varepsilon$ we have

$$\operatorname{Re}F^{-1}(w)<\operatorname{Re}w-1+\frac{\varepsilon}{2}\quad\text{and}\quad|\operatorname{Im}F^{-1}(w)|+\varepsilon\operatorname{Re}F^{-1}(w)<|\operatorname{Im}w|+\varepsilon\operatorname{Re}w-\frac{\varepsilon(1-\varepsilon)}{2};$$

this means that every inverse orbit *must* eventually leave $\tilde H_\varepsilon$.

Coming back to the z-plane, we have thus proved that every (forward) orbit of f must eventually leave any repelling petal. So if $z\in K_f\setminus\{O\}$, where the stable set is obtained working in the neighborhood of the origin constructed in part (i), the orbit of z must eventually land in an attracting petal, and thus z belongs to a basin centered at one of the r attracting directions — and (ii) is proved.

To prove (iii), first of all we notice that we have

$$|F'(w)-1|\le\frac{2^{1+1/r}C}{|w|^{1+1/r}}\tag{3.6}$$

in $\tilde H_\varepsilon$. Indeed, (3.5) says that if $|w|>1/(2r\delta)$ then the function $w\mapsto F(w)-w-1$ sends the disk of center w and radius $|w|/2$ into the disk of center the origin and radius $C/(|w|/2)^{1/r}$; inequality (3.6) then follows from the Cauchy estimates on the derivative.

Now choose $w_0\in\tilde H_\varepsilon$, and set $\tilde\varphi_k(w)=F^k(w)-F^k(w_0)$. Given $w\in\tilde H_\varepsilon$, as soon as $k\in\mathbb{N}$ is so large that $F^k(w)\in H_\delta$ we can apply Lagrange's theorem to the segment from $F^k(w_0)$ to $F^k(w)$ to get a $t_k\in[0,1]$ such that

$$\begin{aligned}\left|\frac{\tilde\varphi_{k+1}(w)}{\tilde\varphi_k(w)}-1\right|&=\left|\frac{F\big(F^k(w)\big)-F^k\big(F^k(w_0)\big)}{F^k(w)-F^k(w_0)}-1\right|=\left|F'\big(t_kF^k(w)+(1-t_k)F^k(w_0)\big)-1\right|\\&\le\frac{2^{1+1/r}C}{\min\{\operatorname{Re}|F^k(w),\operatorname{Re}|F^k(w_0)|\}^{1+1/r}}\le\frac{C'}{k^{1+1/r}},\end{aligned}$$

where we used (3.6) and (3.4), and the constant C' is uniform on compact subsets of $\tilde H_\varepsilon$ (and it can be chosen uniform on H_δ).

As a consequence, the telescopic product $\prod_k\tilde\varphi_{k+1}/\tilde\varphi_k$ converges uniformly on compact subsets of $\tilde H_\varepsilon$ (and uniformly on H_δ), and thus the sequence $\tilde\varphi_k$ converges, uniformly on compact subsets, to a holomorphic function $\tilde\varphi\colon\tilde H_\varepsilon\to\mathbb{C}$. Since we have

$$\tilde\varphi_k\circ F(w)=F^{k+1}(w)-F^k(w_0)=\tilde\varphi_{k+1}(w)+F\big(F^k(w_0)\big)-F^k(w_0)=\tilde\varphi_{k+1}(w)+1+O\big(|F^k(w)|^{-1/r}\big),$$

it follows that

$$\tilde\varphi\circ F(w)=\tilde\varphi(w)+1$$

on $\tilde H_\varepsilon$. In particular, $\tilde\varphi$ is not constant; being the limit of injective functions, by Hurwitz's theorem it is injective.

We now prove that the image of $\tilde\varphi$ contains a right half-plane. First of all, we claim that

$$\lim_{\substack{|w|\to+\infty\\ w\in H_\delta}}\frac{\tilde\varphi(w)}{w}=1.\tag{3.7}$$

Indeed, choose $\eta>0$. Since the convergence of the telescopic product is uniform on H_δ, we can find $k_0\in\mathbb{N}$ such that

$$\left|\frac{\tilde\varphi(w)-\tilde\varphi_{k_0}(w)}{w-w_0}\right|\le\frac{\eta}{2}$$

on H_δ. Furthermore, we have

$$\left|\frac{\tilde\varphi_{k_0}(w)}{w-w_0}-1\right| = \left|\frac{k_0+\sum_{j=0}^{k_0-1} O(|F^j(w)|^{-1/r})+w_0-F^{k_0}(w_0)}{w-w_0}\right| = O(|w|^{-1})$$

on H_δ; therefore we can find $R>0$ such that

$$\left|\frac{\tilde\varphi(w)}{w-w_0}-1\right| < \frac{\eta}{2}$$

as soon as $|w|>R$ in H_δ.

Equality (3.7) clearly implies that $(\tilde\varphi(w)-w^o)/(w-w^o)\to 1$ as $|w|\to+\infty$ in H^δ for any $w^o\in\mathbb{C}$. But this means that if $\operatorname{Re} w^o$ is large enough then the difference between the variation of the argument of $\tilde\varphi - w^o$ along a suitably small closed circle around w^o and the variation of the argument of $w-w^o$ along the same circle will be less than 2π — and thus it will be zero. Then the principle of the argument implies that $\tilde\varphi - w^o$ and $w-w^o$ have the same number of zeroes inside that circle, and thus $w^o\in\tilde\varphi(H_\delta)$, as required.

So setting $\varphi=\tilde\varphi\circ\psi$, we have defined a function φ with the required properties on $\tilde P_j$. To extend it to the whole basin B it suffices to put

$$\varphi(z)=\varphi\bigl(f^k(z)\bigr)-k,$$

where $k\in\mathbb{N}$ is the first integer such that $f^k(z)\in\tilde P_j$. □

Remark 3.1: It is possible to construct petals that cannot be contained in any sector strictly smaller than Σ_j. To do so we need an F-invariant subset $\hat H_\varepsilon$ of $C^*\setminus\mathbb{R}^-$ containing $\tilde H_\varepsilon$ and containing eventually every half-line issuing from the origin (but $\mathbb{R}^-$). For $M>>1$ and $C>0$ large enough, replace the straight lines bounding $\tilde H_\varepsilon$ on the left of $\operatorname{Re} w=-M$ by the curves

$$|\operatorname{Im} w| = \begin{cases} C\log|\operatorname{Re} w| & \text{if } r=1,\\ C|\operatorname{Re} w|^{1-1/r} & \text{if } r>1.\end{cases}$$

Then it is not too difficult to check that the domain $\hat H_\varepsilon$ so obtained is as desired (see [CG]).

So we have a complete description of the dynamics in the neighbourhood of the origin. Actually, Camacho has pushed this argument even further, obtaining a complete topological classification of one-dimensional holomorphic local dynamical systems tangent to the identity:

Theorem 3.2: (Camacho, 1978 [C]; Shcherbakov, 1982 [S]) *Let $f\in\operatorname{End}(\mathbb{C},0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1$ at the fixed point. Then f is topologically locally conjugated to the map*

$$z\mapsto z-z^{r+1}.$$

The formal classification is simple too, though different, and it can be obtained with an easy computation (see, e.g., Milnor [Mi]):

Proposition 3.3: *Let $f\in\operatorname{End}(\mathbb{C},0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1$ at the fixed point. Then f is formally conjugated to the map*

$$g(z)=z-z^{r+1}+\beta z^{2r+1}, \tag{3.8}$$

where β is a formal (and holomorphic) invariant given by

$$\beta=\frac{1}{2\pi i}\int_\gamma\frac{dz}{z-f(z)}, \tag{3.9}$$

where the integral is taken over a small positive loop γ about the origin.

Proof: An easy computation shows that if f is given by (3.8) then β is given by the integral (3.9). Let us now show that the latter integral is a holomorphic invariant. Let φ be a local biholomorphism fixing the origin, and set $F=\varphi^{-1}\circ f\circ\varphi$. Then

$$\frac{1}{2\pi i}\int_\gamma\frac{dz}{z-f(z)}=\frac{1}{2\pi i}\int_{\varphi^{-1}\circ\gamma}\frac{\varphi'(w)\,dw}{\varphi(w)-f\bigl(\varphi(w)\bigr)}=\frac{1}{2\pi i}\int_{\varphi^{-1}\circ\gamma}\frac{\varphi'(w)\,dw}{\varphi(w)-\varphi\bigl(F(w)\bigr)}.$$

Now, we can clearly find M, $M_1 > 0$ such that

$$\begin{aligned}\left|\frac{1}{w-F(w)}-\frac{\varphi'(w)}{\varphi(w)-\varphi\big(\big(F(w)\big)}\right| &= \frac{1}{\big|\varphi(w)-\varphi\big(\big(F(w)\big)\big|}\left|\frac{\varphi(w)-\varphi\big(\big(F(w)\big)}{w-F(w)}-\varphi'(w)\right| \\ &\le M\frac{|w-F(w)|}{\big|\varphi(w)-\varphi\big(\big(F(w)\big)\big|}\le M_1,\end{aligned}$$

in a neighbourhood of the origin, where the last inequality follows from the fact that $\varphi'(0)\neq 0$. This means that the two meromorphic functions $1/\big(w-F(w)\big)$ and $\varphi'(w)/\big(\varphi(w)-\varphi\big(\big(F(w)\big)\big)$ differ by a holomorphic function; so they have the same integral along any small loop surrounding the origin, and

$$\frac{1}{2\pi i}\int_\gamma\frac{dz}{z-f(z)}=\frac{1}{2\pi i}\int_{\varphi^{-1}\circ\gamma}\frac{dw}{w-F(w)},$$

as claimed.

To prove that f is formally conjugated to g, let us first take a local formal change of coordinates φ of the form

$$\varphi(z)=z+\mu z^d+O_{d+1} \tag{3.10}$$

with $\mu\neq 0$, and where we are writing O_{d+1} instead of $O(z^{d+1})$. It follows that $\varphi^{-1}(z)=z-\mu z^d+O_{d+1}$, $(\varphi^{-1})'(z)=1-d\mu z^{d-1}+O_d$ and $(\varphi^{-1})^{(j)}=O_{d-j}$ for all $j\ge 2$. Then using the Taylor expansion of φ^{-1} we get

$$\begin{aligned}\varphi^{-1}\circ f\circ\varphi(z) &= \varphi^{-1}\left(\varphi(z)+\sum_{j\ge r+1}a_j\varphi(z)^j\right)\\ &= z+(\varphi^{-1})'\big(\varphi(z)\big)\sum_{j\ge r+1}a_jz^j(1+\mu z^{d-1}+O_d)^j+O_{d+2r}\\ &= z+[1-d\mu z^{d-1}+O_d]\sum_{j\ge r+1}a_jz^j(1+j\mu z^{d-1}+O_d)+O_{d+2r}\\ &= z+a_{r+1}z^{r+1}+\cdots+a_{r+d-1}z^{r+d-1}+[a_{r+d}+(r+1-d)\mu a_{r+1}]z^{r+d}+O_{r+d+1}.\end{aligned} \tag{3.11}$$

This means that if $d\neq r+1$ we can use a polynomial change of coordinates of the form $\varphi(z)=z+\mu z^d$ to remove the term of degree $r+d$ from the Taylor expansion of f without changing the lower degree terms.

So to conjugate f to g it suffices to use a linear change of coordinates to get $a_{r+1}=-1$, and then apply a sequence of change of coordinates of the form $\varphi(z)=z+\mu z^d$ to kill all the terms in the Taylor expansion of f but the term of degree z^{2r+1}.

Finally, the formula (3.11) also shows that two maps of the form (3.8) with different β cannot be formally conjugated, and we are done. □

The number β given by (3.9) is called *index* of f at the fixed point.

The holomorphic classification is much more complicated: as shown by Voronin [V] and Écalle [É1–2] in 1981, it depends on functional invariants. We shall now try to roughly describe it; see [I2] (and the original papers; see also [K]) for details. Let $f\in\mathrm{End}(\mathbb{C},0)$ be tangent to the identity with multiplicity $r+1$ at the fixed point; up to a linear change of coordinates we can assume that $a_{r+1}=1$. Let $P_1,\ldots,P_{2r}$ be a set of petals as in Theorem 3.1.(i), chosen so that P_{2r} is centered on the positive real semiaxis, and the others are arranged cyclically counterclockwise. Denote by H_j the biholomorphism conjugating $f|_{P_j}$ to the shift $z\mapsto z+1$ in either a right (if j is odd) or left (if j is even) half-plane given by Theorem 3.1.(iii) — applied to f^{-1} for the repelling petals. If we moreover require that

$$H_j(z)=-\frac{1}{rz^r}+\beta\log z+o(1), \tag{3.12}$$

where β is the index of f at the origin, then H_j is uniquely determined. Thus in the sets $H_j(P_j\cap P_{j+1})$ we can consider the composition $\tilde\Phi_j=H_{j+1}\circ H_j^{-1}$. It is easy to check that $\tilde\Phi_j(w+1)=\tilde\Phi_j(w)+1$ for $j=1,\ldots,2r-1$, and thus $\psi_j = \tilde{\Phi}_j - \mathrm{id}$ is a 1-periodic holomorphic function (for $j = 2r$ we need to take $\psi_{2r} = \tilde{\Phi}_{2r} - \mathrm{id} + 2\pi i\beta$ to get a 1-periodic function). Hence each ψ_j can be extended to a suitable upper (if j is odd) or lower (if j is even) half-plane. Furthermore, it is possible to prove that the functions $\psi_1, \ldots, \psi_{2r}$ are exponentially decreasing, that is they are bounded by $\exp(-c|w|)$ as $|\operatorname{Im} w| \to +\infty$, for a suitable $c > 0$ depending on f.

Now, if we replace f by a holomorphic local conjugate $g = h^{-1} \circ f \circ h$, and denote by G_j the corresponding biholomorphisms, it turns out that $H_j \circ G_j^{-1} = \mathrm{id} + a$ for a suitable $a \in \mathbb{C}$ independent of j. This suggests the introduction of an equivalence relation on the set of $2r$-uple of functions of the kind $(\psi_1, \ldots, \psi_{2r})$.

