Bounded slope condition, gradient estimate, and up-to-the-boundary Lipschitz estimates for harmonic functions

Bounded slope condition

The bounded slope condition is a geometric condition on the graph of a function

$$g : \partial\Omega \to \mathbb{R}.$$

Precisely, we have the following

Definizione 1 (Bounded slope condition)**.** *Let Ω be a bounded open set in $\mathbb{R}^d$ and let $g : \partial\Omega \to \mathbb{R}$ be a continuous function. We say that g satisfies the bounded slope condition, if there is a constant $S > 0$ such that: for every $x_0 \in \partial\Omega$ there exist two vectors*

$$\underline{\nu} \in B_1 \qquad e \qquad \overline{\nu} \in B_1,$$

such that

$$S\,\underline{\nu} \cdot (x - x_0) \le g(x) - g(x_0) \le S\,\overline{\nu} \cdot (x - x_0) \quad \textit{for all} \quad x \in \partial\Omega.$$

Proposizione 2. *A function $g : \partial\Omega \to \mathbb{R}$ that satisfies the bounded slope condition is Lipschitz continuous with Lipschitz constant S.*

Dimostrazione. For every $x, y \in \partial\Omega$, we have

$$-S|\underline{\nu}||x - y| \le S\,\underline{\nu} \cdot (x - y) \le g(x) - g(y) \le s\,\overline{\nu} \cdot (x - y) \le S|\overline{\nu}||x - y|.$$

Since $|\underline{\nu}| \le 1$ and $|\overline{\nu}| \le 1$ we have

$$-S|x - y| \le g(x) - g(y) \le S|x - y|.$$

which concludes the proof. □

Esercizio 3. *Let Ω be an open set in $\mathbb{R}^2$ whose boundary $\partial\Omega$ contains the segment $\Gamma := (-1, 1) \times \{0\}$ (for instance, one such domain is the rectangle $(-1, 1) \times (0, 1)$).*

- *Prove that the function $g(x, y) = x^2$ does not satisfy the bounded slope condition on $\partial\Omega$.*
- *Prove that if $g : \partial\Omega \to \mathbb{R}$ satisfies the bounded slope condition, then $g : \Gamma \to \mathbb{R}$ is an affine function.*

Proposizione 4. *Let Ω be a convex bounded open set of class C^2 in $\mathbb{R}^d$. Moreover, suppose that there is a positive constant $c > 0$ such that at every boundary point $x_0 \in \partial\Omega$ the principle curvatures $\kappa_1(x_0), \dots, \kappa_{d-1}(x_0)$ are bounded from below by c. Then, every function $g : \mathbb{R}^d \to \mathbb{R}$ of class C^2 satisfies the bounded slope condition on $\partial\Omega$.*

Dimostrazione. Fix a point $X_0 \in \partial\Omega$ and consider the function

$$h(X) = g(X) - g(X_0) - (X - X_0) \cdot \nabla g(X_0).$$

Since Ω is a bounded C^2 domain, there is a radius $R > 0$ such that for every $X_0 \in \partial\Omega$ there is a function

$$\eta : B'_R \to (-R, R)$$

of class C^2 on the ball $B'_R \subset \mathbb{R}^{d-1}$ such that, up to a rotation and translation,

$$X_0 = 0, \quad \eta(0) = 0\ , \quad \nabla_{x'}\eta(0) = 0\ ,$$

$$\Omega \cap \Big(B'_R \times (-R, R)\Big) = \Big\{(x, y) \in B'_R \times (-R, R)\ :\ y > \eta(x)\Big\}\,,$$

$$\partial\Omega \cap \Big(B'_R \times (-R, R)\Big) = \Big\{(x, \eta(x))\ :\ x \in B'_R\Big\}.$$

Assume for simplicity that $d = 2$. We proceed in several steps.

