The evolution of Loewner's differential equations

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1 Introduction

Ch. Loewner

Charles Loewner was born as Karel Löwner on 29 May 1893 in Lány, Bohemia. He also used the German spelling Karl of his first name; indeed, although he spoke Czech at home, all of his education was in German.

Loewner received his PhD from the University of Prague in 1917 under the supervision of George Pick; then he spent some years at the Universities of Berlin and Cologne. In 1930, he returned to Charles University of Prague as a professor. When the Nazis occupied Prague, he was put in jail. Luckily, after paying the "emigration tax" he was allowed to leave the country with his family and move, in 1939, to the US, where he changed his name to Charles Loewner. Although J. von Neumann promptly arranged a position for him at Louisville University, he had to start his life from scratch. In the United States, he worked at Brown University, Syracuse University and eventually at Stanford University, where he remained until his death on 8 January 1968.

Loewner's work covers wide areas of complex analysis and differential geometry, and displays his deep understanding of Lie theory and his passion for semigroup theory; his papers are nowadays cornerstones of the theory bearing his name.

He began his research in the theory of conformal mappings. His most prominent contribution to this field was the introduction of infinitesimal methods for univalent functions, leading to the Loewner differential equations that are now a classical tool of complex analysis. Loewner's basic idea to consider semigroups related to conformal mappings led him to the general study of semigroups of transformations. In this context he characterised monotone matrix transformations, sets of projective mappings and similar geometric transformation classes.

Here we are mainly interested in Loewner's early research about the composition semigroups of conformal mappings and in the developments (some quite recent) springing from his work. Loewner's most important work in this area is his 1923 paper [41] where he introduced the nowadays well-known Loewner parametric method and the so-called Loewner differential equations, allowing him to prove the first non-elementary case of the celebrated Bieberbach conjecture: if f is a univalent function defined on the unit disc in the complex plane, with expansion at the origin given by

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots ,$$

then $|a_n| \leq n$ for each $n \geq 1$. Loewner was able to prove that $|a_3| \leq 3$. It is well-known that the Bieberbach conjecture was finally proven by L. de Branges [15] in 1985. In his proof, de Branges introduced ideas related to Loewner's but he used only a distant relative of Loewner's equation in connection with his main apparatus, his own rather sophisticated theory of composition operators. However, elaborating on de Branges' ideas, FitzGerald and Pommerenke [18] discovered how to avoid altogether the composition operators and rewrote the proof in classical terms, applying the *bona fide* Loewner equation and providing in this way a direct and powerful application of Loewner's classical method.

The seminal paper [41] has been a source of inspiration for many mathematicians and there have been many further developments and extensions of the results and techniques introduced there. This is especially true for the differential equations he first considered and this note will be a brief tour of the development of Loewner's theory and its several applications and generalisations.

We would like to end this introduction by recalling that one of the last (but definitely not least) contributions to this growing theory was the discovery, by Oded Schramm in 2000 [49], of the stochastic Loewner equation (SLE), also known as the Schramm-Loewner equation. The SLE is a conformally invariant stochastic process; more precisely, it is a family of random planar curves generated by solving Loewner's differential equation with a Brownian motion as the driving term. This equation was studied and developed by Schramm together with Greg Lawler and Wendelin Werner in a series of joint papers that led, among other things, to a proof of Mandelbrot's conjecture about the Hausdorff dimension of the Brownian frontier [36], [37]. This achievement was one of the reasons Werner was awarded the Fields Medal in 2006. Sadly, Oded Schramm, born 10 December 1961 in Jerusalem, died in a tragic hiking accident on 01 September 2008 while climbing Guye Peak, north of Snoqualmie Pass in Washington.

Quite recently, Stanislav Smirnov has also been awarded the Fields Medal (2010) for his outstanding contributions to SLE and the theory of percolation.

2 The slit radial Loewner equation

In his 1923 paper [41], Loewner proved that the class of single-slit mappings (i.e. holomorphic functions mapping univalently the unit disc $\mathbb{D} \subset \mathbb{C}$ onto the complement in $\mathbb{C}$ of a Jordan arc) is a dense subset of the class $\mathcal{S}$ of all univalent mappings f in the unit disc normalised by $f(0) = 0$ and $f'(0) = 1$. He also discovered a method to parametrise single-slit maps. Let g be a single-slit map whose image in $\mathbb{C}$ avoids the Jordan arc $\gamma\colon [0, +\infty) \to \mathbb{C}$. Loewner introduced the family (g_t) of univalent maps in $\mathbb{D}$, indexed by the time $t \in [0, +\infty)$, where $g_0 = g$ and g_t is the Riemann mapping whose image is the complement in $\mathbb{C}$ of the Jordan arc $\gamma|_{[t,+\infty)}$. The family of domains $\{g_t(\mathbb{D})\}$ is increasing, and as time goes to ∞ it fills out the whole complex plane.