Let M_r denote the set of $2r$-uple of holomorphic 1-periodic functions $\psi = (\psi_1, \ldots, \psi_{2r})$, with ψ_j defined in a suitable upper (if j is odd) or lower (if j is even) half-plane, and exponentially decreasing when $|\operatorname{Im} w| \to +\infty$. We shall say that $\psi, \tilde{\psi} \in M_r$ are *equivalent* if there is $a \in \mathbb{C}$ such that $\tilde{\psi}_j = \psi_j \circ (\mathrm{id} + a)$ for $j = 1, \ldots, 2r$. We denote by $\mathcal{M}_r$ the set of all equivalence classes.

The procedure described above allows us to associate to any $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity with multiplicity $r+1$ at the fixed point an element $\mu_f \in \mathcal{M}_r$, called the *sectorial invariant*. Then the holomorphic classification proved by Écalle and Voronin is

Theorem 3.4: (Écalle, 1981 [É1–2]; Voronin, 1981 [V]) *Let f, $g \in \mathrm{End}(\mathbb{C}, 0)$ be two holomorphic local dynamical systems tangent to the identity. Then f and g are holomorphically locally conjugated if and only if they have the same multiplicity, the same index and the same sectorial invariant. Furthermore, for any $r \geq 1$, $\beta \in \mathbb{C}$ and $\mu \in \mathcal{M}_r$ there exists $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity with multiplicity $r + 1$, index β and sectorial invariant μ.*

For a sketch of the proof, together with a more geometrical description of the sectorial invariant, see [I2] and [M1–2].

Remark 3.2: In particular, holomorphic local dynamical systems tangent to the identity give examples of local dynamical systems that are topologically conjugated without being neither holomorphically nor formally conjugated, and of local dynamical systems that are formally conjugated without being holomorphically conjugated.

Finally, if $f \in \mathrm{End}(\mathbb{C}, 0)$ satisfies $a_1 = e^{2\pi i p/q}$, then f^q is tangent to the identity. Therefore we can apply the previous results to f^q and then infer informations about the dynamics of the original f. We list here a few results; see [Mi], [Ma], [C], [É1–2] and [V] for proofs and further details.

Proposition 3.5: *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Then f is holomorphically (or topologically or formally) linearizable if and only if $f^q \equiv \mathrm{id}$.*

Proof: We already remarked that if f is holomorphically (or topologically or formally) linearizable then $f^q \equiv \mathrm{id}$. Conversely, assume that $f^q \equiv \mathrm{id}$ and set

$$\varphi(z) = \frac{1}{q} \sum_{j=0}^{q-1} \frac{f^j(z)}{\lambda^j}.$$

Then it is easy to check that $\varphi'(0) = 1$ and $\varphi \circ f(z) = \lambda \varphi(z)$, and so f is holomorphically (and topologically and formally) linearizable. □

Proposition 3.6: *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$. Then there exist $n \geq 1$ and $c \in \mathbb{C}$ such that f is formally conjugated to*

$$g(z) = \lambda z + z^{nq+1} + c z^{2nq+1}.$$

Proposition 3.7: (Camacho) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$. Then there exist $n \geq 1$ such that f is topologically conjugated to*

$$g(z) = \lambda z + z^{nq+1}.$$

Theorem 3.8: (Leau-Fatou) *Let $f \in \operatorname{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$. Then there exist $n \geq 1$ such that f^q has multiplicity $nq+1$, and f acts on the attracting (respectively, repelling) petals of f^q as a permutation composed by n disjoint cycles. Finally, $K_f = K_{f^q}$.*

4. One complex variable: the elliptic case

We are left with the elliptic case:

$$f(z) = e^{2\pi i\theta} z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\}, \tag{4.1}$$

with $\theta \notin \mathbb{Q}$. It turns out that the local dynamics depends mostly on the numerical properties of θ. More precisely, for a full measure subset B of $\theta \in [0,1] \setminus \mathbb{Q}$ all holomorphic local dynamical systems of the form (4.1) are *holomorphically linearizable*, that is holomorphically locally conjugated to their (common) linear part, the irrational rotation $z \mapsto e^{2\pi i\theta} z$. Conversely, the complement $[0,1] \setminus B$ is a G_δ-dense set, and for all $\theta \in [0,1] \setminus B$ the quadratic polynomial $z \mapsto z^2 + e^{2\pi i\theta} z$ is not holomorphically linearizable. This is the gist of the results due to Cremer, Siegel, Bryuno and Yoccoz we are going to describe in this section.

The first worthwhile observation in this setting is that it is possible to give a topological characterization of the holomorphically linearizable local dynamical systems:

Proposition 4.1: *Let $f \in \operatorname{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier $0 < |\lambda| \leq 1$. Then f is holomorphically linearizable if and only if it is topologically linearizable if and only if 0 is stable for f.*

Proof: If f is holomorphically linearizable it is topologically linearizable, and if it is topologically linearizable (and $|\lambda| \leq 1$) then it is stable. Assume that 0 is stable. If $0 < |\lambda| < 1$, we already saw that f is holomorphically linearizable. If $|\lambda| = 1$, set

$$\varphi_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{f^j(z)}{\lambda^j},$$

so that $\varphi_k'(0) = 1$ and

$$\varphi_k \circ f = \lambda \varphi_{k+1} + \frac{\lambda}{k}(\varphi_{k+1} - \mathrm{id}). \tag{4.2}$$

The stability of 0 implies that there are bounded open sets $V \subset U$ $\{\varphi_k\}$ containing the origin such that $f^k(V) \subset U$ for all $k \in \mathbb{N}$. Since $|\lambda| = 1$, it follows that $\{\varphi_k\}$ is a uniformly bounded family on V, and hence, by Montel's theorem, it admits a converging subsequence. But (4.2) implies that a converging subsequence converges to a conjugation between f and the rotation $z \mapsto \lambda z$, and f is holomorphically linearizable. □

The second important observation is that two elliptic holomorphic local dynamical systems with the same multiplier are always formally conjugated:

Proposition 4.2: *Let $f \in \operatorname{End}(\mathbb{C},0)$ be a holomorphic local dynamical system of multiplier $\lambda = e^{2\pi i\theta} \in S^1$ with $\theta \notin \mathbb{Q}$. Then f is formally conjugated to its linear part.*

Proof: We shall prove that there is a unique formal power series

$$h(z) = z + h_2 z^2 + \cdots \in \mathbb{C}[[z]]$$

such that $h(\lambda z) = f\bigl(h(z)\bigr)$. Indeed we have

$$\begin{aligned} h(\lambda z) - f\bigl(h(z)\bigr) &= \sum_{j\geq 2} \left\{ \bigl[(\lambda^j - \lambda)h_j - a_j\bigr] z^j - \sum_{\ell=1}^{j} \binom{j}{\ell} z^{\ell+j} \left(\sum_{k\geq 2} h_k z^{k-2} \right)^{\ell} \right\} \\ &= \sum_{j\geq 2} \bigl[(\lambda^j - \lambda)h_j - a_j - X_j(h_2,\ldots,h_{j-1})\bigr] z^j, \end{aligned}$$

where X_j is a polynomial in $j-2$ variables. It follows that the coefficients of h are uniquely determined by induction using the formula

$$h_j = \frac{a_j + X_j(h_2,\ldots,h_{j-1})}{\lambda^j - \lambda}, \tag{4.3}$$

where X_j is a polynomial. In particular, h_j depends only on λ, $a_2,\ldots,a_j$. □

The formal power series linearizing f is not converging if its coefficients grow too fast. Thus (4.3) links the radius of convergence of h to the behavior of $\lambda^j - \lambda$: if the latter becomes too small, the series defining h does not converge. This is known as the *small denominators problem* in this context.

It is then natural to introduce the following quantity:

$$\Omega_\lambda(m) = \min_{1\le k\le m} |\lambda^k - 1|,$$

for $\lambda \in S^1$ and $m \ge 1$. Clearly, λ is a root of unity if and only if $\Omega_\lambda(m) = 0$ for all m greater or equal to some $m_0 \ge 1$; furthermore,

$$\lim_{m\to+\infty} \Omega_\lambda(m) = 0$$

for all $\lambda \in S^1$.

The first one to actually prove that there are non-linearizable elliptic holomorphic local dynamical systems has been Cremer, in 1927 [Cr1]. His more general result is the following:

Theorem 4.3: (Cremer, 1938 [Cr2]) *Let $\lambda \in S^1$ be such that*

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\Omega_\lambda(m)} = +\infty. \tag{4.4}$$

Then there exists $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ which is not holomorphically linearizable. Furthermore, the set of $\lambda \in S^1$ satisfying (4.4) contains a G_δ-dense set.

Proof: Choose inductively $a_j \in \{0, 1\}$ so that $|a_j + X_j| \ge 1/2$ for all $j \ge 2$, where X_j is as in (4.3). Then

$$f(z) = \lambda z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\}$$

while (4.4) implies that the radius of convergence of the formal linearization h is 0, and thus f cannot be holomorphically linearizable, as required.

Finally, let $S(q_0) \subset S^1$ denote the set of $\lambda = e^{2\pi i\theta} \in S^1$ such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2^{q!}} \tag{4.5}$$

for some $p/q \in \mathbb{Q}$ in lowest terms with $q \ge q_0$. Then it is not difficult to check that each $S(q_0)$ is a dense open set in S^1, and that all $\lambda \in \mathcal{S} = \bigcap_{q_0\ge1} S(q_0)$ satisfy (4.4). Indeed, if $\lambda = e^{2\pi i\theta} \in \mathcal{S}$ we can find $q \in \mathbb{N}$ arbitrarily large such that there is $p \in \mathbb{N}$ so that (4.5) holds. Now, it is easy to see that

$$|e^{2\pi it} - 1| \le 2\pi|t|$$

for all $t \in [-1/2, 1/2]$. Then let p_0 be the integer closest to $q\theta$, so that $|q\theta - p_0| \le 1/2$. Then we have

$$|\lambda^q - 1| = |e^{2\pi iq\alpha} - e^{2\pi ip_0}| = |e^{2\pi i(q\alpha - p_0)} - 1| \le 2\pi|q\theta - p_0| \le 2\pi|q\theta - p| < \frac{2\pi}{2^{q!-1}}$$

for arbitrarily large q, and (4.4) follows. □

On the other hand, Siegel in 1942 gave a condition on the multiplier ensuring holomorphic linearizability:

Theorem 4.4: (Siegel, 1942 [Si]) *Let $\lambda \in S^1$ be such that there exists $\beta > 1$ and $\gamma > 0$ such that*

$$\forall m \ge 2 \qquad \frac{1}{\Omega_\lambda(m)} \le \gamma\, m^\beta. \tag{4.6}$$

Then all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ are holomorphically linearizable. Furthermore, the set of $\lambda \in S^1$ satisfying (4.6) for some $\beta \ge 1$ and $\gamma > 0$ is of full Lebesgue measure in S^1.

Remark 4.1: It is interesting to notice that for generic (in a topological sense) $\lambda \in S^1$ there is a non-linearizable holomorphic local dynamical system with multiplier λ, while for almost all (in a measure-theoretic sense) $\lambda \in S^1$ every holomorphic local dynamical system with multiplier λ is holomorphically linearizable.

The original proof of Theorem 4.4 was based on the method of majorant series, that requires finding a convergent series whose coefficients are greater than the coefficients of the formal linearization. We shall describe here a different proof, in the spirit of the so-called Kolmogorov-Arnold-Moser (or KAM) method.

Let $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$ be the open disk centered at the origin of radius $r > 0$ in the complex plane. We shall need the following

Lemma 4.5: (i) *Let* $f(z) = \sum_{k=0}^{\infty} f_k z^k$ *be a function holomorphic and bounded on* Δ_r*; then we have* $|f_k| \le \|f\|_0 r^{-k}$ *for* $k \in \mathbb{N}$.
(ii) *If* $\{f_k\} \subset \mathbb{C}$ *is a sequence such that* $|f_k| \le K r^{-k}$ *for some* $K > 0$ *and all* $k \in \mathbb{N}$, *then the function* $f = \sum_k f_k z^k$ *is holomorphic in* Δ_r, *and* $\|f\|_0 \le Kr/\delta$ *on* $\overline{\Delta_{r-\delta}}$ *for all* $0 < \delta < r$.

Proof: (i) Indeed for every $\rho < r$ the Cauchy formula yields

$$|f_k| = \left| \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{k+1}}\, dz \right| \le \frac{1}{2\pi} \int_{|z|=\rho} \left| \frac{f(z)}{z^{k+1}} \right| dz \le \frac{\|f\|_0}{\rho^k}.$$

The assertion follows letting $\rho \to r^-$.

(ii) If $z \in \overline{\Delta_{r-\delta}}$ we have

$$|f(z)| \le K \sum_{k=0}^{\infty} r^{-k}(r-\delta)^k = \frac{Kr}{\delta}.$$

□

Proof of Theorem 4.4: Write $\Lambda(z) = \lambda z$ for the linear part of f. Let us denote by $\mathcal{F}(f,h)$ the operator $\mathcal{F}(f,h) = h^{-1} \circ f \circ h$ acting on $\mathrm{End}(\mathbb{C},0)$; we want to solve the equation $\mathcal{F}(f,h) = \Lambda$. The idea is that we shall get an approximate solution by solving a "linearized" version of the equation; then we shall set up an iteration process converging to the actual conjugation.