Step 1. Bounds on η' and η''. We know that

$$\eta'' \geq 0 \quad \text{in} \quad (-R,R).$$

and that the curvature $\kappa : \partial\Omega \to \mathbb{R}$ of $\partial\Omega$ is given by

$$\kappa\big(x,\eta(x)\big) = \frac{\eta''(x)}{\big(1+(\eta'(x))^2\big)^{3/2}}.$$

By the lower bound on the curvature and the continuity of κ, there are constants $c>0$ and $C>0$ such that

$$0 < c \leq \kappa\big(x,\eta(x)\big) \leq C < +\infty.$$

First we notice that

$$\eta'(0) = 0 \quad \text{and} \quad \eta' \geq 0 \quad \text{on} \quad (0,R).$$

Moreover, if

$$\begin{aligned} \eta' \leq \kappa \quad \text{on} \quad [0,L] \qquad &\Rightarrow \qquad \eta''(x) \leq (1+\kappa^2)^{3/2}\frac{\eta''(x)}{\big(1+(\eta'(x))^2\big)^{3/2}} \leq C(1+\kappa^2)^{3/2} \quad \text{on} \quad [0,L] \\ &\Rightarrow \qquad \eta'(x) \leq LC(1+\kappa^2)^{3/2} \quad \text{on} \quad [0,L]. \end{aligned}$$

Thus, choosing

$$L := \min\Big\{\frac{1}{8C}, R\Big\},$$

we get that

$$\eta' \leq 1 \quad \text{on} \quad [0,L].$$

By the same argument on $[-L,0]$, we get

$$|\eta'| \leq 1 \quad \text{on} \quad [-L,L].$$

In particular, this implies the bounds

$$\frac{\eta''(x)}{\big(1+(\eta'(x))^2\big)^{3/2}} \leq \eta''(x) \leq 2^{3/2}\frac{\eta''(x)}{\big(1+(\eta'(x))^2\big)^{3/2}} \quad \text{for all} \quad x \in [-L,L],$$

so we get

$$c \leq \eta''(x) \leq 2^{3/2}C \quad \text{for all} \quad x \in [-L,L].$$

Step 2. An upper bound on $h(x,\eta(x))$ for $x \in [-L,L]$. We consider the function

$$v : [-L,L] \to \mathbb{R}\ , \qquad v(x) = h(x,\eta(x)),$$

which is such that

$$v(0) = v'(0) = 0$$

and

$$\begin{aligned} v''(x) &= h_{xx}(x,\eta(x)) + 2\eta'(x)h_{xy}(x,\eta(x)) + |\eta'(x)|^2 h_{yy}(x,\eta(x)) + \eta''(x)h_y(x,\eta(x)) \\ &= g_{xx}(x,\eta(x)) + 2\eta'(x)g_{xy}(x,\eta(x)) + |\eta'(x)|^2 g_{yy}(x,\eta(x)) + \eta''(x)\Big(g_y(x,\eta(x)) - g_y(0,0)\Big). \end{aligned}$$

Since g_{xx}, g_{xy}, g_{yy}, g_y are bounded functions, setting

$$\|\nabla^2 g\|_{L^\infty} = \sup\sqrt{g_{xx}^2 + 2g_{xy}^2 + g_{yy}^2} \qquad \text{and} \qquad \|\nabla g\|_{L^\infty} = \sqrt{g_x^2 + g_y^2},$$

we get

$$|v''| \leq 4\|\nabla^2 g\|_{L^\infty} + 2C\|\nabla g\|_{L^\infty}.$$

Thus, setting

$$A := \frac{4\|\nabla^2 g\|_{L^\infty} + 2C\|\nabla g\|_{L^\infty}}{c}$$

we get

$$|v''(x)| \leq A\eta''(x) \quad \text{for all} \quad x \in [-L,L].$$

This implies that

$$|h(x,\eta(x))| = |v(x)| \leq A\eta(x) \quad \text{for all} \quad x \in [-L,L].$$

Step 3. An upper bound on $h : \partial\Omega \to \mathbb{R}$. By the lower bound

$$\eta''(x) \geq \frac{\eta''(x)}{\big(1+(\eta'(x))^2\big)^{3/2}} \geq c,$$

we obtain that

$$\eta(L) \geq \frac{c}{2}L^2 \quad \text{and} \quad \eta(-L) \geq \frac{c}{2}L^2.$$

Thus, setting

$$a := \inf\Big\{L, \frac{c}{2}L^2\Big\},$$

we get that there are points

$$\ell_- \in [-L, 0) \qquad \text{and} \qquad \ell_+ \in (0, L],$$

such that

$$\eta(\ell_-) = \eta(\ell_+) = a.$$

Now, by the convexity of Ω we know that

$$\Omega \cap \big\{(x,y) \in \mathbb{R}^2 \ : \ y \leq a\big\} = \big\{(x,y) \in \mathbb{R}^2 \ : \ \ell_- \leq x \leq \ell_+ \ , \ \eta(x) \leq y \leq a\big\}.$$