Loewner's crucial observation is that the family (g_t) can be described by differential equations. More precisely, with a suitable choice of parametrisation, there exists a continuous function $\kappa\colon [0, +\infty) \to \partial\mathbb{D}$, called the *driving term*, such that (g_t) satisfies

$$\frac{\partial g_t(w)}{\partial t} = w\,\frac{\kappa(t) + w}{\kappa(t) - w}\frac{\partial g_t(w)}{\partial w}\,. \tag{2.1}$$

This equation is usually called the *(slit-radial) Loewner PDE* (and it is the first one of several *evolution equations* we shall see originated by Loewner's ideas). Loewner also remarked (and used) that the associated family of holomorphic self-maps of the unit disc $(\varphi_{s,t}) := (g_t^{-1} \circ g_s)$ for $0 \le s \le t$ gives solutions of the characteristic equation

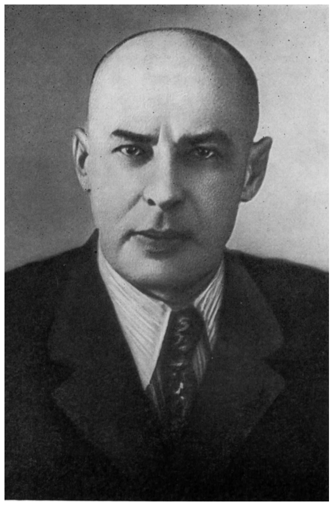
$$\frac{dw}{dt} = -w\,\frac{\kappa(t) + w}{\kappa(t) - w} \tag{2.2}$$

subjected to the initial condition $w(s) = z \in \mathbb{D}$. Equation (2.2) is nowadays known as the *(slit-radial) Loewner ODE*. The adjective "radial" in these names comes from the fact that the image of each $\varphi_{s,t}$ is the unit disc minus a single Jordan arc approaching a sort of radius as t goes to ∞.

The two slit-radial Loewner equations can be studied on their own without any reference to parametrised families of univalent maps. Imposing the initial condition $w(s) = z$, the Loewner ODE (2.2) has a unique solution $w_s^z(t)$ defined for all $t \in [s, +\infty)$. Moreover, $\varphi_{s,t}(z) := w_s^z(t)$ is a holomorphic self-map of the unit disc for all $0 \le s \le t < +\infty$. However, without conditions on the driving term, the solutions of the Loewner ODE are in general not of slit type. For instance, P. P. Kufarev in 1947 gave examples of continuous driving terms such that the solutions to (2.2) have subdomains of the unit disc bounded by hyperbolic geodesics as image, and thus are non-slit maps. The problem of understanding exactly which driving terms produce slit solutions of (2.2) has become, and still is, a basic problem in the theory. Deep and very recent contributions to this question are due to J. R. Lind, D. E. Marshall and S. Rhode [42], [39], [40]. See also [44] and references therein.

3 The general radial Loewner equations

It is not easy to follow the historical development of the parametric method because in the middle of the 20th century a number of papers appeared independently; moreover, some of them were published in the Soviet Union, remaining partially unknown to Western mathematicians. Anyhow, it is widely recognised that the Loewner method was brought to its full power by Pavel Parfen'evich Kufarev (Tomsk, 18 March 1909 – Tomsk, 17 July 1968) and Christian Pommerenke (Copenhagen, 17 December 1933).



P.P. Kufarev

Using slightly different points of view, both Kufarev and Pommerenke merged Loewner's ideas with evolutionary aspects of increasing families of *general* complex domains. Pommerenke's approach was to impose an ordering on the images of univalent mappings of the unit disc, and it seems to have been the first one to use the expression "Loewner chain" for describing the family of increasing univalent mappings in Loewner's theory. Kufarev [31] too studied increasing families of domains and, although they were not exactly Loewner chains in the sense of Pommerenke, his theory bears some resemblance to the one developed by Pommerenke [45].

A *Loewner chain* (in the sense of Pommerenke) is a family (f_t) of univalent mappings of the unit disc whose images form an increasing family of simply connected domains and normalised imposing $f_t(0) = 0$ and $f_t'(0) = e^t$ for all $t \ge 0$ (we notice that as soon as $f_t(0) = 0$ holds, the second normalising condition can always be obtained by means of a reparametrisation in the time variable). The families of single-slit mappings originally considered by Loewner are thus a very particular example of Loewner chains.