We write $h = \mathrm{id} + \psi$ and

$$\mathcal{F}(f,h) = \mathcal{F}(\Lambda, \mathrm{id}) + D_1\mathcal{F}(\Lambda,\mathrm{id})(f - \Lambda) + D_2\mathcal{F}(\Lambda,\mathrm{id})\psi + \mathcal{R}(f,h), \tag{4.7}$$

where $D_j\mathcal{F}$ is the derivative of $\mathcal{F}$ with respect to the j-th argument, and $\mathcal{R}$ is of second order in $f - \Lambda$ and $\psi = h - \mathrm{id}$. Now, $\mathcal{F}(\cdot, \mathrm{id}) = \mathrm{id}$, and so $D_1\mathcal{F}(\Lambda,\mathrm{id}) = \mathrm{id}$. On the other hand,

$$\begin{aligned} D_2\mathcal{F}(\Lambda,\mathrm{id})\psi &= \lim_{t\to 0} \frac{1}{t}\big[\mathcal{F}(\Lambda, \mathrm{id} + t\psi) - \mathcal{F}(\Lambda,\mathrm{id})\big] \\ &= \lim_{t\to 0} \frac{1}{t}\big[(\mathrm{id} + t\psi)^{-1} \circ \Lambda \circ (\mathrm{id} + t\psi) - \Lambda\big] \\ &= \lim_{t\to 0} \frac{1}{t}\big[(\mathrm{id} - t\psi) \circ \Lambda \circ (\mathrm{id} + t\psi) - \Lambda + O(t^2)\big] \\ &= \lim_{t\to 0} \frac{1}{t}\big[t(\Lambda \circ \psi - \psi \circ \Lambda) + O(t^2)\big] = \Lambda \circ \psi - \psi \circ \Lambda. \end{aligned}$$

So h is a solution of $\mathcal{F}(f,h) = \Lambda$ if and only if ψ satisfies

$$f - \Lambda + (\Lambda \circ \psi - \psi \circ \Lambda) + \mathcal{R}(f, \mathrm{id} + \psi) = O.$$

Setting $u = f - \Lambda$, we shall study the "linearized" equation

$$-D_2\mathcal{F}(\Lambda,\mathrm{id})\hat{\psi} = u, \tag{4.8}$$

and consider $\hat{h} = \mathrm{id} + \hat{\psi}$ and $f_1 = \mathcal{F}(f,\hat{h}) = \Lambda + \mathcal{R}(f,\hat{h})$. Since $u = O(z^2)$ it will turn out that $f_1 - \Lambda = o(f - \Lambda)$, and thus repeating this process we shall get better and better approximations. Of course, we need to prove that this process converges somewhere…

But first let us prove that we can solve (4.8), or, in other words, that $D_2\mathcal{F}(\Lambda,\mathrm{id})$ is invertible.

Lemma 4.6: *Let* $\varphi(z) = \sum_{k=2}^{\infty} \varphi_k z^k$ *be holomorphic and bounded in* Δ_r *for some* $r > 0$, *and for* $k \ge 2$ *set*

$$\psi_k = \frac{\varphi_k}{\lambda^k - \lambda},$$

where λ satisfies (4.6). *Then $\psi(z)=\sum_{k=2}^\infty \psi_k z^k$ is holomorphic in Δ_r, satisfies*

$$D_2\mathcal{F}(\Lambda,\mathrm{id})\psi=-\varphi$$

and there is $c_1>0$ depending only on λ such that for all $0<\eta<1$ we have

$$\|\psi\|_{0,\Delta_{r(1-\eta)}}\le\frac{c_1}{\eta^{d+1}}\|\varphi\|_0.$$

Proof: First of all, by Lemma 4.5.(i) and (4.6) we get

$$|\psi_k|\le\frac{\gamma\|\varphi\|_0}{r^k}(k-1)^\beta<\frac{\gamma\|\varphi\|_0}{c_0 r^k}k^\beta.$$

So $\chi(z)=\sum_{k\ge2}k^{-\beta}\psi_k z^k$ is holomorphic on Δ_r by Lemma 4.5.(ii). But then

$$z\chi'(z)=\sum_{k\ge2}k^{-\beta+1}\psi_k z^k$$

is still holomorphic on Δ_r, and by induction on β we see that ψ is holomorphic on Δ_r. On $\Delta_{r(1-\eta)}$ we have

$$|\psi(z)|\le\gamma\|\varphi\|_0\sum_{k=2}^\infty k^\beta(1-\eta)^k\le\frac{c_1\|\varphi\|_0}{\eta^{d+1}},$$

where the last inequality holds since it is easy to prove by induction that for $0<x<1$ we have

$$\sum_{k=0}^\infty k^\beta x^k=\frac{p_\beta(x)}{(1-x)^{\beta+1}}$$

where $p_\beta(x)$ is a monic polynomial of degree β.

We are left to show that ψ satisfies $D_2\mathcal{F}(\Lambda,\mathrm{id})\psi=-\varphi$. But indeed

$$D_2\mathcal{F}(\Lambda,\mathrm{id})\psi(z)=\sum_{k=2}^\infty[\lambda-\lambda^k]\psi_k z^k=-\sum_{k=2}^\infty\varphi_k z^k=-\varphi(z).$$

□

Now we can start the iterative procedure. Since $u=f-\Lambda=O(z^2)$, for any $\varepsilon>0$ we can choose $1>\rho=\rho_\varepsilon>0$ so that

$$\|u'\|_{0,\Delta_\rho}\le\varepsilon. \tag{4.9}$$

Now apply Lemma 4.6 with $\varphi=u$ and $r=\rho$, recalling that (4.9) implies $\|u\|_{0,\Delta_\rho}\le\varepsilon\rho$; we get

$$\|\psi\|_{0,\Delta_{\rho(1-\eta)}}\le\frac{c_1\varepsilon\rho}{\eta^{\beta+1}}.$$

The same argument applied to $zu'(z)$ yields

$$\|\psi'\|_{0,\Delta_{\rho(1-\eta)}}\le\frac{c_1\varepsilon\rho}{\eta^{\beta+1}}. \tag{4.10}$$

Set $h=\mathrm{id}+\psi$; we need to estimate the size of the image of h for ε and η small enough.

Lemma 4.7: *Choose $0<\eta<1$ and $\varepsilon>0$ so that*

$$0<\eta<\frac{1}{4} \qquad \text{and} \qquad c_1\varepsilon\le\eta^{\beta+2}.$$

Then $h\bigl(\Delta_{\rho(1-4\eta)}\bigr)\subset\Delta_{\rho(1-3\eta)}$ and $\Delta_{\rho(1-2\eta)}\subset h(\Delta_{\rho(1-\eta)})$, where $\rho=\rho_\varepsilon$. Furthermore, h is injective on $\Delta_{\rho(1-\eta)}$.

Proof: If $|z|<\rho(1-4\eta)$ we get

$$|h(z)|\le|z|+|\psi(z)|\le\rho(1-4\eta)+\eta\rho=\rho(1-3\eta),$$

and thus $h\bigl(\Delta_{\rho(1-4\eta)}\bigr)\subset\Delta_{\rho(1-3\eta)}$.

On the other hand, the choice of ε implies that $|h(z)-z|=|\psi(z)|\le\rho\eta$. In particular, if $|z|=\rho(1-\eta)$ we have $|z|-\rho(1-2\eta)=\rho\eta\ge|\psi(z)|$. Therefore if $|z|=\rho(1-\eta)$ we get $|h(z)|=|z+\psi(z)|\ge|z|-|\psi(z)|\ge\rho(1-2\eta)$, and since $h(0)=0$ we have proved the second inclusion.

Finally, the choice of ε and η implies $\|\psi'\|_{0,\Delta_{\rho(1-\eta)}}\le\eta$; therefore if z_1 and z_2 are distinct points of $\Delta_{\rho(1-\eta)}$ we have $|\psi(z_1)-\psi(z_2)|<|z_1-z_2|$, and then $h(z_1)\ne h(z_2)$. □

So set $f_1=h^{-1}\circ f\circ h=\mathcal{F}(f,h)$, and write $f_1=\Lambda+u_1$. We now prove that f_1 is defined on $\Delta_{\rho(1-4\eta)}$ and gives an estimate on u_1'.

Lemma 4.8: *Choose $\varepsilon>0$ and $0<\eta<1$ so that*

$$0<\varepsilon<\eta<\frac{1}{5} \qquad \text{and} \qquad c_1\varepsilon\le\eta^{\beta+2}. \tag{4.11}$$

Then f_1 is defined on $\Delta_{\rho(1-4\eta)}$, and

$$\|u_1'\|_{0,\Delta_{\rho(1-5\eta)}}\le\frac{5c_1}{4\eta^{\beta+2}}\,\varepsilon^2.$$

Proof: We have seen that $h(\Delta_{\rho(1-4\eta)})\subset\Delta_{\rho(1-3\eta)}$. Since $|\lambda|\le1$, on $\Delta_{\rho(1-3\eta)}$ we have

$$|f(z)|\le|\lambda||z|+|u(z)|\le\rho(1-3\eta)+\varepsilon\rho\le\rho(1-2\eta),$$

that is $f\bigl(\Delta_{\rho(1-3\eta)}\bigr)\subset\Delta_{\rho(1-2\eta)}$. The second inclusion in Lemma 4.7 then shows that f_1 is defined on $\Delta_{\rho(1-4\eta)}$, as claimed.

To estimate u_1' we rewrite $h\circ f_1=f\circ h$ as

$$\lambda z+u_1(z)+\psi\bigl(\lambda z+u_1(z)\bigr)=\lambda\bigl(z+\psi(z)\bigr)+u\bigl(h(z)\bigr)$$

or, recalling the relation between u and ψ, as

$$u_1(z)=\psi(\lambda z)-\psi\bigl(\lambda z+u_1(z)\bigr)+u\bigl(h(z)\bigr)-u(z).$$

Now, since $\rho<1$ by definition, on $\Delta_{\rho(1-4\eta)}$ we have

$$\begin{aligned}\bigl|\psi(\lambda z)-\psi\bigl(\lambda z+u_1(z)\bigr)\bigr|&\le\|\psi'\|_{0,\Delta_{\rho(1-\eta)}}\|u_1\|_{0,\Delta_{\rho(1-4\eta)}}\\&\le\frac{c_1\varepsilon\rho}{\eta^{\beta+1}}\|u_1\|_{0,\Delta_{\rho(1-4\eta)}}<\frac{\|u_1\|_{0,\Delta_{\rho(1-4\eta)}}}{5}.\end{aligned}$$

So

$$\frac{4}{5}\|u_1\|_{0,\Delta_{\rho(1-4\eta)}}\le\|u\circ h-u\|_{0,\Delta_{\rho(1-4\eta)}}\le\|u'\|_{0,\Delta_{\rho(1-3\eta)}}\|\psi\|_{0,\Delta_{\rho(1-4\eta)}}\le\frac{c_1\varepsilon^2\rho}{\eta^{\beta+1}}.$$

Then Lemma 4.5.(i) applied to disks with center in $\Delta_{\rho(1-5\eta)}$ and radius $\rho\eta$ yields

$$\|u_1'\|_{0,\Delta_{\rho(1-5\eta)}}\le\frac{5c_1}{4\eta^{\beta+2}}\,\varepsilon^2.$$

□

We are ready to apply these estimates in the inductive process. Put

$$c_2 = 20^{\beta+2}\,\frac{5c_1}{4}\,.$$

Fix $0 < \eta < 1/5$ and choose $\varepsilon = \varepsilon_0 < c_2^{-2}$ and satisfying (4.11). Set

$$\rho_n = \rho\frac{1+2^{-n}}{2} > \frac{\rho}{2},$$

where $\rho = \rho_{\varepsilon_0}$, and $\eta_n = 1/10(2^n+1)$ so that $\rho_{n+1} = \rho_n(1-5\eta_n)$. Furthermore define inductively ε_n by

$$\varepsilon_{n+1} = \frac{5c_1}{4\eta_n^{\beta+2}}\,\varepsilon_n^2.$$

We have $\varepsilon_{n+1} = \frac{5}{4}10^{\beta+2}(2^n+1)^{\beta+2}c_1\varepsilon_n^2 \le c_2^{n+1}\varepsilon_n^2$; so $\varepsilon'_n = c_2^{n+1}\varepsilon_n$ satisfies

$$\varepsilon'_{n+1} = c_2^{n+2}\varepsilon_{n+1} \le c_2^{n+2}c_2^{n+1}\varepsilon_n^2 = c_2{\varepsilon'_n}^2,$$

that is $\varepsilon'_n \le (c_2\varepsilon'_0)^{2^n}/c_2$. Since $c_2\varepsilon'_0 = c_2^2\varepsilon_0 < 1$, we see that ε'_n and ε_n go to zero superexponentially. In particular, it is clear that choosing ε_0 small enough we can verify (4.11) for all $n \ge 0$.

Now assume we have $f_n = \Lambda + u_n$ defined on Δ_{ρ_n} and satisfying $\|u'_n\|_{0,\Delta_{\rho_n}} \le \varepsilon_n$. Apply Lemma 4.6 to get ψ_n; then $h_n = \mathrm{id} + \psi_n$ is defined and injective on $\Delta_{\rho_n(1-\eta_n)}$, by Lemma 4.7, and thus $f_{n+1} = h_n^{-1}\circ f_n \circ h_n$ is defined on $\Delta_{\rho_n(1-4\eta_n)}$ and, by Lemma 4.8, satisfies $\|u'_{n+1}\|_{0,\Delta_{\rho_{n+1}}} \le \varepsilon_{n+1}$.

Set $k_n = h_0\circ\cdots\circ h_{n-1}$, so that $f_n = k_n^{-1}\circ f\circ k_n$; everything is well-defined on $\Delta_{\rho/2}$. Since $u_n(0) = 0$ for all $n\in\mathbb{N}$ and $\|u'_n\|_{0,\Delta_{\rho/2}} \to 0$, if we prove that $\{k_n\}$ converges uniformly on $\Delta_{\rho/2}$ to h (necessarily injective, by Hurwitz's theorem, because $k'_n(0) = 1$ for all $n\in\mathbb{N}$) we shall get that $f_n \to \Lambda = h^{-1}\circ f\circ h$, as wanted.

