Trace spaces of counterexamples to Naĭmark's Problem

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Outline

1. Naĭmark's Problem
2. Counterexamples to Naĭmark's Problem
3. Trace spaces of counterexamples to Naĭmark's Problem

Naĭmark's Problem

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Definition

A *counterexample to Naĭmark's Problem* is a C*-algebra with only one irreducible representation up to unitarily equivalence which is not isomorphic to any $K(H)$.

Consistency of a counterexample to Naĭmark's Problem

The Diamond Principle ($\diamondsuit$)

There exists a sequence of sets $\{S_\beta\}_{\beta<\aleph_1}$ such that $S_\beta \subseteq \beta$, and for any $S \subseteq \aleph_1$ the set $\{\beta : S \cap \beta = S_\beta\}$ is stationary.

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Theorem (Akemann-Weaver, 2004)

Assume $\diamondsuit$. There exists a counterexample to Naĭmark's Problem.

Characterizing counterexamples to Naĭmark's Problem

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- A can't be separable (Rosenberg, 1953)

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- A can't be separable (Rosenberg, 1953)

Remark

A counterexample to Naĭmark's Problem would also guarantee the failure of Glimm's Theorem on type I C-algebras in the nonseparable setting.*

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$$\begin{aligned} \Psi : U(A) \times S(A) &\to S(A) \\ (u, \phi) &\mapsto \phi \circ \mathsf{Ad}\, u \end{aligned}$$

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Question

How big can $T(A)$ be?

The main result

Theorem

Assume $\diamondsuit$, and let X be a metrizable Choquet simplex.

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1. *There exists a counterexample to Naĭmark's Problem A such that $T(A) \cong X$.*
2. *There exists a counterexample to Naĭmark's Problem whose trace space is nonseparable.*

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We want a nonseparable simple unital C*-algebra A such that $f \sim g$ for all $f, g \in P(A)$. Build a sequence of separable simple unital C*-algebras and pure states

$$(A_0, f_0) \subseteq (A_1, f_1) \subseteq \cdots \subseteq (A_\beta, f_\beta) \subseteq \cdots \subseteq (A = \cup_{\beta<\aleph_1} A_\beta, f)$$

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- β limit: $A_\beta = \overline{\cup_{\gamma<\beta} A_\gamma}$ and f_β is the only extension of all f_γ's
- $\beta + 1$: pick a "certain" $g_\beta \in P(A_\beta)$ such that $g_\beta \nsim f_\beta$ and build $A_{\beta+1}$ so that g' and $f_{\beta+1}$ are the unique extensions respectively of g_β and f_β and $g' \sim f_{\beta+1}$

Kishimoto-Ozawa-Sakai Theorem

Theorem (Kishimoto-Ozawa-Sakai, 2003)

Let A be a separable simple unital C-algebra. If f and g are two pure states on A, there is an asymptotically inner automorphism α (i.e there is a path of unitaries $(u_t)_{t\in[0,\infty)}$ such that $\alpha(a) = \lim_{t\to\infty} Adu_t(a)$ for all $a \in A$) such that $f = g \circ \alpha$.*

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Lemma

Let A be a separable simple unital C-algebra, and let f and g be two inequivalent pure states on A. There exists a separable simple unital C*-algebra B which unitally contains A such that f and g have unique equivalent extensions to B.*

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Lemma

Let A be a separable simple unital C-algebra, and let f and g be two inequivalent pure states on A. There exists a separable simple unital C*-algebra B which unitally contains A such that f and g have unique equivalent extensions to B.*

The idea is to put $B = A \rtimes_\alpha \mathbb{Z}$, where α is the automorphism given by KOS-AW Theorem such that $f \sim g \circ \alpha$.

The trace space of a counterexample to Naĭmark's Problem

Proposition

Given a counterexample to Naĭmark's Problem $A = \bigcup_{\beta<\aleph_1} A_\beta$ *from the Akemann-Weaver's construction, there is an embedding* $e : T(A_0) \to T(A)$*.*

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Let B be a C*-algebra and $\tau \in T(B)$.

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Let B be a C*-algebra and $\tau \in T(B)$. If $\alpha \in \mathrm{Aut}(B)$, and τ is α-invariant ($\tau(\alpha(a)) = \tau(a)$ for all $a \in B$), then

$$\tau' \left(\sum_{n \in \mathbb{Z}} a_n u_\alpha^n \right) = \tau(a_0)$$

is a trace of $B \rtimes_\alpha \mathbb{Z}$ extending τ.

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Corollary

Assume $\diamondsuit$. Given any metrizable Choquet simplex X, there is a counterexample to Naĭmark's Problem A such that X can be embedded in $T(A)$.

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Is there a counterexample to Naĭmark's Problem A such that $T(A)$ is nonseparable?

The trace space of a crossed product

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$$u_\alpha^\tau(a\xi_\tau) = \alpha(a)(\xi_\tau)$$

is such that $\mathrm{Ad}u_\alpha^\tau = \alpha$ on $\pi_\tau(B)$.

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Theorem (Thomsen, 1995)

Consider the crossed product $B \rtimes_\alpha \mathbb{Z}$, B being separable unital. Suppose furthermore that α is approximately inner. The following are equivalent:

1. *The restriction map $r : T(B \rtimes_\alpha \mathbb{Z}) \to T(B)$ is an homeomorphism.*
2. *$\alpha_\tau^k \restriction \pi_\tau(B)''$ is outer for all extremal traces τ and all $k \in \mathbb{Z}$.*

Two variants of Kishimoto-Ozawa-Sakai Theorem

Theorem

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1. *if A is nuclear $\alpha_\tau^k \restriction \pi_\tau(A)''$ is **outer** for all $k \in \mathbb{Z}$ and all $\tau \in \partial T(A)$.*
2. *$\alpha_\tau \restriction \pi_\tau(A)''$ is **inner** for some $\tau \in \partial T(A)$.*

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Bibliography

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