Again, to a Loewner chain (f_t) we can associate a family $(\varphi_{s,t}) := (f_t^{-1} \circ f_s)$ for $0 \le s \le t$ of holomorphic self-maps of the unit disc, and again both (f_t) and $(\varphi_{s,t})$ can be recovered as solutions of differential equations. In fact, (f_t) satisfies the (general) *radial Loewner PDE*

$$\frac{\partial f_t(w)}{\partial t} = w\,p(w, t)\frac{\partial f_t(w)}{\partial w}\,, \tag{3.1}$$

where $p\colon \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a normalised parametric *Herglotz function*, i.e. it satisfies the following conditions:
- $p(0, \cdot) \equiv 1$.
- $p(\cdot, t)$ is holomorphic for all $t \ge 0$.
- $p(z, \cdot)$ is measurable for all $z \in \mathbb{D}$.
- $\mathrm{Re}\, p(z, t) \ge 0$ for all $t \ge 0$ and $z \in \mathbb{D}$.

Analogously, $(\varphi_{s,t})$ satisfies the so-called (general) *radial Loewner ODE*

$$\frac{dw}{dt} = -w\,p(w, t), \quad w(s) = z\,, \tag{3.2}$$

where p again is a normalised parametric Herglotz function.

Since

$$\mathrm{Re}\,\frac{\kappa(t) + w}{\kappa(t) - w} \ge 0$$

for all w and any driving term κ, equations (2.1) and (2.2) are particular cases of (3.1) and (3.2); furthermore, in contrast with the slit case, the general radial Loewner equations yield a one-to-one correspondence between Loewner chains and normalised parametric Herglotz functions.

Ch. Pommerenke

Roughly speaking, differential equations such as (3.1) and (3.2) are important because they allow one to get estimates and growth bounds for f_t and $\varphi_{s,t}$ starting from the well-known estimates and growth bounds for maps, such as $p(z,t)$, having image in the right half-plane. For instance, in this way Pommerenke [45], and also [19], [27], solved the "embedding problem", showing that for any $f \in \mathcal{S}$ it is possible to find a Loewner chain (f_t) such that $f_0 = f$. More general questions of embeddability in Loewner chains satisfying specific properties are still open and this is an active area of research.

Loewner's and Pommerenke's approaches to the parametric method work under the essential assumption that all the elements of the chain fix a given point of the unit disc, usually the origin. This is a natural hypothesis if one deals with increasing sequences of simply connected domains, and it also yields (up to a reparametrisation) good regularity in the time parameter – a basic fact necessary for the derivation of the associated differential equations. However, in certain situations related to some concrete physical and stochastic processes we will discuss later, there is no fixed point in the interior, and a similar role has to be played by a point on the boundary of the unit disc. Because the geometry of the boundary of the unit disc, the "infinity" of hyperbolic geometry, is quite different from the geometry inside the unit disc, new phenomena appear, and it does not seem possible to deal with this case by somehow appealing to the classical case. As a consequence, new extensions of Loewner's theory have been provided.

4 The chordal equation

In 1946, Kufarev [32] proposed an evolution equation in the upper half-plane analogous to the one introduced by Loewner in the unit disc. In 1968, Kufarev, Sobolev and Sporysheva [33] established a parametric method, based on this equation, for the class of univalent functions in the upper half-plane, which is known to be related to physical problems in hydrodynamics. Moreover, during the second half of the past century, the Soviet school intensively studied Kufarev's equation. We ought to cite here at least the contributions of I. A. Aleksandrov [2], S. T. Aleksandrov and V. V. Sobolev [4], V. V. Goryainov and I. Ba [22, 23]. However, this work was mostly unknown to many Western mathematicians, mainly because some of it appeared in journals not easily accessible outside the Soviet Union. In fact, some of Kufarev's papers were not even reviewed by Mathematical Reviews. Anyhow, we refer the reader to [3], which contains a complete bibliography of his papers.

In order to introduce Kufarev's equation properly, let us fix some notation. Let γ be a Jordan arc in the upper half-plane $\mathbb{H}$ with starting point $\gamma(0) = 0$. Then there exists a unique conformal map $g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$ with the normalisation

$$g_t(z) = z + \frac{c(t)}{z} + O\left(\frac{1}{z^2}\right) .$$

After a reparametrisation of the curve γ, one can assume that $c(t) = 2t$. Under this normalisation, one can show that g_t satisfies the following differential equation:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - h(t)}, \qquad g_0(z) = z. \tag{4.1}$$

The equation is valid up to a time $T_z \in (0, +\infty]$ that can be characterised as the first time t such that $g_t(z) \in \mathbb{R}$ and where h is a continuous real-valued function. Conversely, given a continuous function $h\colon [0,+\infty) \to \mathbb{R}$, one can consider the following initial value problem for each $z \in \mathbb{H}$:

$$\frac{dw}{dt} = \frac{2}{w - h(t)}, \qquad w(0) = z\ . \tag{4.2}$$

Let $t \mapsto w^z(t)$ denote the unique solution of this Cauchy problem and let $g_t(z) := w^z(t)$. Then g_t maps holomorphically a (not necessarily slit) subdomain of the upper half-plane $\mathbb{H}$ onto $\mathbb{H}$. Equation (4.2) is nowadays known as the *chordal Loewner differential equation* with the function h as the driving term. The name is due to the fact that the curve $\gamma[0,t]$ evolves in time as t tends to infinity into a sort of chord joining two boundary points. This kind of construction can be used to model evolutionary aspects of decreasing families of domains in the complex plane.