So we are left to prove the convergence of $\{k_n\}$. By definition,

$$k'_n = \prod_{l=0}^{n-1} h'_l \circ (h_{l+1}\circ\cdots\circ h_{n-1}) = \prod_{l=0}^{n-1}\bigl(1+\psi'_l\circ(h_{l+1}\circ\cdots\circ h_{n-1})\bigr).$$

Now, (4.10) yields $\|\psi'_l\|_{0,\Delta_{\rho/2}} \le c_3^l\varepsilon_l$ for a suitable $c_3 > 0$. But then

$$\prod_{l=0}^{\infty}(1+\|\psi'_l\|_{0,\Delta_{\rho/2}}) \le c_4 < +\infty,$$

and so

$$\|k_{n+1}-k_n\|_{0,\Delta_{\rho/2}} \le c_4\|h_n - \mathrm{id}\,\|_{0,\Delta_{\rho/2}} = c_4\|\psi_n\|_{0,\Delta_{\rho/2}} \le c_4c_5^n\varepsilon_n$$

again for a suitable $c_5 > 0$, and thus $\{k_n\}$ is a Cauchy sequence.

We leave to the reader the verification that the set of $\lambda\in S^1$ verifying (4.6) for some $\gamma > 0$ and $\beta \ge 1$ is of full measure in S^1. □

A bit of terminology is now useful: if $f\in\mathrm{End}(\mathbb{C},0)$ is elliptic, we shall say that the origin is a *Siegel point* if f is holomorphically linearizable; otherwise it is a *Cremer point.*

Theorem 4.4 suggests the existence of a number-theoretical condition on λ ensuring that the origin is a Siegel point for any holomorphic local dynamical system of multiplier λ. And indeed this is the content of the celebrated *Bryuno-Yoccoz theorem:*

Theorem 4.9: (Bryuno, 1965 [Bry1–3], Yoccoz, 1988 [Y1–2]) *Let $\lambda\in S^1$. Then the following statements are equivalent:*

(i) *the origin is a Siegel point for the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$;*
(ii) *the origin is a Siegel point for all $f\in\mathrm{End}(\mathbb{C},0)$ with multiplier λ;*

(iii) *the number λ satisfies Bryuno's condition*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_\lambda(2^{k+1})} < +\infty. \tag{4.12}$$

Bryuno, using majorant series as in Siegel's proof of Theorem 4.4 (see also [He] and references therein) has proved that condition (iii) implies condition (ii). Yoccoz, using a more geometric approach based on conformal and quasi-conformal geometry, has proved that (i) is equivalent to (ii), and that (ii) implies (iii), that is that if λ does not satisfy (4.12) then the origin is a Cremer point for some $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ — and hence it is a Cremer point for the quadratic polynomial $f_\lambda(z)$. See also [P9] for related results.

Remark 4.2: Condition (4.12) is usually expressed in a different way. Write $\lambda = e^{2\pi i\theta}$, and let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions. Then (4.12) is equivalent to

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty,$$

while (4.6) is equivalent to $q_{n+1} = O(q_n^\beta)$, and (4.4) is equivalent to

$$\limsup_{k\to+\infty} \frac{1}{q_k} \log q_{k+1} = +\infty.$$

See [He], [Y2] and references therein for details.

The proof of Theorem 4.9 is beyond the scope of these notes. We shall limit ourselves to report two of the easiest results of [Y2], and to illustrate what is the connection between condition (4.12) and the radius of convergence of the formal linearizing map.

The first result is a weaker (but with a much easier proof) version of Theorem 4.4:

Proposition 4.10: *The origin is a Siegel point of $f_\lambda(z) = \lambda z + z^2$ for almost every $\lambda \in S^1$.*

Proof: (Yoccoz [Y2]) The idea is to study the radius of convergence of the inverse of the linearization of $f_\lambda(z) = \lambda z + z^2$ when $\lambda \in \Delta^*$. Theorem 2.1 says that there is a unique map φ_λ defined in some neighbourhood of the origin such that $\varphi'_\lambda(0) = 1$ and $\varphi_\lambda \circ f = \lambda\varphi_\lambda$. Let ρ_λ be the radius of convergence of φ_λ^{-1}; we want to prove that φ_λ is defined in a neighbourhood of the unique critic point $-\lambda/2$ of f_λ, and that $\rho_\lambda = |\varphi_\lambda(-\lambda/2)|$.

Let $\Omega_\lambda \subset\subset \mathbb{C}$ be the basin of attraction of the origin. Notice that setting $\varphi_\lambda(z) = \lambda^{-k}\varphi_\lambda\bigl(f_\lambda(z)\bigr)$ we can extend φ_λ to the whole of Ω_λ. Moreover, since the image of φ_λ^{-1} is contained in Ω_λ, which is bounded, necessarily $\rho_\lambda < +\infty$. Let $U_\lambda = \varphi_\lambda^{-1}(\Delta_{\rho_\lambda})$. Since we have

$$(\varphi'_\lambda \circ f)f' = \lambda\varphi'_\lambda \tag{4.13}$$

and φ_λ is invertible in U_λ, the function f cannot have critic points in U_λ.

If $z = \varphi_\lambda^{-1}(w) \in U$, we have $f(z) = \varphi_\lambda^{-1}(\lambda w) \in \varphi_\lambda^{-1}(\Delta_{|\lambda|\rho_\lambda}) \subset\subset U_\lambda$; therefore

$$f(\overline{U_\lambda}) \subseteq \overline{f(U_\lambda)} \subset\subset U_\lambda \subseteq \Omega_\lambda,$$

which implies that $\partial U \subset \Omega_\lambda$. So φ_λ is defined on ∂U_λ, and clearly $|\varphi_\lambda(z)| = \rho_\lambda$ for all $z \in \partial U_\lambda$.

If f had no critic points in ∂U_λ, (4.13) would imply that φ_λ has no critic points in ∂U_λ. But then φ_λ would be locally invertible in ∂U_λ, and thus φ_λ^{-1} would extend across $\partial\Delta_{\rho_\lambda}$, impossible. Therefore $-\lambda/2 \in \partial U_\lambda$, and $|\varphi_\lambda(-\lambda/2)| = \rho_\lambda$, as claimed.

(Up to here it was classic; let's now start Yoccoz's argument.) Put $\eta(\lambda) = \varphi_\lambda(-\lambda/2)$. From the proof of Theorem 2.1 one easily see that φ_λ depends holomorphically on λ; so $\eta\colon \Delta^* \to \mathbb{C}$ is holomorphic. Furthermore, since $\Omega_\lambda \subseteq \Delta_2$, Schwarz's lemma applied to $\varphi_\lambda^{-1}\colon \Delta_{\rho_\lambda} \to \Delta_2$ yields

$$1 = |(\varphi_\lambda^{-1})'(0)| \le 2/\rho_\lambda,$$

that is $\rho_\lambda \le 2$. So η is bounded, and thus it extends holomorphically to the origin.

So $\eta\colon \Delta \to \Delta_2$ is a bounded holomorphic function not identically zero; Fatou's theorem then implies that

$$\rho(\lambda_0) := \limsup_{r\to 1^-} |\eta(r\lambda_0)| > 0$$

for almost every $\lambda_0 \in S^1$. This means that we can find $0 < \rho_0 < \rho(\lambda_0)$ and a sequence $\{\lambda_j\} \subset \Delta$ such that $\lambda_j \to \lambda_0$ and $|\eta(\lambda_j)| > \rho_0$. This means that $\varphi_{\lambda_j}^{-1}$ is defined in Δ_{ρ_0} for all $j \ge 1$; up to a subsequence, we can assume that $\varphi_{\lambda_j}^{-1} \to \psi\colon \Delta_{\rho_0} \to \Delta_2$. But then we have $\psi'(0) = 1$ and

$$f_{\lambda_0}\big(\psi(z)\big) = \psi(\lambda_0 z)$$

in Δ_{ρ_0}, and thus the origin is a Siegel point for f_{λ_0}. □

The second result we would like to present is the implication (i) $\Longrightarrow$ (ii) in Theorem 4.9. The proof depends on the following result of Douady and Hubbard, obtained using the theory of quasiconformal maps:

Theorem 4.11: (Douady-Hubbard, 1985 [DH]) *Given $\lambda \in \mathbb{C}^*$, let $f_\lambda(z) = \lambda z + z^2$ be a quadratic polynomial. Then there exists a universal constant $C > 0$ such that for every holomorphic function $\psi\colon \Delta_{3|\lambda|/2} \to \mathbb{C}$ with $\psi(0) = \psi'(0) = 0$ and $|\psi(z)| \le C|\lambda|$ for all $z \in \Delta_{3|\lambda|/2}$ the function $f = f_\lambda + \psi$ is quasiconformally (and hence topologically) conjugated to f_λ in $\Delta_{|\lambda|}$.*

Then

Theorem 4.12: (Yoccoz [Y2]) *Let $\lambda \in S^1$ be such that the origin is a Siegel point for $f_\lambda(z) = \lambda z + z^2$. Then the origin is a Siegel point for every $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ.*

Proof: Write $f(z) = \lambda z + \sum_{k\ge 2} a_k z^k$; up to a linear change of coordinates we can assume $|a_k| \le (C/4)2^{-k}$ for all $k \ge 2$, where C is the constant given by Theorem 4.11 (actually, $C \ge 10^{-2}/2$); in particular, f is defined on Δ_2. Set now $f^b(z) = f(z) + bz^2$. Since λ is not a root of unity, there exists a unique formal power series $\hat h^b \in \mathbb{C}[[z]]$ tangent to the identity such that $f^b \circ \hat h^b(z) = \hat h^b(\lambda z)$, and we can write

$$\hat h^b(z) = z + \sum_{k\ge 2} h_k(b) z^k,$$

where $h_k(b)$ is a polynomial in b (see the proof of Proposition 4.2). In particular, by the maximum principle we have

$$|h_k(0)| \le \max_{|b|=1} |h_k(b)| \tag{4.14}$$

for all $k \ge 2$. Now, if $|b| = 1$ then (possibly after conjugating with a rotation) we have $f^b(z) = f_\lambda(z) + \psi(z)$ where $\psi(0) = \psi'(0) = 0$ and $|\psi(z)| < C$ for all $z \in \Delta_{3/2}$. Theorem 4.11 then implies that f^b is topologically conjugated to f_λ. Now, by assumption, f_λ is linearizable; therefore (Proposition 4.1) 0 is stable for f_λ. But stability is a topological property; therefore the origin is stable for f^b too, and thus f^b is holomorphically linearizable. This implies that $\hat h^b$ is convergent, and thus there exist constants M, $r > 0$ such that $|h_k(b)| \le M r^{-k}$ for all b of modulus 1. Recalling (4.14), it follows that $|h_k(0)| \le M r^{-k}$ for all $k \ge 2$, and so $\hat h^0$ is convergent, that is f is holomorphically linearizable. □

Finally, we introduce the connection between condition (4.12) and linearization. From the function theoretical side, given $\theta \in [0, 1)$ set

$$r(\theta) = \inf\{r(f) \mid f \in \mathrm{End}(\mathbb{C}, 0) \text{ has multiplier } e^{2\pi i\theta} \text{ and it is defined and injective in } \Delta\},$$

where $r(f) \ge 0$ is the radius of convergence of the unique formal linearization of f tangent to the identity.

From the number theoretical side, given an irrational $\theta \in [0, 1)$ let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions, and put

$$\alpha_n = -\frac{q_n\theta - p_n}{q_{n-1}\theta - p_{n-1}}, \qquad \alpha_0 = \theta,$$
$$\beta_n = (-1)^n (q_n\theta - p_n), \qquad \beta_{-1} = 1.$$

The *Bryuno function* $B\colon [0,1)\setminus\mathbb{Q}\to(0,+\infty]$ is then defined by

$$B(\theta)=\sum_{n=0}^{\infty}\beta_{n-1}\log\frac{1}{\alpha_n}.$$

Then Theorem 4.9 is consequence of what we have seen and the following

Theorem 4.13: (Yoccoz [Y2]) (i) *$B(\theta)<+\infty$ if and only if $\lambda=e^{2\pi i\theta}$ satisfies Bryuno's condition* (4.12)*;*
(ii) *if $B(\theta)=+\infty$ then there exists a non-linearizable $f\in\mathrm{End}(\mathbb{C},0)$ with multiplier $e^{2\pi i\theta}$;*
(iii) *there exists a universal constant $C>0$ such that*

$$|\log r(\theta)+B(\theta)|\le C$$

for all $\theta\in[0,1)\setminus\mathbb{Q}$ such that $B(\theta)<+\infty$.

If 0 is a Siegel point for $f\in\mathrm{End}(\mathbb{C},0)$, the local dynamics of f is completely clear, and simple enough. On the other hand, if 0 is a Cremer point of f, then the local dynamics of f is very complicated and not yet completely understood. Pérez-Marco (in [P2, 4–7]) has studied the topology and the dynamics of the stable set in this case. Some of his results are summarized in the following

Theorem 4.14: (Pérez-Marco, 1995 [P6, 7]) *Assume that 0 is a Cremer point for an elliptic holomorphic local dynamical system $f\in\mathrm{End}(\mathbb{C},0)$. Then:*
(i) *The stable set K_f is compact, connected, full (i.e., $\mathbb{C}\setminus K_f$ is connected), it is not reduced to $\{0\}$, and it is not locally connected at any point distinct from the origin.*
(ii) *Any point of $K_f\setminus\{0\}$ is recurrent (that is, a limit point of its orbit).*
(iii) *There is an orbit in K_f which accumulates at the origin, but no non-trivial orbit converges to the origin.*

Remark 4.3: So, if $\lambda\in S^1$ is not a root of unity and does not satisfy Bryuno's condition (4.12), we can find f_1, $f_2\in\mathrm{End}(\mathbb{C},0)$ with multiplier λ such that f_1 is holomorphically linearizable while f_2 is not. Then f_1 and f_2 are formally conjugated without being neither holomorphically nor topologically locally conjugated.

Remark 4.4: As far as I know, there are neither a topological nor a holomorphic complete classification of elliptic holomorphic dynamical systems with a Cremer point. Actually, the holomorphic classification might be unfeasible: indeed, Yoccoz [Y2] has proved that if $\lambda\in S^1$ is not a root of unity and does not satisfy Bryuno's condition (4.12) then there is an uncountable family of germs in $\mathrm{End}(\mathbb{C},O)$ with multiplier λ which are not holomorphically conjugated nor holomorphically conjugated to any entire function.

