Trace spaces of counterexamples to Naǐmark’s Problem

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Outline

1. Naïmark’s Problem
2. Counterexamples to Naïmark’s Problem
3. Trace spaces of counterexamples to Naïmark’s Problem
Naĭmark’s Problem

Every irreducible representation of the algebra of compact operators $K(H)$ is unitarily equivalent to the identity representation.
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Problem (Naĭmark, 1951)

If $A$ is a C*-algebra with only one irreducible representation up to unitary equivalence, is $A$ necessarily isomorphic to $K(H)$ for some Hilbert space $H$?
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**Problem (Naĭmark, 1951)**

If $A$ is a C*-algebra with only one irreducible representation up to unitary equivalence, is $A$ necessarily isomorphic to $K(H)$ for some Hilbert space $H$?

**Definition**

A *counterexample to Naĭmark’s Problem* is a C*-algebra with only one irreducible representation up to unitarily equivalence which is not isomorphic to any $K(H)$. 
Consistency of a counterexample to Naĭmark’s Problem

The Diamond Principle (◊)

There exists a sequence of sets \( \{ S_\beta \}_{\beta < \aleph_1} \) such that \( S_\beta \subseteq \beta \), and for any \( S \subseteq \aleph_1 \) the set \( \{ \beta : S \cap \beta = S_\beta \} \) is stationary.
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The Diamond Principle is true in Gödel’s constructible universe and implies CH, hence it is independent from ZFC.
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Theorem (Akemann-Weaver, 2004)

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- $A$ must be simple

Remark: A counterexample to Na̧ımark’s Problem would also guarantee the failure of Glimm’s Theorem on type I C*-algebras in the nonseparable setting.
Characterizing counterexamples to Naĭmark’s Problem

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- $A$ can’t be type I (Kaplanski, 1951)

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Remark

A counterexample to Naǐmark’s Problem would also guarantee the failure of Glimm’s Theorem on type I $C^*$-algebras in the nonseparable setting.
The action of $U(A)$ on $S(A)$

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$$\Psi : U(A) \times S(A) \rightarrow S(A)$$

$$(u, \phi) \mapsto \phi \circ \text{Ad} u$$

Question: How big can $T(A)$ be?
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How big can $T(A)$ be?
The main result

**Theorem**

Assume ♦, and let $X$ be a metrizable Choquet simplex.

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Assume \[\diamondsuit\], and let \( X \) be a metrizable Choquet simplex.

1. There exists a counterexample to Naïmark’s Problem A such that \( T(A) \cong X \).
2. There exists a counterexample to Naïmark’s Problem whose trace space is nonseparable.
The Akemann-Weaver’s Theorem

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We want a nonseparable simple unital C*-algebra $A$ such that $f \sim g$ for all $f, g \in P(A)$. Build a sequence of separable simple unital C*-algebras and pure states

$$(A_0, f_0) \subseteq (A_1, f_1) \subseteq \cdots \subseteq (A_\beta, f_\beta) \subseteq \cdots \subseteq (A = \bigcup_{\beta < \aleph_1} A_\beta, f)$$
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- **β limit:** $A_\beta = \overline{\bigcup_{\gamma < \beta} A_\gamma}$ and $f_\beta$ is the only extension of all $f_\gamma$’s
- **β + 1:** pick a “certain” $g_\beta \in P(A_\beta)$ such that $g_\beta \not\sim f_\beta$ and build $A_{\beta+1}$ so that $g'$ and $f_{\beta+1}$ are the unique extensions respectively of $g_\beta$ and $f_\beta$ and $g' \sim f_{\beta+1}$
Kishimoto-Ozawa-Sakai Theorem

Theorem (Kishimoto-Ozawa-Sakai, 2003)

Let $A$ be a separable simple unital $C^*$-algebra. If $f$ and $g$ are two pure states on $A$, there is an asymptotically inner automorphism $\alpha$ (i.e. there is a path of unitaries $(u_t)_{t \in [0, \infty)}$ such that $\alpha(a) = \lim_{t \to \infty} A u_t(a)$ for all $a \in A$) such that $f = g \circ \alpha$. 
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**Lemma**

Let $A$ be a separable simple unital $C^*$-algebra, and let $f$ and $g$ be two inequivalent pure states on $A$. There exists a separable simple unital $C^*$-algebra $B$ which unitally contains $A$ such that $f$ and $g$ have unique equivalent extensions to $B$. 