This implies the bound

$$|h(x,y)| \leq Ay \quad \text{for all} \quad (x,y) \in \partial\Omega \cap \big\{(x,y) \in \mathbb{R}^2 \ : \ y \leq a\big\}.$$

On the other hand, it is immediate to check that

$$|h(x,y)| \leq \frac{\|g\|_{L^\infty(\partial\Omega)}}{a} y \quad \text{for all} \quad (x,y) \in \partial\Omega \cap \big\{(x,y) \in \mathbb{R}^2 \ : \ y \geq a\big\}.$$

Thus, there is a constant K, that depends on Ω and g, such that:

$$|h(x,y)| \leq Ky \quad \text{for all} \quad (x,y) \in \partial\Omega. \tag{1}$$

Step 4. Conclusion. We recall that by the the definition of h, we get that

$$g(x,y) - g(0,0) = h(x,y) + (x,y) \cdot \nabla g(0,0).$$

Using the estimate (1), we get that

$$-Ky + (x,y) \cdot \nabla g(0,0) \leq g(x,y) - g(0,0) \leq Ky + (x,y) \cdot \nabla g(0,0),$$

which can also be written as

$$(x,y) \cdot \Big(-Ke_2 + \nabla g(0,0)\Big) \leq g(x,y) - g(0,0) \leq (x,y) \cdot \Big(Ke_2 + \nabla g(0,0)\Big),$$

Finally, setting

$$S := K + \|\nabla g\|_{L^\infty(\partial\Omega)},$$

we get that g satisfies the bounded slope condition with slope S. $\square$

Lipschitz regularity up to the boundary

Let Ω be a bounded open set in $\mathbb{R}^d$. Thanks to Proposition 2 we know that if $g : \partial\Omega \to \mathbb{R}$ is a function that satisfies the bounded slope condition on $\partial\Omega$, then it is Lipschitz continuous on $\partial\Omega$. Thus, it can be extended to a Lipschitz continuous function $\widetilde{g} : \mathbb{R}^d \to \mathbb{R}$. As a consequence, there is a unique weak solution $h \in H^1(\Omega)$ of the problem

$$\Delta h = 0 \quad \text{in} \quad \Omega \ , \qquad h = g \quad \text{on} \quad \partial\Omega,$$

in the sense that

$$h - \widetilde{g} \in H^1_0(\Omega),$$

and

$$\int_\Omega \nabla h \cdot \nabla\varphi \, dx = 0 \qquad \text{for every} \qquad \varphi \in C^\infty_c(\mathbb{R}^d).$$

We already know that h is C^∞ in Ω and that

$$\Delta h(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

In this section we will prove that h is Lipschitz continuous on $\overline{\Omega}$ (with Lipschitz constant depending only on the slope of g and on the dimension of the space). We will only need that $g : \partial\Omega \to \mathbb{R}$ satisfies the bounded slope condition as the proof does not require any further assumptions on the geometry or on the regularity of Ω. A key ingredient of the proof will be the following general result for harmonic functions.

Lemma 5 (Gradient estimate)**.** *Let $u \in C^\infty(B_R)$ be a harmonic function in a ball $B_R \subset \mathbb{R}^d$. Then*

$$\|\nabla u\|_{L^\infty(B_{R/2})} \le \frac{2d}{R}\|u\|_{L^\infty(B_R)}.$$

Dimostrazione. For every $i = 1, \dots, d$, the partial derivative $\partial_i u : B_R \to \mathbb{R}$ is a harmonic function in B_R. In particular, for every constant unit vector $V = (v_1, \dots, v_d) \in \mathbb{R}^d$, the function

$$V \cdot \nabla u : B_R \to \mathbb{R}\ , \quad V \cdot \nabla u(x) = \sum_{j=1}^d v_j \partial_j u(x),$$

is harmonic in B_R and so it satisfies the mean value property

$$V \cdot \nabla u(x_0) = \fint_{B_r(x_0)} V \cdot \nabla u(y)\, dy,$$

for every ball $B_r(x_0) \subset B_R$. Since we have the inclusion

$$B_{R/2}(x_0) \subset B_R \quad \text{for every} \quad x_0 \in B_{R/2},$$

we get that

$$V \cdot \nabla u(x_0) = \fint_{B_{R/2}(x_0)} V \cdot \nabla u(x)\, dx = \frac{2^d}{\omega_d R^d} \int_{B_{R/2}(x_0)} V \cdot \nabla u(x)\, dx \quad \text{for every} \quad x_0 \in B_{R/2}.$$