See also [P1, 3] for other results on the dynamics about a Cremer point.

5. Several complex variables: the hyperbolic case

Now we start the discussion of local dynamics in several complex variables. In this case the theory is much less complete than its one-variable counterpart.

Let $f\in\mathrm{End}(\mathbb{C}^n,O)$ be a holomorphic local dynamical system at $O\in\mathbb{C}^n$, with $n\ge2$. We can write f using a *homogeneous expansion*

$$f(z)=P_1(z)+P_2(z)+\cdots\in\mathbb{C}_0\{z_1,\dots,z_n\}^n,$$

where P_j is an n-uple of homogeneous polynomials of degree j. In particular, P_1 is the differential df_O of f at the origin, and f is locally invertible if and only if P_1 is invertible.

We have seen that in dimension one the multiplier (i.e., the derivative at the origin) plays a main rôle. When $n>1$, a similar rôle is played by the eigenvalues of the differential. Thus we introduce the following definitions:

– if all eigenvalues of df_O have modulus less than 1, we say that the fixed point O is *attracting;*
– if all eigenvalues of df_O have modulus greater than 1, we say that the fixed point O is *repelling;*
– if all eigenvalues of df_O have modulus different from 1, we say that the fixed point O is *hyperbolic* (notice that we allow the eigenvalue zero);

– if all eigenvalues of df_O are roots of unity, we say that the fixed point O is *parabolic;* in particular, if $df_O = \mathrm{id}$ we say that f is *tangent to the identity;*
– if all eigenvalues of df_O have modulus 1 but none is a root of unity, we say that the fixed point O is *elliptic;*
– if $df_O = O$, we say that the fixed point O is *superattracting*.

Other cases are clearly possible, but for our aims this list is enough. In this survey we shall be mainly concerned with hyperbolic and parabolic fixed points; however, in the last section we shall also present some results valid in other cases.

Let us begin assuming that the origin is a hyperbolic fixed point for an $f \in \mathrm{End}(\mathbb{C}^n, O)$ not necessarily invertible. We then have a canonical splitting

$$\mathbb{C}^n = E^s \oplus E^u,$$

where E^s (respectively, E^u) is the direct sum of the generalized eigenspaces associated to the eigenvalues of df_O with modulus less (respectively, greater) than 1. Then the first main result in this subject is the famous *stable manifold theorem* (originally due to Perron [Pe] and Hadamard [H]; see [FHY, HK, HPS, Pes, Sh] for proofs in the C^∞ category, Wu [Wu] for a proof in the holomorphic category, and [A3] for a proof in the non-invertible case):

Theorem 5.1: *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system with a hyperbolic fixed point at the origin. Then:*

(i) *the stable set K_f is an embedded complex submanifold of (a neighbourhood of the origin in) $\mathbb{C}^n$, tangent to E^s at the origin;*
(ii) *there is an embedded complex submanifold W_f of (a neighbourhood of the origin in) $\mathbb{C}^n$, called the unstable set of f, tangent to E^u at the origin, such that $f|_{W_f}$ is invertible, $f^{-1}(W_f) \subseteq W_f$, and $z \in W_f$ if and only if there is a sequence $\{z_{-k}\}_{k\in\mathbb{N}}$ in the domain of f such that $z_0 = z$ and $f(z_{-k}) = z_{-k+1}$ for all $k \geq 1$. Furthermore, if f is invertible then W_f is the stable set of f^{-1}.*

The proof is too involved to be summarized here; it suffices to say that both K_f and W_f can be recovered, for instance, as fixed points of a suitable contracting operator in an infinite dimensional space (see the references quoted above for details).

Remark 5.1: If the origin is an attracting fixed point, then $E^s = \mathbb{C}^n$, and K_f is an open neighbourhood of the origin, its *basin of attraction*. However, as we shall discuss below, this does not imply that f is holomorphically linearizable, not even when it is invertible. Conversely, if the origin is a repelling fixed point, then $E^u = \mathbb{C}^n$, and $K_f = \{O\}$. Again, not all holomorphic local dynamical systems with a repelling fixed point are holomorphically linearizable.

If a point in the domain U of a holomorphic local dynamical system with a hyperbolic fixed point does not belong either to the stable set or to the unstable set, it escapes both in forward time (that is, its orbit escapes) and in backward time (that is, it is not the end point of an infinite orbit contained in U). In some sense, we can think of the stable and unstable sets (or, as they are usually called in this setting, stable and unstable *manifolds*) as skewed coordinate planes at the origin, and the orbits outside these coordinate planes follow some sort of hyperbolic path, entering and leaving any neighbourhod of the origin in finite time.

Actually, this idea of straightening stable and unstable manifolds can be brought to fruition (at least in the invertible case), and it yields one of the possible proofs (see [HK, Sh, A3] and references therein) of the *Grobman-Hartman theorem:*

Theorem 5.2: (Grobman, 1959 [G1–2]; Hartman, 1960 [Har]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point. Then f is topologically locally conjugated to its differential df_O.*

Thus, at least from a topological point of view, the local dynamics about an invertible hyperbolic fixed point is completely clear. This is definitely not the case if the local dynamical system is not invertible in a neighbourhood of the fixed point. For instance, already Hubbard and Papadopol [HP] noticed that a Böttcher-type theorem for superattracting points in several complex variables is just not true: there are holomorphic local dynamical systems with a superattracting fixed point which are not even topologically locally conjugated to the first non-vanishing term of their homogeneous expansion. Recently, Favre and Jonsson [FJ] have begun a very detailed study of superattracting fixed points in $\mathbb{C}^2$, study that should lead to their topological classification.

The holomorphic and even the formal classification are not as simple as the topological one. The main problem is that, if we denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ the eigenvalues of df_O, then it may happen that

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0 \tag{5.1}$$

for some $1 \le j \le n$ and some $k_1, \ldots, k_n \in \mathbb{N}$ with $k_1 + \cdots + k_n \ge 2$; a relation of this kind is called a *resonance* of f. When $n = 1$ there is a resonance if and only if the multiplier is a root of unity, or zero; but if $n > 1$ resonances may occur in the hyperbolic case too. Anyway, a computation completely analogous to the one yielding Proposition 4.2 proves the following

Proposition 5.3: *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a (locally invertible) holomorphic local dynamical system with a hyperbolic fixed point and no resonances. Then f is formally conjugated to its differential df_O.*

In presence of resonances, even the formal classification is not that easy. Let us assume, for simplicity, that df_O is in Jordan form, that is

$$P_1(z) = (\lambda_1 z, \epsilon_2 z_1 + \lambda_2 z_2, \ldots, \epsilon_n z_{n-1} + \lambda_n z_n),$$

with $\epsilon_1, \ldots, \epsilon_{n-1} \in \{0, 1\}$. We shall say that a monomial $z_1^{k_1} \cdots z_n^{k_n}$ in the j-th coordinate of f is *resonant* if $k_1 + \cdots + k_n \ge 2$ and $\lambda_1^{k_1} \cdots \lambda_n^{k_n} = \lambda_j$. Then the Proposition 5.3 can be generalized to

Proposition 5.4: *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point. Then it is formally conjugated to a $g \in \mathbb{C}_0[[z_1, \ldots, z_n]]^n$ such that dg_O is in Jordan normal form, and g has only resonant monomials.*

The formal series g is called *Poincaré-Dulac normal form* of f; see Arnold [Ar] for a proof of Proposition 5.4.

The problem with Poincaré-Dulac normal forms is that they are not unique. In particular, one may wonder whether it could be possible to have such a normal form including *finitely many* resonant monomials only (as happened, for instance, in Proposition 3.3). This is indeed the case (see, e.g., Reich [R1]) when df_O belongs to the so-called *Poincaré domain*, that is when df_O is invertible and O is either attracting or repelling (when df_O is still invertible but does not belong to the Poincaré domain, we shall say that it belongs to the *Siegel domain*). As far as I know, the problem of finding canonical formal normal forms when df_O belongs to the Siegel domain (and f is hyperbolic) is still open.

It should be remarked that, in the hyperbolic case, the problem of formal linearization is equivalent to the problem of smooth linearization. This has been proved by Sternberg [St1–2] and Chaperon [Ch]:

Theorem 5.5: (Sternberg, 1957 [St1–2]; Chaperon, 1986 [Ch]) *Let f, $g \in \mathrm{End}(\mathbb{C}^n, O)$ be two holomorphic local dynamical systems, and assume that f is locally invertible and with a hyperbolic fixed point at the origin. Then f and g are formally conjugated if and only if they are smoothly locally conjugated. In particular, f is smoothly linearizable if and only if it is formally linearizable. Thus if there are no resonances then f is smoothly linearizable.*

Even without resonances, the holomorphic linearizability is not guaranteed. The easiest positive result is due to Poincaré [Po] who, using majorant series, proved the following

Theorem 5.6: (Poincaré, 1893 [Po]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system with an attracting or repelling fixed point. Then f is holomorphically linearizable if and only if it is formally linearizable. In particular, if there are no resonances then f is holomorphically linearizable.*

Reich [R2] describes holomorphic normal forms when df_O belongs to the Poincaré domain and there are resonances (see also [ÉV]); Pérez-Marco [P8] discusses the problem of holomorphic linearization in the presence of resonances.

When df_O belongs to the Siegel domain, even without resonances, the formal linearization might diverge. To describe the known results, let us introduce the following quantity:

$$\Omega_{\lambda_1,\ldots,\lambda_n}(m) = \min\bigl\{|\lambda_1^{k_1}\cdots\lambda_n^{k_n} - \lambda_j| \bigm| k_1,\ldots,k_n \in \mathbb{N},\, 2 \le k_1 + \cdots + k_n \le m,\, 1 \le j \le n\bigr\}$$

for $m \ge 2$ and $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$. In particular, if $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of df_O, we shall write $\Omega_f(m)$ for $\Omega_{\lambda_1,\ldots,\lambda_n}(m)$.

It is clear that $\Omega_f(m) \ne 0$ for all $m \ge 2$ if and only if there are no resonances. It is also not difficult to prove that if df_O belongs to the Siegel domain then

$$\lim_{m\to+\infty} \Omega_f(m) = 0,$$

which is the reason why, even without resonances, the formal linearization might be diverging, exactly as in the one-dimensional case. As far as I know, the best positive and negative results in this setting are due to Bryuno [Bry2–3], and are a natural generalization of their one-dimensional counterparts:

Theorem 5.7: (Bryuno, 1971 [Bry2–3]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system such that df_O belongs to the Siegel domain, is linearizable and has no resonances. Assume moreover that*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_f(2^{k+1})} < +\infty. \tag{5.2}$$

Then f is holomorphically linearizable.

Theorem 5.8: *Let $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ be without resonances and such that*

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\Omega_{\lambda_1,\ldots,\lambda_n}(m)} = +\infty.$$

Then there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$, with $df_O = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, not holomorphically linearizable.

Remark 5.2: These theorems hold even without hyperbolicity assumptions.

It should be remarked that, contrarily to the one-dimensional case, it is not known whether condition (5.2) is necessary for the holomorphic linearizability of all holomorphic local dynamical systems with a given linear part belonging to the Siegel domain. See also Pöschel [Pö] for a generalization of Theorem 5.7, and Il'yachenko [I1] for an important result related to Theorem 5.8. Finally, in [DG] are discussed results in the spirit of Theorem 5.7 without assuming that the differential is diagonalizable.

6. Several complex variables: the parabolic case

A first natural question in the several complex variables parabolic case is whether a result like the Leau-Fatou flower theorem holds, and, if so, in which form. To present what is known on this subject in this section we shall restrict our attention to holomorphic local dynamical systems tangent to the identity; consequences on dynamical systems with a more general parabolic fixed point can be deduced taking a suitable iterate (but see also the end of this section for results valid when the differential at the fixed point is not diagonalizable).

So we are interested in the local dynamics of a holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z) = z + P_\nu(z) + P_{\nu+1}(z) + \cdots \in \mathbb{C}_0\{z_1,\ldots,z_n\}^n,$$

where P_ν is the first non-zero term in the homogeneous expansion of f; the number $\nu \ge 2$ is the *order* of f.

The two main ingredients in the statement of the Leau-Fatou flower theorem were the attracting directions and the petals. Let us first describe a several variables analogue of attracting directions.

Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent at the identity and of order ν. A *characteristic direction* for f is a non-zero vector $v \in \mathbb{C}^n \setminus \{O\}$ such that $P_\nu(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. If $P_\nu(v) = O$ (that is, $\lambda = 0$) we shall say that v is a *degenerate* characteristic direction; otherwise, (that is, if $\lambda \ne 0$) we shall say that v is *non-degenerate*.

There is an equivalent definition of characteristic directions that shall be useful later on. The n-uple of ν-homogeneous polynomial P_ν induces a meromorphic self-map of $\mathbb{P}^{n-1}(\mathbb{C})$, still denoted by P_ν. Then, under the canonical projection $\mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ that we shall denote by $v \mapsto [v]$, the non-degenerate characteristic directions correspond exactly to fixed points of P_ν, and the degenerate characteristic directions correspond exactly to indeterminacy points of P_ν. By the way, using Bezout's theorem it is easy to prove (see, e.g., [AT]) that the number of characteristic directions of f, counted according to a suitable multiplicity, is given by $(\nu^n - 1)/(\nu - 1)$.

Remark 6.1: The characteristic directions are *complex* directions; in particular, it is easy to check that f and f^{-1} have the same characteristic directions. Later we shall see how to associate to (most) characteristic directions $\nu - 1$ petals, each one in some sense centered about a *real* attracting direction corresponding to the same complex characteristic direction.

The notion of characteristic directions has a dynamical origin. We shall say that an orbit $\{f^k(z_0)\}$ converges to the origin *tangentially* to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ if $f^k(z_0) \to O$ in $\mathbb{C}^n$ and $[f^k(z_0)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$. Then

Proposition 6.1: *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic dynamical system tangent to the identity. If there is an orbit of f converging to the origin tangentially to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, then v is a characteristic direction of f.*

Sketch of proof: ([Ha2]) For simplicity let us assume $\nu = 2$; a similar argument works for $\nu > 2$.