For later use, we remark that using the Cayley transform we may assume (working in an increasing context) that the chordal Loewner equation in the unit disc takes the form

$$\frac{dz}{dt} = (1-z)^2 p(z,t), \qquad z(0) = z, \tag{4.3}$$

where $\operatorname{Re} p(z,t) \geq 0$ for all $t \geq 0$ and $z \in \mathbb{D}$.

O. Schramm

In 2000 Schramm [49] had the simple but very effective idea of replacing the function h in (4.2) by a Brownian motion, and of using the resulting chordal Loewner equation, nowadays known as the SLE (*stochastic Loewner equation*) to understand critical processes in two dimensions, relating probability theory to complex analysis in a completely novel way. In fact, the SLE was discovered by Schramm as a conjectured scaling limit of the planar uniform spanning tree and the planar loop-erased random walk probabilistic processes. Moreover, this tool also turned out to be very important for the proofs of conjectured scaling limit relations on some other models from statistical mechanics, such as self-avoiding random walks and percolation.

5 Semigroups of holomorphic mappings

To each Loewner chain (f_t) one can associate a family of holomorphic self-maps of the unit disc $(\varphi_{s,t}) := (f_t^{-1} \circ f_s)$, sometimes called *transition functions* or the *evolution family*. By the very construction, an evolution family satisfies the algebraic property

$$\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u} \tag{5.1}$$

for all $0 \leq s \leq u \leq t < +\infty$.

Special but important cases of evolution families are semigroups of holomorphic self-maps of the unit disc. A family of holomorphic self-maps of the unit disc (ϕ_t) is a *(continuous) semigroup* if $\phi\colon (\mathbb{R}^+, +) \to \mathsf{Hol}(\mathbb{D}, \mathbb{D})$ is a continuous homomorphism between the semigroup of non-negative real numbers and the semigroup of holomorphic self-maps of the disc with respect to composition, endowed with the topology of uniform convergence on compact sets. In other words: $\phi_0 = \mathsf{id}_{\mathbb{D}}$; $\phi_{t+s} = \phi_s \circ \phi_t$ for all $s, t \geq 0$; and ϕ_t converges to ϕ_{t_0} uniformly on compact sets as t goes to t_0.

Setting $\varphi_{s,t} := \phi_{t-s}$ for $0 \leq s \leq t < +\infty$, it can be checked that $\varphi_{s,t}$ satisfies (5.1) and semigroups of holomorphic maps provide examples of evolution families in the sense of Section 6.

Semigroups of holomorphic maps are a classical subject of study, both as (local/global) flows of continuous dynamical systems and from the point of view of "fractional iteration", the problem of embedding the discrete set of iterates generated by a single self-map into a one-parameter family (a problem that is still open even in the disc). It is difficult to exactly date the birth of this notion but it seems that the first paper dealing with semigroups of holomorphic maps and their asymptotic behaviour is due to F. Tricomi in 1917 [53]. Semigroups of holomorphic maps also appear in connection with the theory of Galton-Watson processes (branching processes) started in the 40s by A. Kolmogorov and N. A. Dmitriev [28]. Furthermore, they are an important tool in the theory of strongly continuous semigroups of operators between spaces of analytic functions (see, for example, [51]).

A very important contribution to the theory of semigroups of holomorphic self-maps of the unit disc is due to E. Berkson and H. Porta [9]. They proved that a semigroup of holomorphic self-maps of the unit disc (ϕ_t) is in fact real-analytic in the variable t, and is the solution of the Cauchy problem

$$\frac{\partial \phi_t(z)}{\partial t} = G(\phi_t(z)), \quad \phi_0(z) = z\,, \tag{5.2}$$

where the map G, the *infinitesimal generator* of the semigroup, has the form

$$G(z) = (z - \tau)(\overline{\tau} z - 1)p(z) \tag{5.3}$$

for some $\tau \in \overline{\mathbb{D}}$ and a holomorphic function $p\colon \mathbb{D} \to \mathbb{C}$ with $\mathsf{Re}\, p \geq 0$.