Now, by the divergence theorem

$$\int_\Omega \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu\ ,$$

applied to the domain $\Omega := B_{R/2}(x_0)$ and to the vector field $X(x) := u(x)V$, we get

$$\int_{B_{R/2}(x_0)} V \cdot \nabla u(x)\, dx = \int_{\partial B_{R/2}(x_0)} u(x) V \cdot \nu(x)\, d\mathcal{H}^{d-1}(x),$$

where $\nu(x) := x/R$ is the exterior normal to $\partial B_{R/2}(x_0)$. Finally, using that

$$|u(x) V \cdot \nu(x)| \le |u(x)| \quad \text{on} \quad \partial B_{R/2}(x_0),$$

and putting the above estimates together, we get

$$\begin{aligned}
|V \cdot \nabla u(x_0)| &= \frac{2^d}{\omega_d R^d} \left| \int_{B_{R/2}(x_0)} V \cdot \nabla u(y)\, dy \right| \\
&= \frac{2^d}{\omega_d R^d} \left| \int_{\partial B_{R/2}(x_0)} u\, V \cdot \nu\, d\mathcal{H}^{d-1} \right| \\
&\le \frac{2d}{R} \|u\|_{L^\infty(\partial B_{R/2}(x_0))} \\
&\le \frac{2d}{R} \|u\|_{L^\infty(B_R)}\ .
\end{aligned}$$

Since $x_0 \in B_{R/2}$ and $V \in \partial B_1$ is arbitrary, this concludes the proof. □

Teorema 6 (Bounded slope condition and Lipschitz regularity up to the boundary)**.** *Let Ω be a bounded open set in $\mathbb{R}^d$ and let $g : \partial\Omega \to \mathbb{R}$ be a function that satisfies the bounded slope condition with slope $S > 0$ on $\partial\Omega$. Let $h \in H^1(\Omega)$ be the weak solution to the problem*

$$\Delta h = 0 \quad \text{in} \quad \Omega\ , \qquad h = g \quad \text{su} \quad \partial\Omega.$$

Then, there is a dimensional constant C_d such that

$$|\nabla h| \le C_d S \qquad \text{on} \qquad \Omega,$$

and such that the function

$$\widetilde{h} : \overline{\Omega} \to \mathbb{R}\ , \qquad \widetilde{h}(x) := \begin{cases} h(x) & \text{if} \quad x \in \Omega, \\ g(x) & \text{if} \quad x \in \partial\Omega, \end{cases}$$

is Lipschitz continuous on $\overline{\Omega}$ with Lipschitz constant $C_d S$.

Dimostrazione. We know that h is C^∞ in Ω. We will first prove that $|\nabla h|$ is bounded in Ω. Consider a point $x_0 \in \Omega$ and let y_0 be a projection of x_0 on $\partial\Omega$. Setting

$$R_0 := |x_0 - y_0|$$

we get that $B_{R_0}(x_0) \subset \Omega$. Then, by the gradient estimate in $B_{R_0}(x_0)$ applied to the harmonic function

$$x \mapsto h(x) - g(y_0)$$

we get that

$$|\nabla h(x_0)| \le \frac{2d}{R_0} \|h - g(y_0)\|_{L^\infty(B_{R_0}(x_0))}.$$

Now, since g satisfied the bounded slope condition on $\partial\Omega$ we have that

$$S\,\underline{\nu} \cdot (y - y_0) \le g(y) - g(y_0) \le S\,\overline{\nu} \cdot (y - y_0) \quad \text{for all} \quad y \in \partial\Omega.$$

for some vectors $|\underline{\nu}| \le 1$ and $|\overline{\nu}| \le 1$. Since the functions

$$x \mapsto S\,\underline{\nu} \cdot (x - y_0) \qquad \text{and} \qquad x \mapsto S\,\overline{\nu} \cdot (x - y_0)$$