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**Kishimoto-Ozawa-Sakai Theorem**
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Let $A$ be a separable simple unital C*-algebra. If $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ are two sequences of inequivalent pure states on $A$, there is an asymptotically inner automorphism $\alpha$ (i.e. there is a path of unitaries $(u_t)_{t \in [0, \infty)}$ such that $\alpha(a) = \lim_{t \to \infty} A u_t(a)$ for all $a \in A$) such that $f_n \sim g_n \circ \alpha$ for all $n \in \mathbb{N}$.

Lemma

Let $A$ be a separable simple unital C*-algebra, and let $f$ and $g$ be two inequivalent pure states on $A$. There exists a separable simple unital C*-algebra $B$ which unitally contains $A$ such that $f$ and $g$ have unique equivalent extensions to $B$.

The idea is to put $B = A \rtimes_\alpha \mathbb{Z}$, where $\alpha$ is the automorphism given by KOS-AW Theorem such that $f \sim g \circ \alpha$. 
The trace space of a counterexample to Naǐmark’s Problem

**Proposition**

Given a counterexample to Naǐmark’s Problem \( A = \bigcup_{\beta < \aleph_1} A_\beta \) from the Akemann-Weaver’s construction, there is an embedding \( e : T(A_0) \to T(A) \).
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Given a counterexample to Našmark’s Problem $A = \bigcup_{\beta < \aleph_1} A_\beta$ from the Akemann-Weaver’s construction, there is an embedding $e : T(A_0) \to T(A)$.

Proof:
Let $B$ be a C*-algebra and $\tau \in T(B)$. 

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**Proof:**

Let \( B \) be a C*-algebra and \( \tau \in T(B) \). If \( \alpha \in \text{Aut}(B) \), and \( \tau \) is \( \alpha \)-invariant (\( \tau(\alpha(a)) = \tau(a) \) for all \( a \in B \)), then

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\tau' \left( \sum_{n \in \mathbb{Z}} a_n u_\alpha^n \right) = \tau(a_0)
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is a trace of \( B \rtimes_\alpha \mathbb{Z} \) extending \( \tau \).
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is a trace of $B \rtimes_{\alpha} \mathbb{Z}$ extending $\tau$. Since every trace is invariant for inner automorphisms, it is also invariant for asymptotically inner automorphisms. It is thus possible to iteratively extend any $\tau \in T(A_0)$ to a trace on $A$. 
Corollary

Assume ♦. Given any metrizable Choquet simplex $X$, there is a counterexample to Naïmark’s Problem $A$ such that $X$ can be embedded in $T(A)$. 
Corollary

Assume ◊. Given any metrizable Choquet simplex $X$, there is a counterexample to Naïmark’s Problem A such that $X$ can be embedded in $T(A)$.

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\beta \subseteq \cdots \subseteq A = \bigcup_{\beta < \omega_1} A_\beta$$
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$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\beta \subseteq \cdots \subseteq A = \bigcup_{\beta < \aleph_1} A_\beta$$

$$T(A_0) \overset{e_{0,1}}{\leftrightarrow} T(A_1) \overset{e_{1,2}}{\leftrightarrow} \cdots T(A_\beta) \overset{e_{\beta,\beta+1}}{\leftrightarrow} \cdots \overset{e}{\leftrightarrow} T(A)$$
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Assume \( \diamond \). Given any metrizable Choquet simplex \( X \), there is a counterexample to Naǐmark’s Problem A such that \( X \) can be embedded in \( T(A) \).

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A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\beta \subseteq \cdots \subseteq A = \bigcup_{\beta < \aleph_1} A_\beta
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- if \( \beta \) is limit ordinal then \( A_\beta = \overline{\bigcup_{\gamma < \beta} A_\gamma} \)
- \( A_{\beta+1} = A_\beta \asymp_\alpha \mathbb{Z} \) for an asymptotically inner \( \alpha \)
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Assume ♦. Given any metrizable Choquet simplex \( X \), there is a counterexample to Naïmark’s Problem \( A \) such that \( X \) can be embedded in \( T(A) \).

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Question

Given any metrizable Choquet simplex \( X \), is there a counterexample to Naïmark’s Problem \( A \) such that \( T(A) \cong X \)?
Corollary

Assume ◇. Given any metrizable Choquet simplex $X$, there is a counterexample to Naĭmark’s Problem A such that $X$ can be embedded in $T(A)$.