The dynamics of the semigroup (ϕ_t) are governed by the analytical properties of the infinitesimal generator G. For instance, the semigroup has a common fixed point at τ (in the sense of non-tangential limit if τ belongs to the boundary of the unit disc) and asymptotically tends to τ, which can thus be considered a sink point of the dynamical system generated by G.

When $\tau = 0$, it is clear that (5.3) is a particular case of (3.2), because the infinitesimal generator G is of the form $-wp(w)$, where p is a (autonomous, and not necessarily normalised) Herglotz function. As a consequence, when the semigroup has a fixed point in the unit disc (which, up to a conjugation by an automorphism of the disc, amounts to taking $\tau = 0$), *once differentiability in t is proved* Berkson-Porta's theorem can be easily deduced from Loewner's theory. However, when the semigroup has no common fixed points in the interior of the unit disc, Berkson-Porta's result is really a new advance in the theory.

We have already remarked that semigroups give rise to evolution families; they also provide examples of Loewner chains. Indeed, M. H. Heins [29] and A. G. Siskasis [50] have independently proved that if (ϕ_t) is a semigroup of holomorphic self-maps of the unit disc then there exists a (unique, when suitably normalised) holomorphic function $h\colon \mathbb{D} \to \mathbb{C}$, the *Königs function* of the semigroup, such that $h(\phi_t(z)) = m_t(h(z))$ for all $t \geq 0$, where m_t is an affine map (in other words, the semigroup is semiconjugated to a semigroup of affine maps). Then it is easy to see that the maps $f_t(z) := m_t^{-1}(h(z))$, for $t \geq 0$, form a Loewner chain (in the sense explained in the next section).

The theory of semigroups of holomorphic self-maps has been extensively studied and generalised: to Riemann surfaces (in particular, Heins [29] has shown that Riemann surfaces with non-Abelian fundamental group admit no non-trivial semigroup of holomorphic self-maps); to several complex variables; and to infinitely dimensional complex Banach spaces, by I. N. Baker, C. C. Cowen, M. Elin, V. V. Goryainov, P. Poggi-Corradini, Ch. Pommerenke, S. Reich, D. Shoikhet, A. G. Siskakis, E. Vesentini and many others. We refer to [10] and the books [1] and [47] for references and more information on the subject.

6 A general Loewner's theory

Comparing the radial Loewner equation (3.2), the chordal Loewner equation (4.3) and the Berkson-Porta decomposition (5.3) for infinitesimal generators of semigroups, one realises that for all fixed $t \geq 0$ the maps appearing in Loewner's theory are infinitesimal generators of semigroups of holomorphic self-maps of the unit disc. Therefore, one is tempted to consider a general Loewner equation of the following form:

$$\frac{dz}{dt} = G(z, t), \qquad z(0) = z, \tag{6.1}$$

with $G(\cdot, t)$ being an infinitesimal generator for almost all fixed $t \geq 0$, as well as the associated general Loewner PDE:

$$\frac{\partial f_t(z)}{\partial t} = -G(z, t)\frac{\partial f_t(z)}{\partial z}. \tag{6.2}$$

Thanks to (5.3), when assuming $G(0, t) \equiv 0$ or $G(z, t)$ of the special chordal form, these equations coincide with those we have already discussed, and hence they can be viewed as general and unified Loewner equations (see, for example, [11] and [14]).

As we have seen, Loewner introduced his theory to deal with univalent normalised functions. Hence he put more emphasis on the concept of Loewner chains than on evolution families, as did Pommerenke. An intrinsic study of evolution families and of their relationship with other aspects of the theory has not been carried out until recently; let us describe the approach proposed in [11] and [14]. An *evolution family of order* $d \in [1, +\infty]$ is a family $(\varphi_{s,t})_{0 \leq s \leq t < +\infty}$ of holomorphic self-maps of the unit disc such that $\varphi_{s,s} = \mathsf{id}_{\mathbb{D}}$, $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$ and such that for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0, T], \mathbb{R})$ satisfying

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi)\, d\xi \tag{6.3}$$

for all $0 \leq s \leq u \leq t \leq T$.

If (f_t) is a normalised Loewner chain (in the sense of Pommerenke) then the family $(\varphi_{s,t}) := (f_t^{-1} \circ f_s)$ is an evolution family of order $+\infty$: the regularity condition (6.3) (with $d = +\infty$) holds because $\varphi'_{s,t}(0) = e^{s-t}$ [45, Lemma 6.1]. Similarly, one can show that the solutions of the chordal Loewner ordinary differential equations satisfy (6.3). Hence, this concept of evolution families of order d is a natural generalisation of the evolution families appearing in the classical Loewner theory. We also remark that, although it is not assumed in the definition, it turns out that maps belonging to an evolution family are always univalent.