If v is a degenerate characteristic direction, there is nothing to prove. If not, up to a linear change of coordinates we can write

$$\begin{cases} f_1(z) = z_1 + p_2^1(z_1, z') + p_3^1(z_1, z') + \cdots, \\ f'(z) = z' + p_2'(z_1, z') + p_3'(z_1, z') + \cdots, \end{cases}$$

where $z' = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$, $f = (f_1, f')$, $P_j = (p_j^1, p_j')$ and so on, with $v = (1, v')$ and $p_2^1(1, v') \neq 0$. Making the substitution

$$\begin{cases} w_1 = z_1, \\ z' = w' z_1, \end{cases} \tag{6.1}$$

which is a change of variable outside the hyperplane $z_1 = 0$, the map f becomes

$$\begin{cases} \tilde{f}_1(w) = w_1 + p_2^1(1, w') w_1^2 + p_3^1(1, w') w_1^3 + \cdots, \\ \tilde{f}'(w) = w' + r(w') w_1 + O(w_1^2), \end{cases} \tag{6.2}$$

where $r(w')$ is a polynomial such that $r(v') = O$ if and only if $(1, v')$ is a characteristic direction of f with $p_2^1(1, v') \neq 0$.

Now, the hypothesis is that there exists an orbit $\{f^k(z_0)\}$ converging to the origin and such that $[f^k(z_0)] \to [v]$. Writing $\tilde{f}^k(w_0) = \big(w_1^k, (w')^k\big)$, this implies that $w_1^k \to 0$ and $(w')^k \to v'$. Then it is not difficult to prove that

$$\lim_{k \to +\infty} \frac{1}{k w_1^k} = -p_2^1(1, v')$$

and then that $(w')^{k+1} - (w')^k$ is of the order of $r(v')/k$, which implies $r(v') = O$, as claimed. $\square$

Remark 6.2: There are (unfortunately?) examples of germs $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity with an orbit converging to the origin which is not tangent to any direction: for instance

$$f(z, w) = \big(z + \alpha z w, h(w)\big)$$

with $\alpha \in \mathbb{C}^*$ and $h \in \mathrm{End}(\mathbb{C}, 0)$ such that $h'(0) = 1$, $h''(0) \neq 0$, α and $\mathrm{Re}\big(\alpha/h''(0)\big) = 1$ (see [Ri1]).

The several variables analogue of a petal is instead given by the notion of parabolic curve. A *parabolic curve* for $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity is an injective holomorphic map $\varphi\colon \Delta \to \mathbb{C}^n \setminus \{O\}$ satisfying the following properties:

(a) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;

(b) φ is continuous at the origin, and $\varphi(0) = O$;
(c) $\varphi(\Delta)$ is f-invariant, and $(f|_{\varphi(\Delta)})^k \to O$ uniformly on compact subsets as $k \to +\infty$.

Furthermore, if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$ as $\zeta \to 0$ in Δ, we shall say that the parabolic curve φ is *tangent* to the direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$.

Then the first main generalization of the Leau-Fatou flower theorem to several complex variables is

Theorem 6.2: (Écalle, 1985 [É3]; Hakim, 1998 [Ha2]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Then for any non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ there exist (at least) $\nu - 1$ parabolic curves for f tangent to $[v]$.*

Sketch of proof: Écalle proof is based on his theory of resurgence of divergent series; we shall describe here the ideas behind Hakim's proof, which depends on more standard arguments.

For the sake of simplicity, let us assume $n = 2$; without loss of generality we can also assume $[v] = [1 : 0]$. Then after a linear change of variables and a transformation of the kind (6.1) we are reduced to prove the existence of a parabolic curve at the origin for a map of the form

$$\begin{cases} f_1(z) = z_1 - z_1^\nu + O(z_1^{\nu+1}, z_1^\nu z_2), \\ f_2(z) = z_2\bigl(1 - \lambda z_1^{\nu-1} + O(z_1^\nu, z_1^{\nu-1} z_2)\bigr) + z_1^\nu \psi(z), \end{cases}$$

where ψ is holomorphic with $\psi(O) = 0$, and $\lambda \in \mathbb{C}$. Given $\delta > 0$, set $D_{\delta,\nu} = \{\zeta \in \mathbb{C} \mid |\zeta^{\nu-1} - \delta| < \delta\}$; this open set has $\nu - 1$ connected components, all of them satisfying condition (a) on the domain of a parabolic curve. Furthermore, if u is a holomorphic function defined on one of these connected components, of the form $u(\zeta) = \zeta^2 u_o(\zeta)$ for some bounded holomorphic function u_o, and such that

$$u\bigl(f_1(\zeta, u(\zeta))\bigr) = f_2\bigl(\zeta, u(\zeta)\bigr), \tag{6.3}$$

then it is not difficult to verify that $\varphi(\zeta) = \bigl(\zeta, u(\zeta)\bigr)$ is a parabolic curve for f tangent to $[v]$.

So we are reduced to finding a solution of (6.3) in each connected component of $D_{\delta,\nu}$, with δ small enough. For any holomorphic $u = \zeta^2 u_o$ defined in such a connected component, let $f_u(\zeta) = f_1\bigl(\zeta, u(\zeta)\bigr)$, put

$$H(z) = z_2 - \frac{z_1^\lambda}{f_1(z)^\lambda} f_2(z),$$

and define the operator T by setting

$$(Tu)(\zeta) = \zeta^\lambda \sum_{k=0}^{\infty} \frac{H\bigl(f_u^k(\zeta), u\bigl(f_u^k(\zeta)\bigr)\bigr)}{f_u^k(\zeta)^\lambda}.$$

Then, if $\delta > 0$ is small enough, it is possible to prove that T is well-defined, that u is a fixed point of T if and only if it satisfies (6.3), and that T is a contraction of a closed convex set of a suitable complex Banach space — and thus it has a fixed point. In this way if $\delta > 0$ is small enough we get a unique solution of (6.3) for each connected component of $D_{\delta,\nu}$, and hence $\nu - 1$ parabolic curves tangent to $[v]$. □

A set of $\nu - 1$ parabolic curves obtained in this way will be called a *Fatou flower* for f tangent to $[v]$.

Remark 6.3: It should be remarked that a similar result for 2-dimensional maps with $\lambda \notin \mathbb{N}^*$ has been obtained by Weickert [W] too; the computations needed in the proof for the case $\lambda \in \mathbb{N}^*$ are considerably harder, and were not carried out by him.

Remark 6.4: When there is a one-dimensional f-invariant complex submanifold passing through the origin tangent to a characteristic direction $[v]$, the previous theorem is just a consequence of the usual one-dimensional theory. But it turns out that in most cases such an f-invariant complex submanifold does not exist: see [Ha2] for a concrete example, and [É3] for a general discussion.

We can also have f-invariant complex submanifolds of dimension strictly greater than one still attracted by the origin. Given a holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity of order $\nu \geq 2$, and a non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, the eigenvalues $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ of the linear operator $d(P_\nu)_{[v]} - \mathrm{id}\colon T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}) \to T_{[v]}\mathbb{P}^{n-1}(\mathbb{C})$ will be called the *directors* of $[v]$. Then, using a more elaborate version of her proof of Theorem 6.2, Hakim has been able to prove the following:

Theorem 6.3: (Hakim, 1997 [Ha3]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction, with directors $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$. Furthermore, assume that $\mathrm{Re}\,\alpha_1, \ldots, \mathrm{Re}\,\alpha_d > 0$ and $\mathrm{Re}\,\alpha_{d+1}, \ldots, \mathrm{Re}\,\alpha_{n-1} \leq 0$ for a suitable $d \geq 0$. Then:*

(i) *There exists an f-invariant $(d+1)$-dimensional complex submanifold M of $\mathbb{C}^n$, with the origin in its boundary, such that the orbit of every point of M converges to the origin tangentially to $[v]$;*
(ii) *$f|_M$ is holomorphically conjugated to the translation $\tau(w_0, w_1, \ldots, w_d) = (w_0 + 1, w_1, \ldots, w_d)$ defined on a suitable right half-space in $\mathbb{C}^{d+1}$.*

Remark 6.5: In particular, if all the directors of $[v]$ have positive real part, there is an open domain attracted by the origin. However, the condition given by Theorem 6.3 is not necessary for the existence of such an open domain; see Rivi [Ri1] for an easy example, and Ushiki [Us] for a more elaborate example with an open domain attracted by the origin where f cannot be conjugate to a translation.

In his monumental work [É3] Écalle has given a complete set of formal invariants for holomorphic local dynamical systems tangent to the identity with at least one non-degenerate characteristic direction. For instance, he has proved the following

Theorem 6.4: (Écalle, 1985 [É3]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Assume that*

(a) *f has exactly $(\nu^n - 1)/(\nu - 1)$ distinct non-degenerate characteristic directions and no degenerate characteristic directions;*
(b) *the directors of any non-degenerate characteristic direction are irrational and mutually independent over $\mathbb{Z}$.*

Choose a non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, and let $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ be its directors. Then there exist a unique $\rho \in \mathbb{C}$ and unique (up to dilations) formal series $R_1, \ldots, R_n \in \mathbb{C}[[z_1, \ldots, z_n]]$, where each R_j contains only monomial of total degree at least $\nu+1$ and of partial degree in z_j at most $\nu - 2$, such that f is formally conjugated to the time-1 map of the formal vector field

$$X = \frac{1}{(\nu-1)(1+\rho z_n^{\nu-1})} \left\{ [-z_n^\nu + R_n(z)] \frac{\partial}{\partial z_n} + \sum_{j=1}^{n-1} [-\alpha_j z_n^{\nu-1} z_j + R_j(z)] \frac{\partial}{\partial z_j} \right\}.$$

Other approaches to the formal classification, at least in dimension 2, are described in [BM] and in [AT2].

Furthermore, using his theory of resurgence, and always assuming the existence of at least one non-degenerate characteristic direction, Écalle has also provided a set of holomorphic invariants for holomorphic local dynamical systems tangent to the identity, in terms of differential operators with formal power series as coefficients. Moreover, if the directors of all non-degenerate characteristic direction are irrational and satisfy a suitable diophantine condition, then these invariants become a complete set of invariants. See [É4] for a description of his results, and [É3] for the details.

Now, all these results beg the question: what happens when there are no non-degenerate characteristic directions? For instance, this is the case for

$$\begin{cases} f_1(z) = z_1 + b z_1 z_2 + z_2^2, \\ f_2(z) = z_2 - b^2 z_1 z_2 - b z_2^2 + z_1^3, \end{cases}$$

for any $b \in \mathbb{C}^*$, and it is easy to build similar examples of any order. At present, the theory in this case is satisfactorily developed for $n = 2$ only. In particular, in [A2] is proved the following

Theorem 6.5: (Abate, 2001 [A2]) *Every holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity, with an isolated fixed point, admits at least one Fatou flower tangent to some direction.*

Remark 6.6: Bracci and Suwa have proved a version of Theorem 6.5 for $f \in \mathrm{End}(M, p)$ where M is a *singular* variety with not too bad a singularity at p; see [BrS] for details.

Let us describe the main ideas in the proof of Theorem 6.5, because they provide some insight on the dynamical structure of holomorphic local dynamical systems tangent to the identity, and on how to deal with it.

The first idea is to exploit in a systematic way the transformation (6.1), following a procedure borrowed from algebraic geometry. If p is a point in a complex manifold M, there is a canonical way to build a complex manifold $\tilde{M}$, called the *blow-up* of M at p, provided with a holomorphic projection $\pi\colon \tilde{M} \to M$, and such that $E = \pi^{-1}(p)$, the *exceptional divisor* of the blow-up, is canonically biholomorphic to $\mathbb{P}(T_pM)$, and $\pi|_{\tilde{M}\setminus E}\colon \tilde{M} \setminus E \to M \setminus \{p\}$ is a biholomorphism. In suitable local coordinates, the map π is exactly given by (6.1). Furthermore, if $f \in \mathrm{End}(M, p)$ is tangent to the identity, there is a unique way to lift f to a map $\tilde{f} \in \mathrm{End}(\tilde{M}, E)$ such that $\pi \circ \tilde{f} = f \circ \pi$, where $\mathrm{End}(\tilde{M}, E)$ is the set of holomorphic maps defined in a neighbourhood of E with values in $\tilde{M}$ and which are the identity on E. In particular, the characteristic directions of f become points in the domain of $\tilde{f}$.

This approach allows to determine which characteristic directions are dynamically meaningful. Take $f = (f_1, f_2) \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity; if $\ell = \gcd(f_1 - z_1, f_2 - z_2)$, we can write

$$f_j(z) = z_j + \ell(z) g_j(z)$$

for $j = 1, 2$, with g_1 and g_2 relatively prime in $\mathbb{C}\{z_1, z_2\}$. We shall say that O is a *singular point* for f if $g_1(O) = g_2(O) = 0$. Clearly, if O is an isolated fixed point of f then it is singular; but if O is not an isolated fixed point (i.e., $\ell \not\equiv 1$) it might not be singular. Only singular points are dynamically meaningful, because a not too difficult computation (see [A2], and [AT1] for an n-dimensional generalization) yields the following

Proposition 6.6: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a holomorphic local dynamical system tangent to the identity. If the fixed point O is not singular, then K_f reduces to the fixed point set of f.*

Now, if $\tilde{M}$ is the blow-up of $\mathbb{C}^2$ at the origin, then the lift $\tilde{f}$ of f belongs to $\mathrm{End}(\tilde{M}, [v])$ for any direction $[v] \in \mathbb{P}^1(\mathbb{C}) = E$. We shall then say that $[v] \in \mathbb{P}^1(\mathbb{C})$ is a *singular direction* of f if it is a singular point for $\tilde{f}$. It turns out that non-degenerate characteristic directions are always singular (but the converse does not necessarily hold), and that singular directions are always characteristic (but the converse does not necessarily hold): the singular directions are the dynamically interesting characteristic directions.