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- if $\beta$ is limit ordinal then $A_\beta = \bigcup_{\gamma < \beta} A_\gamma$
- $A_{\beta+1} = A_\beta \rtimes_\alpha Z$ for an asymptotically inner $\alpha$

Question

Given any metrizable Choquet simplex $X$, is there a counterexample to Naĭmark’s Problem A such that $T(A) \cong X$?

Question

Is there a counterexample to Naĭmark’s Problem A such that $T(A)$ is nonseparable?
The trace space of a crossed product

Consider $\tau \in T(B)$ and let $(\pi_\tau, H_\tau, \xi_\tau)$ be the GNS representation associated to $\tau$. 

Theorem (Thomsen, 1995)

Consider the crossed product $B \rtimes Z$, $B$ being separable unital. Suppose furthermore that $\alpha$ is approximately inner. The following are equivalent:

1. The restriction map $r: T(B \rtimes Z) \to T(B)$ is an homeomorphism.
2. $\alpha^k \tau \upharpoonright \pi_\tau(B)'$ is outer for all extremal traces $\tau$ and all $k \in Z$. 

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Consider $\tau \in T(B)$ and let $(\pi_\tau, H_\tau, \xi_\tau)$ be the GNS representation associated to $\tau$. If $\alpha \in \text{Aut}(B)$ and $\tau$ is $\alpha$-invariant, then the unique unitary $u^\tau_\alpha$ on $H_\tau$ defined as follows, given a $a \in B$

$$u^\tau_\alpha(a\xi_\tau) = \alpha(a)(\xi_\tau)$$

is such that $\text{Ad} u^\tau_\alpha = \alpha$ on $\pi_\tau(B)$. 

Thus $\alpha$ can be extended to all $B(\mathcal{H}_\tau)$. We will denote such extension by $\alpha^\tau$. 

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Consider \( \tau \in T(B) \) and let \((\pi_\tau, H_\tau, \xi_\tau)\) be the GNS representation associated to \( \tau \). If \( \alpha \in \text{Aut}(B) \) and \( \tau \) is \( \alpha \)-invariant, then the unique unitary \( u_\alpha^\tau \) on \( H_\tau \) defined as follows, given a \( a \in B \)

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is such that \( \text{Ad} u_\alpha^\tau = \alpha \) on \( \pi_\tau(B) \). Thus \( \alpha \) can be extended to all \( B(H_\tau) \). We will denote such extension by \( \alpha_\tau \).
The trace space of a crossed product

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Consider the crossed product $B \rtimes_\alpha \mathbb{Z}$, $B$ being separable unital. Suppose furthermore that $\alpha$ is approximately inner. The following are equivalent:

1. The restriction map $r : T(B \rtimes_\alpha \mathbb{Z}) \to T(B)$ is an homeomorphism.
2. $\alpha_k^\tau \upharpoonright \pi_\tau(B)''$ is outer for all extremal traces $\tau$ and all $k \in \mathbb{Z}$. 
Two variants of Kishimoto-Ozawa-Sakai Theorem

**Theorem**

Let $A$ be a separable simple unital C*-algebra. If $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ are two sequences of inequivalent pure states on $A$. Then there is an asymptotically inner automorphism $\alpha$ such that $f_n \sim g_n \circ \alpha$ for all $n \in \mathbb{N}$.

1. If $A$ is nuclear, $\alpha_k \mid_{\pi(\tau)(A)''}$ is outer for all $k \in \mathbb{Z}$ and all $\tau \in \partial T(A)$.

2. $\alpha_{\tau} \mid_{\pi(\tau)(A)''}$ is inner for some $\tau \in \partial T(A)$.
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Let $A$ be a separable simple unital C*-algebra. If $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ are two sequences of inequivalent pure states on $A$. Then there is an asymptotically inner automorphism $\alpha$ such that $f_n \sim g_n \circ \alpha$ for all $n \in \mathbb{N}$ and one of the following holds:

1. **if $A$ is nuclear** $\alpha_k^\tau \upharpoonright \pi_\tau(A)''$ is **outer** for all $k \in \mathbb{Z}$ and all $\tau \in \partial T(A)$.
2. **$\alpha_\tau \upharpoonright \pi_\tau(A)''$ is **inner** for some $\tau \in \partial T(A)$.
The main result (again)

Theorem

Assume ♦, and let $X$ be a metrizable Choquet simplex.

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Thank you!
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