Associated to evolution families of order d there are *Herglotz vector fields of order* $d \in [1, +\infty]$. These are time-dependent vector fields $G(z, t)$ that are measurable in t for all fixed z, are holomorphic infinitesimal generators of semigroups for almost all fixed t and are such that for each compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ so that

$$|G(z, t)| \leq k_{K,T}(t)$$

for all $z \in K$ and almost all $t \in [0, T]$. Once again, the vector fields introduced in classical Loewner theory satisfy these conditions, with $d = +\infty$.

In [11] it is proved that there is a one-to-one correspondence between evolution families $(\varphi_{s,t})$ of order d and Herglotz vector fields $G(z, t)$ of order d, and the bridge producing such a correspondence is precisely (6.1), namely,

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z), t), \quad \varphi_{s,s}(z) = z. \tag{6.4}$$

Moreover, a Herglotz vector field $G(z, t)$ admits a Berkson-Porta-like decomposition. Namely, there exists a function $p\colon \mathbb{D}\times [0, +\infty) \to \mathbb{C}$ satisfying

– $z \mapsto p(z, t)$ is holomorphic for all $t \in [0, +\infty)$.
– $\operatorname{Re} p(z, t) \geq 0$ for all $z \in \mathbb{D}$ and almost all $t \in [0, +\infty]$.
– $t \mapsto p(z, t) \in L^d_{\text{loc}}([0, +\infty], \mathbb{C})$ for all $z \in \mathbb{D}$.

and a measurable function $\tau\colon [0, +\infty] \to \overline{\mathbb{D}}$ such that

$$G(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t) \,. \tag{6.5}$$

Conversely, any vector field of the form (6.5) is a Herglotz vector field. Notice that when $\tau \equiv 0$ (respectively, $\tau \equiv 1$) equation (6.4) (via (6.5)) reduces to (3.2) (respectively, to (4.3)) and when $G(z, t)$ does not depend on t it reduces to the semigroup equation (5.2).

In the classical theory, every Loewner chain can be obtained from a normalised evolution family $(\varphi_{s,t})$ by taking

$$f_s(z) := \lim_{t\to\infty} e^t \varphi_{s,t}(z). \tag{6.6}$$

In [14], a new definition of Loewner chains was introduced, allowing one to reproduce the relationship between Loewner chains and evolution families in this more general context. A family of univalent maps (f_t) in the unit disc is said to be a *Loewner chain of order d* if the ranges $f_t(\mathbb{D})$ form an increasing family of complex domains and for any compact set $K \subset \mathbb{D}$ and any $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi) d\xi$$

for all $z \in K$ and all $0 \leq s \leq t \leq T$. Exploiting the (classical) parametric representation of univalent maps (6.6), it can be proved that there is a one-to-one (up to composition with biholomorphisms) correspondence between evolution families of order d and Loewner chains of the same order, related by the equation

$$f_s = f_t \circ \varphi_{s,t}. \tag{6.7}$$

An alternative functorial method to create Loewner chains from evolution families, which also works on abstract complex manifolds, has been introduced in [5].

Once the previous correspondences are established, given a Loewner chain (f_t) of order d, the general Loewner PDE (6.2) follows by differentiating the structural equation (6.7). Conversely, given a Herglotz vector field $G(z, t)$ of order d, one can build the associated Loewner chain (of the same order d), solving (6.2) by means of the associated evolution family.

The Berkson-Porta decomposition (6.5) of a Herglotz vector field $G(z, t)$ also gives information on the dynamics of the associated evolution family. For instance, when $\tau(t) \equiv \tau \in \mathbb{D}$, the point τ is a (common) fixed point of $(\varphi_{s,t})$ for all $0 \leq s \leq t < +\infty$. Moreover, it can be proved that, in such a case, there exists a unique locally absolutely continuous function $\lambda\colon [0, +\infty) \to \mathbb{C}$ with $\lambda' \in L^d_{loc}([0, +\infty), \mathbb{C})$, $\lambda(0) = 0$ and $\operatorname{Re} \lambda(t) \geq \operatorname{Re} \lambda(s) \geq 0$ for all $0 \leq s \leq t < +\infty$ such that for all $s \leq t$

$$\varphi'_{s,t}(\tau) = \exp(\lambda(s) - \lambda(t)).$$

A similar characterisation holds when $\tau(t) \equiv \tau \in \partial\mathbb{D}$ [11].

7 Applications and extension of Loewner's theory

Loewner's theory has been used to prove several deep results in various branches of mathematics, even apparently unrelated to complex analysis. In this last section we briefly highlight some of these applications and extensions, referring to the bibliography for more information and details. Necessarily, the list of topics we have chosen to present is rather incomplete and only reflects our personal tastes and, certainly, we have not tried to give an exhaustive picture of them. For a more comprehensive view of the theory we strongly recommend the monographs [16] and [46].