The important feature of the blow-up procedure is that even though the underlying manifold becomes more complex, the lifted maps become simpler. Indeed, using an argument similar to one (described, for instance, in [MM]) used in the study of singular holomorphic foliations of 2-dimensional complex manifolds, in [A2] it is shown that after a finite number of blow-ups our original holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^2, O)$ can be lifted to a map $\tilde{f}$ whose singular points (are finitely many and) satisfy one of the following conditions:

(o) they are *dicritical*, that is with infinitely many singular directions; or,

($\star$) in suitable local coordinates centered at the singular point we can write

$$\begin{cases} \tilde{f}_1(z) = z_1 + \ell(z)\bigl(\lambda_1 z_1 + O(\|z\|^2)\bigr), \\ \tilde{f}_2(z) = z_2 + \ell(z)\bigl(\lambda_2 z_2 + O(\|z\|^2)\bigr), \end{cases}$$

with

($\star_1$) $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \notin \mathbb{N}$, or

($\star_2$) $\lambda_1 \neq 0$, $\lambda_2 = 0$.

Remark 6.7: This "reduction of the singularities" statement holds only in dimension 2, and it is not clear how to replace it in higher dimensions.

It is not too difficult to prove that Theorem 6.2 (actually, the "easy" case of this theorem) can be applied both to dicritical and to ($\star_1$) singularities; therefore as soon as this blow-up procedure produces such a singularity, we get a Fatou flower for the original dynamical system f.

So to end the proof of Theorem 6.5 it suffices to prove that any such blow-up procedure *must* produce at least one dicritical or ($\star_1$) singularity. To get such a result, we need a completely new ingredient.

Let E be a compact Riemann surface inside a 2-dimensional complex manifold (for instance, E can be the exceptional divisor of the blow-up of a point p), and take $f \in \mathrm{End}(M, E)$ tangent to the identity to all points of E (this happens, for instance, if f is the lifting of a map tangent at the identity at p). Given $q \in E$, choose local coordinates (z_1, z_2) in M centered at q and such that E is locally given by $\{z_2 = 0\}$. Then the function

$$k(z_1) = \lim_{z_2 \to 0} \frac{f_2(z) - z_2}{z_2(f_1(z) - z_1)}$$

is either a meromorphic function defined in a neighbourhood of q, or identically ∞. It turns out that:

- if k is identically ∞ at one point $q \in E$, it is identically ∞ at all points of E; in this case we shall say that f is *not tangential* to E;
- if f is tangential to E (this happens, for instance, if f is obtained blowing up a non-dicritical singularity), then the residue of k at q is independent of the local coordinates used to define k, and it is called the *index* $\iota_q(f, E)$ of f at q along E;
- if f is tangential to E, and $q \in E$ is not singular for f, then $\iota_q(f, E) = 0$; in particular, $\iota_q(f, E) \neq 0$ only for a finite number of points of E.

Then following an argument suggested by Camacho and Sad [CS] in their study of the separatrices of holomorphic foliations it is possible to prove the following *index theorem:*

Theorem 6.7: (Abate, 2001 [A2]) *Let E be a compact Riemann surface inside a 2-dimensional complex manifold M. Take $f \in \mathrm{End}(M, E)$ such that f is tangent to the identity at all points of E, and assume that f is tangential to E. Then*

$$\sum_{q \in E} \iota_q(f, E) = c_1(N_E),$$

where $c_1(N_E)$ is the first Chern class of the normal bundle N_E of E in M.

Remark 6.8: If f is the lift to the blow-up of a map tangent to the identity, and $[v] \in E$ is a non-degenerate characteristic direction with non-zero director α, then $\iota_{[v]}(f, E) = 1/\alpha$.

Remark 6.9: Theorem 6.7 is only a very particular case of a much more general index theorem, valid for holomorphic self-maps of complex manifolds of any dimension fixing pointwise a smooth complex submanifold of any codimension, or a hypersurface even with singularities; see [BrT], [Br] and [ABT], where some applications to dynamics are also discussed. In particular, in [ABT] is introduced a canonical section of a suitable vector bundle describing the local dynamics in an infinitesimal neighbourhood of the submanifold, providing in particular a more intrinsic description of the index.

Now, a combinatorial argument (again inspired by Camacho and Sad [CS]) shows that if we have $f \in \mathrm{End}(\mathbb{C}^2, O)$ with an isolated fixed point, and such that applying the blow-up procedure to the lifted map $\tilde f$ starting from a singular direction $[v] \in \mathbb{P}^1(\mathbb{C}) = E$ we end up with only $(\star_2)$ singularities, then the index of $\tilde f$ at $[v]$ along E must be a non-negative rational number. But the first Chern class of N_E is -1, and so there must be at least one singular directions whose index is not a non-negative rational number, and thus the blow-up procedure must yield at least one dicritical or $(\star_1)$ singularity, and hence a Fatou flower for our map f, completing the proof of Theorem 6.5.

Actually, we have proved the following slightly more precise result:

Theorem 6.8: (Abate, 2001 [A2]) *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a holomorphic local dynamical system tangent to the identity and with an isolated fixed point at the origin. Let $[v] \in \mathbb{P}^1(\mathbb{C})$ be a singular direction such that $\iota_{[v]}\big(\tilde f, \mathbb{P}^1(\mathbb{C})\big) \notin \mathbb{Q}^+$, where $\tilde f$ is the lift of f to the blow-up of the origin. Then f has a Fatou flower tangent to $[v]$.*

Remark 6.10: To be even more precise, Theorem 6.8 is more a statement on the lifted map $\tilde f$ than on the original f. Indeed, the argument used to prove Theorem 6.8 (or a similar argument along the lines of [Ca]) can be used to prove the following: *let E be a (not necessarily compact) Riemann surface inside a 2-dimensional complex manifold M, and take $f \in \mathrm{End}(M, E)$ tangent to the identity at all points of E and tangential to E. Let $p \in E$ be a singular point of f such that $\iota_p(f, E) \notin \mathbb{Q}^+$. Then there exist parabolic curves for f at p.* This latter statement has been recently generalized in two ways. Degli Innocenti [DI] has proved that we can allow E to be singular at p (but irreducible; in the reducible case one has to impose conditions on the indeces of f along all irreducible components of E passing through p). Molino [Mo], on the other hand, has proved that the statement still holds assuming only $\iota_p(f, E) \neq 0$, at least for f of order 2 (and E smooth at p); it is natural to conjecture that this should be true for f of any order.

As already remarked, the reduction of singularities via blow-ups seem to work only in dimension 2. This leaves open the problem of the validity of something like Theorem 6.5 in dimension $n \geq 3$; see [AT1] for some partial results.

Furthermore, as far as I know, it is completely open, even in dimension 2, the problem of describing the stable set of a holomorphic local dynamical system tangent to the identity, as well as the more general problem of the topological classification of such dynamical systems. Some results in the case of a dicritical singularity are presented in [BM].

We end this section with a couple of words on holomorphic local dynamical systems with a parabolic fixed point where the differential is not diagonalizable. Particular examples are studied in detail in [CD], [A4] and [GS]. In [A1] it is described a canonical procedure for lifting an $f \in \mathrm{End}(\mathbb{C}^n, O)$ whose differential at the origin is not diagonalizable to a map defined in a suitable iterated blow-up of the origin (obtained blowing-up not only points but more general submanifolds) with a canonical fixed point where the differential is diagonalizable. Using this procedure it is for instance possible to prove the following

Corollary 6.9: (Abate, 2001 [A2]) *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be a holomorphic local dynamical system with $df_O = J_2$, the canonical Jordan matrix associated to the eigenvalue 1, and assume that the origin is an isolated fixed point. Then f admits at least one parabolic curve tangent to $[1:0]$ at the origin.*

7. Several complex variables: other cases

Outside the hyperbolic and parabolic cases, there are not that many general results. Theorems 5.7 and 5.8 apply to the elliptic case too, but, as already remarked, it is not known whether the Bryuno condition is still necessary for holomorphic linearizability, that is, if any analogue of Theorem 4.9.(ii) holds in several variables. However, another result in the spirit of Theorem 5.8 is the following:

Theorem 7.1: (Yoccoz, 1995 [Y2]) *Let $A \in GL(n, \mathbb{C})$ be an invertible matrix such that its eigenvalues have no resonances and such that its Jordan normal form contains a non-trivial block associated to an eigenvalue of modulus one. Then there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$ with $df_O = A$ which is not holomorphically linearizable.*

A case that has received some attention is the so-called semi-attractive case: a holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ is said *semi-attractive* if the eigenvalues of df_O are either equal to 1 or with modulus strictly less than 1. The dynamics of semi-attractive dynamical systems has been studied in detail by Fatou [F4], Nishimura [N], Ueda [U1–2], Hakim [H1] and Rivi [Ri–2]. Their results more or less say that the eigenvalue 1 yields the existence of a "parabolic manifold" M — in the sense of Theorem 6.3.(ii) — of a suitable dimension, while the eigenvalues with modulus less than one ensure, roughly speaking, that the orbits of points in the normal bundle of M close enough to M are attracted to it. For instance, Rivi proved the following

Theorem 7.2: (Rivi, 1999 [Ri1–2]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system. Assume that 1 is an eigenvalue of (algebraic and geometric) multiplicity $q \geq 1$ of df_O, and that the other eigenvalues of df_O have modulus less than 1. Then:*

(i) *We can choose local coordinates $(z, w) \in \mathbb{C}^q \times \mathbb{C}^{n-q}$ such that f expressed in these coordinates becomes*

$$\begin{cases} f_1(z, w) = A(w)z + P_{2,w}(z) + P_{3,w}(z) + \cdots, \\ f_2(z, w) = G(w) + B(z, w)z, \end{cases}$$

where: $A(w)$ is a $q \times q$ matrix with entries holomorphic in w and $A(O) = I_q$; the $P_{j,w}$ are q-uples of homogeneous polynomials in z of degree j whose coefficients are holomorphic in w; G is a holomorphic self-map of $\mathbb{C}^{n-q}$ such that $G(O) = O$ and the eigenvalues of dG_O are the eigenvalues of df_O with modulus strictly less than 1; and $B(z, w)$ is an $(n-q) \times q$ matrix of holomorphic functions vanishing at the origin. In particular, $f_1(z, O)$ is tangent to the identity.

(ii) *If $v \in \mathbb{C}^q \subset \mathbb{C}^m$ is a non-degenerate characteristic direction for $f_1(z, O)$, and the latter map has order ν, then there exist $\nu - 1$ disjoint f-invariant $(n-q+1)$-dimensional complex submanifolds M_j of $\mathbb{C}^n$, with the origin in their boundary, such that the orbit of every point of M_j converges to the origin tangentially* *to $\mathbb{C}v \oplus E$, where $E \subset \mathbb{C}^n$ is the subspace generated by the generalized eigenspaces associated to the eigenvalues of df_O with modulus less than one.*

Rivi also has results in the spirit of Theorem 6.3, and results when the algebraic and geometric multiplicities of the eigenvalue 1 differ; see [Ri1, 2] for details.

As far as I know, there are no results on the formal or holomorphic classification of semi-attractive holomorphic local dynamical systems. However, building on work done by Canille Martins [CM] in dimension 2, and using Theorem 3.2 and general results on normally hyperbolic dynamical systems due to Palis and Takens [PT], Di Giuseppe has obtained the topological classification when the eigenvalue 1 has multiplicity 1 and the other eigenvalues are not resonant:

Theorem 7.3: (Di Giuseppe, 2004 [Di]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system such that df_O has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, where λ_1 is a primitive q-root of unity, and $|\lambda_j| \neq 0, 1$ for $j = 2, \ldots, n$. Assume moreover that $\lambda_2^{r_2} \cdots \lambda_n^{r^n} \neq 1$ for all multi-indeces $(r_2, \ldots, r_n) \in \mathbb{N}^{n-1}$ such that $r_2 + \cdots + r_n \geq 2$. Then f is topologically locally conjugated either to df_O or to the map*

$$z \mapsto (\lambda_1 z_1 + z_1^{kq+1}, \lambda_2 z_2, \ldots, \lambda_n z_n)$$

for a suitable $k \in \mathbb{N}^*$.

We end this survey by recalling a recent result by Bracci and Molino. Assume that $f \in \mathrm{End}(\mathbb{C}^2, O)$ is a holomorphic local dynamical system such that the eigenvalues of df_O are 1 and $e^{2\pi i\theta} \neq 1$. If $e^{2\pi i\theta}$ satisfies the Bryuno condition, Pöschel [Pö] proved the existence of a 1-dimensional f-invariant holomorphic disk containing the origin where f is conjugated to the irrational rotation of angle θ. On the other hand, Bracci and Molino give sufficient conditions (depending on f but not on $e^{2\pi i\theta}$, expressed in terms of two new holomorphic invariants, and satisfied by generic maps) for the existence of parabolic curves tangent to the eigenspace of the eigenvalue 1; see [BrM] for details.

References

[A1] M. Abate: *Diagonalization of non-diagonalizable discrete holomorphic dynamical systems*. Amer. J. Math. **122** (2000), 757–781.

[A2] M. Abate: *The residual index and the dynamics of holomorphic maps tangent to the identity*. Duke Math. J. **107** (2001), 173–207.

[A3] M. Abate: **An introduction to hyperbolic dynamical systems.** I.E.P.I. Pisa, 2001.

[A4] M. Abate: *Basins of attraction in quadratic dynamical systems with a Jordan fixed point*. Nonlinear Anal. **51** (2002), 271–282.

[AT1] M. Abate, F. Tovena: *Parabolic curves in $\mathbb{C}^3$*. Abstr. Appl. Anal. **2003** (2003), 275–294.

[AT2] M. Abate, F. Tovena: *Formal classification of holomorphic maps tangent to the identity*. Preprint, arXiv: math.DS/0411085, 2004.

[ABT] M. Abate, F. Bracci, F. Tovena: *Index theorems for holomorphic self-maps*. Ann. of Math. **159** (2004), 819–864.