are harmonic on Ω, the (weak) maximum principle gives

$$S\,\underline{\nu} \cdot (x - y_0) \le h(x) - g(y_0) \le S\,\overline{\nu} \cdot (x - y_0) \quad \text{for all} \quad y \in \partial\Omega,$$

so that

$$|h(x) - g(y_0)| \le S\,|x - y_0| \quad \text{for all} \quad x \in \Omega.$$

In particular, for every $x \in B_{R_0}(x_0)$ we have

$$|h(x) - g(y_0)| \le S\,|x - y_0| \le S2R_0,$$

which combined with the gradient estimate provides

$$|\nabla h(x_0)| \le \frac{2d}{R_0} \|h - g(y_0)\|_{L^\infty(B_{R_0}(x_0))} \le \frac{2d}{R_0} 2SR_0 = 4dS.$$

Since $x_0 \in \Omega$ was arbitrary, we get

$$|\nabla h| \le 2dS \quad \text{in} \quad \Omega. \tag{2}$$

We next prove that h is Lipschitz continuous up to the boundary. Let x_1 and x_2 be two points in $\overline{\Omega}$, let

$$\delta_1 := \mathrm{dist}(x_1, \partial\Omega)\ , \quad \delta_2 := \mathrm{dist}(x_2, \partial\Omega)\ ,$$

and let y_1 and y_2 be two points on $\partial\Omega$ that realize the distances δ_1 and δ_2 respectively. Without loss of generality we can suppose that

$$|x_1 - y_1| = \delta_1 \le \delta_2 = |x_2 - y_2|.$$

We notice that by construction

$$\delta_2 \le |x_2 - y_1| \le |x_1 - x_2| + \delta_1.$$

We consider two cases.

Case 1. $|x_1 - x_2| \le 10\delta_1$**.** Then, the ball of center x_1 and radius $r = 2|x_1 - x_2|$ is contained in Ω. Applying the estimate (2) in $B_r(x_1)$ we get that

$$|h(x_1) - h(x_2)| \le \|\nabla h\|_{L^\infty(B_r)} |x_1 - x_2| \le 4dS|x_1 - x_2|.$$

Case 2. $|x_1 - x_2| \ge 10\delta_1$**.** Then, the ball with center y_1 and radius $R = 2|x_1 - x_2|$ contains both x_1 and x_2. Indeed,

$$|x_1 - y_1| = \delta_1 \le |x_1 - x_2|$$

and

$$|x_2 - y_1| \le |x_1 - x_2| + |x_1 - y_1| \le |x_1 - x_2| + \delta_1 \le 2|x_1 - x_2|.$$

Now, we use again the bounded slope condition on g and the maximum principle. By the bounded slope condition, there are vectors $\underline{\nu}$ and $\overline{\nu}$ such that

$$S\,\underline{\nu} \cdot (y - y_1) \le g(y) - g(y_1) \le S\,\overline{\nu} \cdot (y - y_1) \quad \text{for all} \quad y \in \partial\Omega.$$

Since the functions

$$x \mapsto S\,\underline{\nu} \cdot (x - y_1) \qquad \text{and} \qquad x \mapsto S\,\overline{\nu} \cdot (x - y_1)$$

are harmonic on Ω, the (weak) maximum principle gives

$$S\,\underline{\nu}\cdot(x-y_1)\le h(x)-g(y_1)\le S\,\overline{\nu}\cdot(x-y_1)\quad\text{for all}\quad y\in\partial\Omega.$$

Combining the estimates for $g-g(y_1)$ on the boundary and $h-g(y_1)$ in Ω, we get

$$|\widetilde{h}(x)-g(y_1)|\le S\,|x-y_1|\quad\text{for all}\quad x\in\overline{\Omega}.$$

Thus,

$$\begin{aligned}|\widetilde{h}(x_2)-\widetilde{h}(x_1)| &\le |\widetilde{h}(x_1)-g(y_1)|+|\widetilde{h}(x_2)-g(y_1)|\\ &\le S\,|x_1-y_1|+S\,|x_2-y_1|\\ &\le 3S|x_1-x_2|.\end{aligned}$$

In conclusion, since $4d\ge 3$, combining the two cases above, we get that

$$|\widetilde{h}(x_2)-\widetilde{h}(x_1)|\le 4dS|x_1-x_2|,$$

which concludes the proof. □