Extremal problems

After Loewner and E. Peschl, the first to apply Loewner's method to extremal problems in the theory of univalent functions was G. M. Goluzin, obtaining in an elegant way several new and sharp estimates. The most important of them is the sharp estimate for the so-called rotation theorem (estimate of the argument of the derivative – see [20], [21]).

As already recalled, the main conjecture solved with the help of Loewner's theory is the Bieberbach conjecture. Loewner himself proved the case $n = 3$; P. R. Garabedian and M. Schiffer in 1955 solved the case $n = 4$; M. Ozawa in 1969 and R. N. Pederson in 1968 solved the case $n = 6$; and Pederson and Schiffer in 1972 solved $n = 5$. Finally, in 1985, L. de Branges [15] proved the full conjecture and, as already remarked, FitzGerald and Pommerenke [18] gave an alternative proof explicitly based on Loewner's method. In both cases, the main point was proving the validity of the Milin conjecture. Previously, Milin had shown, using the Lebedev-Milin inequality, that his conjecture implied the Bieberbach conjecture (see, for example, [16] for details). This is just a short and incomplete list of the many mathematicians who have worked on this and related problems; Loewner's method is by now an important analytical device, which generates a number of sharp inequalities not accessible by other means (see, for example, [16]).

Univalence criteria

To obtain practical criteria ensuring univalence of conformal maps is a basic and fundamental problem in complex analysis. Perhaps the most famous criterion of this type is due to Z. Nehari [43]. He showed that an estimate on the Schwarzian derivative $(f''/f')' - \frac{1}{2}(f''/f')^2$ implies the univalence of f in the unit disc. Later, P. L. Duren, H. S. Shapiro and A. L. Shields observed that an estimate on the pre-Schwarzian f''/f' implies Nehari's estimate and therefore implies univalence. Then, J. Becker [8] found a totally different approach based on Loewner's equation to show that a weaker estimate on the pre-Schwarzian implies univalence. In the same paper, Becker also applied Loewner's equation to give an independent derivation of Nehari's criterion. In fact, many univalence criteria have later been reproved using Loewner's method; and this approach sometimes provided further insight. We refer the reader to [25, Chapter 3] and [46] for further information.

Optimisation and Loewner chains

The variational method is a standard way to deal with extremal problems in a given class of functions. Roughly speaking, this means that one can try and get information on an extremal function by comparing it with nearby elements in the given class. The larger the family of perturbations, the more relevant the information one obtains. One example of this kind is the class of normalised univalent functions in the unit disc, which, as we already know, are "reachable" by the class of Loewner chains, and this approach has been applied to optimal control theory.

Variational methods were pioneered in the late 1930s by M. Schiffer and independently by G. M. Goluzin. Schiffer wrote a paper in 1945 that applied a variational method to Loewner's equation, the first introduction of a technique later refined as "optimal control". In particular, coefficient extremal problems for univalent functions as optimal control problems for finite-dimensional control systems have been treated by I. A. Aleksandrov and V. I. Popov, G. S. Goodman, S. Friedland and M. Schiffer, and D. V. Prokhorov, and later developed in an infinite dimensional setting by O. Roth [48].

Stochastic Loewner equation

As mentioned in the introduction, this equation was introduced by Schramm in 2000, replacing the driving term in the radial and chordal Loewner equation with a Brownian motion. In particular, the *(chordal) stochastic Loewner evolution* with parameter $k \geq 0$ (SLE_k) starting at a point $x \in \mathbb{R}$ is the random family of maps (g_t) obtained from the chordal Loewner equation (4.1) by letting $h(t) = \sqrt{k}B_t$, where B_t is a standard one dimensional Brownian motion such that $\sqrt{k}B_0 = x$. Similarly, one can define a radial stochastic Loewner evolution.

The SLE_k depends on the choice of the Brownian motion and it comes in several flavours depending on the type of Brownian motion exploited. For example, it might start at a fixed point or start at a uniformly distributed point, or might have a built in drift and so on. The parameter k controls the rate of diffusion of the Brownian motion and the behaviour of the SLE_k critically depends on the value of k.

The SLE_2 corresponds to the loop-erased random walk and the uniform spanning tree. The $SLE_{8/3}$ is conjectured to be the scaling limit of self-avoiding random walks. The SLE_3 is conjectured to be the limit of interfaces for the Ising model, while the SLE_4 corresponds to the harmonic explorer and the Gaussian free field. The SLE_6 was used by Lawler, Schramm and Werner in 2001 [36], [37] to prove the conjecture of Mandelbrot (1982) that the boundary of planar Brownian motion has fractal dimension $4/3$. Moreover, Smirnov [52] proved the SLE_6 is the scaling limit of critical site percolation on the triangular lattice. This result follows from his celebrated proof of Cardy's formula.