[Ar] V.I. Arnold: **Geometrical methods in the theory of ordinary differential equations.** Springer-Verlag, Berlin, 1988.

[B] L.E. Böttcher: *The principal laws of convergence of iterates and their application to analysis*. Izv. Kazan. Fiz.-Mat. Obshch. **14** (1904), 155–234.

[Br] F. Bracci: *The dynamics of holomorphic maps near curves of fixed points*. Ann. Scuola Norm. Sup. Pisa **2** (2003), 493–520.

[BrM] F. Bracci, L. Molino: *The dynamics near quasi-parabolic fixed points of holomorphic diffeomorphisms in $\mathbb{C}^2$*. Amer. J. Math. **126** (2004), 671–686.

[BrS] F. Bracci, T. Suwa: *Residues for singular pairs and dynamics of biholomorphic maps of singular surfaces*. Internat. J. Math. **15** (2004), 443–466.

[BrT] F. Bracci, F. Tovena: *Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces*. Math. Z. **242** (2002), 481–490.

[BM] F.E. Brochero Martínez: *Groups of germs of analytic diffeomorphisms in* $(\mathbb{C}^2, O)$. J. Dynamic. Control Systems **9** (2003), 1–32.

[Bry1] A.D. Bryuno: *Convergence of transformations of differential equations to normal forms*. Dokl. Akad. Nauk. USSR **165** (1965), 987–989.

[Bry2] A.D. Bryuno: *Analytical form of differential equations, I*. Trans. Moscow Math. Soc. **25** (1971), 131–288.

[Bry3] A.D. Bryuno: *Analytical form of differential equations, II*. Trans. Moscow Math. Soc. **26** (1972), 199–239.

[C] C. Camacho: *On the local structure of conformal mappings and holomorphic vector fields*. Astérisque **59–60** (1978), 83–94.

[CS] C. Camacho, P. Sad: *Invariant varieties through singularities of holomorphic vector fields*. Ann. of Math. **115** (1982), 579–595.

[CM] J.C. Canille Martins: *Holomorphic flows in* $(\mathbb{C}^3, O)$ *with resonances*. Trans. Am. Math. Soc. **329** (1992), 825–837.

[Ca] J. Cano: *Construction of invariant curves for singular holomorphic vector fields*. Proc. Am. Math. Soc. **125** (1997), 2649–2650.

[CG] S. Carleson, F. Gamelin: **Complex dynamics.** Springer, Berlin, 1994.

[Ch] M. Chaperon: **Géométrie différentielle et singularités des systèmes dynamiques.** Astérisque **138–139,** 1986.

[CD] D. Coman, M. Dabija: *On the dynamics of some diffeomorphisms of* $\mathbb{C}^2$ *near parabolic fixed points*. Houston J. Math. **24** (1998), 85–96.

[Cr1] H. Cremer: *Zum Zentrumproblem*. Math. An.. **98** (1927), 151–163.

[Cr2] H. Cremer: *Über die Häufigkeit der Nichtzentren*. Math. Ann. **115** (1938), 573–580.

[DI] F. Degli Innocenti: *Holomorphic dynamics near germs of singular curves*. Preprint, 2004.

[DG] D. DeLatte, T. Gramchev: *Biholomorphic maps with linear parts having Jordan blocks: linearization and resonance type phenomena*. Math. Phys. El. J. **8** (2002), 1–27.

[Di] P. Di Giuseppe: *Topological classification of holomorphic, semi-hyperbolic germs in quasi-absence of resonances*. Preprint, 2004.

[DH] A. Douady, J.H. Hubbard: *On the dynamics of polynomial-like mappings*. Ann. Sc. Éc. Norm. Sup. **18** (1985), 287–343.

[É1] J. Écalle: **Les fonctions résurgentes. Tome I: Les algèbres de fonctions résurgentes.** Publ. Math. Orsay **81-05,** Université de Paris-Sud, Orsay, 1981.

[É2] J. Écalle: **Les fonctions résurgentes. Tome II: Les fonctions résurgentes appliquées à l'itération.** Publ. Math. Orsay **81-06,** Université de Paris-Sud, Orsay, 1981.

[É3] J. Écalle: **Les fonctions résurgentes. Tome III: L'équation du pont et la classification analytique des objects locaux.** Publ. Math. Orsay **85-05,** Université de Paris-Sud, Orsay, 1985.

[É4] J. Écalle: *Iteration and analytic classification of local diffeomorphisms of* $\mathbb{C}^\nu$. In **Iteration theory and its functional equations,** Lect. Notes in Math. **1163,** Springer-Verlag, Berlin, 1985, pp. 41–48.

[ÉV] J. Écalle, B. Vallet: *Correction and linearization of resonant vector fields and diffeomorphisms*. Math. Z. **229** (1998), 249–318.

[F1] P. Fatou: *Sur les équations fonctionnelles, I*. Bull. Soc. Math. France **47** (1919), 161–271.

[F2] P. Fatou: *Sur les équations fonctionnelles, II*. Bull. Soc. Math. France **48** (1920), 33–94.

[F3] P. Fatou: *Sur les équations fonctionnelles, III*. Bull. Soc. Math. France **48** (1920), 208–314.

[F4] P. Fatou: *Substitutions analytiques et équations fonctionnelles à deux variables*. Ann. Sc. Ec. Norm. Sup. **40** (1924), 67–142.

[FJ] C. Favre, M. Jonsson: **The valuative tree.** Lect. Notes in Math. 1853, Springer Verlag, Berlin, 2004.

[FHY] A. Fathi, M. Herman, J.-C. Yoccoz: *A proof of Pesin's stable manifold theorem.* In **Geometric Dynamics,** Lect Notes in Math. 1007, Springer Verlag, Berlin, 1983, pp. 177-216.

[GS] V. Gelfreich, D. Sauzin: *Borel summation and splitting of separatrices for the Hénon map.* Ann. Inst. Fourier Grenoble **51** (2001), 513–567.

[G1] D.M. Grobman: *Homeomorphism of systems of differential equations.* Dokl. Akad. Nauk. USSR **128** (1959), 880–881.

[G2] D.M. Grobman: *Topological classification of neighbourhoods of a singularity in n-space.* Math. Sbornik **56** (1962), 77–94.

[H] J.S. Hadamard: *Sur l'itération et les solutions asymptotyques des équations différentielles.* Bull. Soc. Math. France **29** (1901), 224–228.

[Ha1] M. Hakim: *Attracting domains for semi-attractive transformations of $\mathbb{C}^p$.* Publ. Matem. **38** (1994), 479–499.

[Ha2] M. Hakim: *Analytic transformations of $(\mathbb{C}^p, 0)$ tangent to the identity.* Duke Math. J. **92** (1998), 403–428.

[Ha3] M. Hakim: *Transformations tangent to the identity. Stable pieces of manifolds.* Preprint, 1997.

[Har] P. Hartman: *A lemma in the theory of structural stability of differential equations.* Proc. Am. Math. Soc. **11** (1960), 610–620.

[HK] B. Hasselblatt, A. Katok: **Introduction to the modern theory of dynamical systems.** Cambridge Univ. Press, Cambridge, 1995.

[He] M. Herman: *Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point.* In **Proc.** 8^{th} **Int. Cong. Math. Phys.,** World Scientific, Singapore, 1986, pp. 138–198.

[HPS] M. Hirsch, C.C. Pugh, M. Shub: **Invariant manifolds.** Lect. Notes Math. **583,** Springer-Verlag, Berlin, 1977.

[HP] J.H. Hubbard, P. Papadopol: *Superattractive fixed points in $\mathbb{C}^n$.* Indiana Univ. Math. J. **43** (1994), 321–365.

[I1] Yu.S. Il'yashenko: *Divergence of series reducing an analytic differential equation to linear normal form at a singular point.* Funct. Anal. Appl. **13** (1979), 227–229.

[I2] Yu.S. Il'yashenko: *Nonlinear Stokes phenomena.* In **Nonlinear Stokes phenomena,** Adv. in Soviet Math. **14,** Am. Math. Soc., Providence, 1993, pp. 1–55.

[K] T. Kimura: *On the iteration of analytic functions.* Funk. Eqvacioj **14** (1971), 197–238.

[Kœ] G. Kœnigs: *Recherches sur les integrals de certain equations fonctionelles.* Ann. Sci. Éc. Norm. Sup. **1** (1884), 1–41.

[L] L. Leau: *Étude sur les equations fonctionelles à une ou plusieurs variables.* Ann. Fac. Sci. Toulouse **11** (1897), E1–E110.

[M1] B. Malgrange: *Travaux d'Écalle et de Martinet-Ramis sur les systèmes dynamiques.* Astérisque **92-93** (1981/82), 59–73.

[M2] B. Malgrange: *Introduction aux travaux de J. Écalle.* Ens. Math. **31** (1985), 261–282.

[Ma] S. Marmi: **An introduction to small divisors problems.** I.E.P.I., Pisa, 2000.

[MM] J.F. Mattei, R. Moussu: *Holonomie et intégrales premières.* Ann. Scient. Ec. Norm. Sup. **13** (1980), 469–523.

[Mi] J. Milnor: **Dynamics in one complex variable.** Vieweg, Braunschweig, 2000.

[Mo] L. Molino: *The dynamics of maps tangent to the identity and with non-vanishing index.* Preprint, 2004.

[N] Y. Nishimura: *Automorphismes analytiques admettant des sousvariétés de point fixes attractives dans la direction transversale.* J. Math. Kyoto Univ. **23** (1983), 289–299.

[PT] J. Palis, F. Takens: *Topological equivalence of normally hyperbolic dynamical systems*. Topology **16** (1977), 335–345.

[P1] R. Pérez-Marco: *Sur les dynamiques holomorphes non linéarisables et une conjecture de V.I. Arnold*. Ann. Sci. École Norm. Sup. **26** (1993), 565–644.

[P2] R. Pérez-Marco: *Topology of Julia sets and hedgehogs*. Preprint, Université de Paris-Sud, 94-48, 1994.

[P3] R. Pérez-Marco: *Non-linearizable holomorphic dynamics having an uncountable number of symmetries*. Invent. Math. **199** (1995), 67–127.

[P4] R. Pérez-Marco: *Holomorphic germs of quadratic type*. Preprint, 1995.

[P5] R. Pérez-Marco: *Hedgehogs dynamics*. Preprint, 1995.

[P6] R. Pérez-Marco: *Sur une question de Dulac et Fatou*. C.R. Acad. Sci. Paris **321** (1995), 1045–1048.

[P7] R. Pérez-Marco: *Fixed points and circle maps*. Acta Math. **179** (1997), 243–294.

[P8] R. Pérez-Marco: *Linearization of holomorphic germs with resonant linear part*. Preprint, arXiv: math.DS/0009030, 2000.

[P9] R. Pérez-Marco: *Total convergence or general divergence in small divisors*. Comm. Math. Phys. **223** (2001), 451–464.

[Pe] O. Perron: *Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen*. Math. Z. **29** (1928), 129–160.

[Pes] Ja.B. Pesin: *Families of invariant manifolds corresponding to non-zero characteristic exponents*. Math. USSR Izv. **10** (1976), 1261–1305.

[Po] H. Poincaré: **Œuvres, Tome I.** Gauthier-Villars, Paris, 1928, pp. XXXVI–CXXIX.

[Pö] J. Pöschel: *On invariant manifolds of complex analytic mappings near fixed points*. Exp. Math. **4** (1986), 97–109.

[R1] L. Reich: *Das Typenproblem bei formal-biholomorphien Abbildungen mit anziehendem Fixpunkt*. Math. Ann. **179** (1969), 227–250.

[R2] L. Reich: *Normalformen biholomorpher Abbildungen mit anziehendem Fixpunkt*. Math. Ann. **180** (1969), 233–255.

[Ri1] M. Rivi: **Local behaviour of discrete dynamical systems.** Ph.D. Thesis, Università di Firenze, 1999.

[Ri2] M. Rivi: *Parabolic manifolds for semi-attractive holomorphic germs*. Mich. Math. J. **49** (2001), 211–241.

[S] A.A. Shcherbakov: *Topological classification of germs of conformal mappings with identity linear part*. Moscow Univ. Math. Bull. **37** (1982), 60–65.

[Sh] M. Shub: **Global stability of dynamical systems.** Springer, Berlin, 1987.

[Si] C.L. Siegel: *Iteration of analytic functions*. Ann. of Math. **43** (1942), 607–612.

[St1] S. Sternberg: *Local contractions and a theorem of Poincaré*. Amer. J. Math. **79** (1957), 809–824.

[St2] S. Sternberg: *The structure of local homomorphisms, II*. Amer. J. Math. **80** (1958), 623–631.

[U1] T. Ueda: *Local structure of analytic transformations of two complex variables, I*. J. Math. Kyoto Univ. **26** (1986), 233–261.

[U2] T. Ueda: *Local structure of analytic transformations of two complex variables, II*. J. Math. Kyoto Univ. **31** (1991), 695–711.

[Us] S. Ushiki: *Parabolic fixed points of two-dimensional complex dynamical systems*. Sūrikaisekikenkyūsho Kōkyūroku **959** (1996), 168–180.

[V] S.M. Voronin: *Analytic classification of germs of conformal maps* $(\mathbb{C},0) \to (\mathbb{C},0)$ *with identity linear part*. Func. Anal. Appl. **15** (1981), 1–17.

[Y1] J.-C. Yoccoz: *Linéarisation des germes de difféomorphismes holomorphes de* $(\mathbb{C},0)$. C.R. Acad. Sci. Paris **306** (1988), 55–58.

[Y2] J.-C. Yoccoz: *Théorème de Siegel, nombres de Bryuno et polynômes quadratiques*. Astérisque **231** (1995), 3–88.

[W] B.J. Weickert: *Attracting basins for automorphisms of* $\mathbb{C}^2$. Invent. Math. **132** (1998), 581–605.

[Wu] H. Wu: *Complex stable manifolds of holomorphic diffeomorphisms*. Indiana Univ. Math. J. **42** (1993), 1349–1358.