Also worthy of mention is the work of L. Carleson and N. G. Makarov [13] studying growth processes motivated by DLA (diffusion-limited aggregation) via Loewner's equations.

The expository paper [35] is perhaps the best option to start an exploration of this fascinating branch of mathematics.

Hele-Shaw flows

One of the most influential works in fluid dynamics at the end of the 19th century was that of Henry Selby Hele-Shaw. A Hele-Shaw cell is a tool for studying the two-dimensional flow of a viscous fluid in a narrow gap between two parallel plates. Nowadays the Hele-Shaw cell is used as a powerful tool in several fields of natural sciences and engineering, in particular soft condensed matter physics, material sciences, crystal growth and, of course, fluid mechanics.

In 1945, P. I. Polubarinova-Kochina and L. A. Galin introduced an evolution equation for conformal mappings related to Hele-Shaw flows. Kufarev and Vinogradov in 1948 reformulated this equation in the form of a non-linear (even non-quasilinear) integro-differential equation of Loewner type. Despite apparent differences, these two equations have some evident geometric connections and the properties of Loewner's equations play a fundamental role in the study of Polubarinova-Galin's equation. Moreover, this close relationship has suggested the interesting problem of analysing when the solutions of Loewner's equations admit quasiconformal extensions beyond the closed unit disc, a problem studied by J. Becker, V. Ya. Gutlyanskiĭ, A. Vasil'ev and others [8], [26, Chapters 2, 3 and 4].

A nice monograph on Hele-Shaw flows from the point of view of complex analysis, discussing in particular their connection with the Loewner method, is [26] (see also [54]).

Extensions to multiply connected domains

I. Komatu, in 1943 [30], was the first to generalise Loewner's parametric representation to univalent holomorphic functions defined in a circular annulus and with images in the exterior of a disc. Later, G. M. Goluzin [20] gave a much simpler way to establish Komatu's results. With the same techniques, E. P. Li [38] considered a slightly different case, when the image of the annulus is the complex plane with two slits (ending at infinity and at the origin, respectively).

Another way of adapting Loewner's method to multiply connected domains was developed by P. P. Kufarev and M. P. Kuvaev [34]. They obtained a differential equation satisfied by automorphic functions realising conformal covering mappings of the unit disc onto multiply connected domains with a gradually erased slit. Roughly speaking, these results can be considered a version for multiply connected domains of the slit-radial Loewner equation.

Recently, and in a similar way, R. O. Bauer and R. M. Friedrich have developed a slit-chordal theory for multiply connected domains. Moreover, they have even dealt with stochastic versions of both the radial and the chordal cases. In this framework the situation is more subtle than in the simply connected case, because moduli spaces enter the picture [6], [7].

Extension to several complex variables

As far as we know, the first to propose a Loewner theory in several complex variables was J. Pfaltzgraff, who in 1974 extended the basic Loewner theory to $\mathbb{C}^n$ with the aim of obtaining bounds and growth estimates for some classes of univalent mappings defined in the unit ball of $\mathbb{C}^n$. The theory was later developed by T. Poreda, I. Graham, G. Kohr, M. Kohr, H. Hamada and others [25], [24], [17], [12].

In [12], using an equation similar to (6.4), it is proved that there is a one-to-one correspondence between evolution families of order ∞ and Herglotz vector fields of order ∞ on complete hyperbolic complex manifolds whose Kobayashi distance is smooth enough (as it happens, for instance, in bounded strongly convex domains of $\mathbb{C}^n$).

A clear description of Loewner chains in several complex variables is not yet available. Most of the literature in higher dimensions is devoted to the radial Loewner equation (and its consequences) on the unit ball of complex Banach spaces, mainly $\mathbb{C}^n$. The theory is definitely much more complicated than in dimension one; for instance, the class of normalised univalent mappings on the unit ball of $\mathbb{C}^n$ is not compact, and thus one is forced to restrict attention to suitable compact subclasses. Anyway, many natural cases, such as spiral-like maps, can be treated efficiently and many applications and estimates can be obtained [25].

However, in general, there is not yet a satisfactory answer to the question of whether it is possible to associate to an evolution family (or a Herglotz vector field) on the unit ball of $\mathbb{C}^n$ a Loewner chain with image in $\mathbb{C}^n$ solving a Loewner PDE. Keeping in mind the interpretation of Loewner chains as "time-dependent linearisation" for evolution families, it is clear that resonances among eigenvalues of the differentials at the common fixed point(s) have to play a role.

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