C^*-algebras and B-names for Complex Numbers

Andrea Vaccaro

University of Pisa

Gargnano, Scuola Estiva di Logica 2016

Outline

1. Set Theory
2. Independent problems and Forcing
3. Boolean Valued Models
4. Generic Absoluteness
5. B-names for Complex Numbers and C^*-algebras

First Order Logic

Fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

First Order Logic

Fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

An $\mathcal{L}$-**structure** is a tuple

$$\mathcal{M} = \langle M, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K \rangle$$

First Order Logic

Fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

An $\mathcal{L}$-**structure** is a tuple

$$\mathcal{M} = \langle M, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K\rangle$$

where:

- M is a non-empty set;
- $R_i^{\mathcal{M}}$ is a subset of M^{n_i};
- $f_j^{\mathcal{M}}$ is a function from M^{m_j} to M;
- $c_k^{\mathcal{M}}$ is a element of M.

Tarski's Semantic

The truth value of atomic formulas is determined by the interpretation of the symbols in $\mathcal{L}$.

Tarski's Semantic

The truth value of atomic formulas is determined by the interpretation of the symbols in $\mathcal{L}$. For instance, if R is an n-ary relation symbol:

- $\mathcal{M} \models R(a_1, \ldots, a_n) \Leftrightarrow (a_1, \ldots, a_n) \in R^{\mathcal{M}} \subseteq M^n$.

Tarski's Semantic

The truth value of atomic formulas is determined by the interpretation of the symbols in $\mathcal{L}$. For instance, if R is an n-ary relation symbol:

- $\mathcal{M} \models R(a_1, \ldots, a_n) \Leftrightarrow (a_1, \ldots, a_n) \in R^{\mathcal{M}} \subseteq M^n$.

The definition for other formulas is given by induction:

- $\mathcal{M} \models \phi \wedge \psi \Leftrightarrow \mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$;
- $\mathcal{M} \models \neg\phi \Leftrightarrow \mathcal{M} \not\models \phi$;
- $\mathcal{M} \models \exists x \phi(x) \Leftrightarrow$ there is $a \in M$ such that $\mathcal{M} \models \phi(a)$.

Tarski's Semantic

The truth value of atomic formulas is determined by the interpretation of the symbols in $\mathcal{L}$. For instance, if R is an n-ary relation symbol:

- $\mathcal{M} \models R(a_1, \ldots, a_n) \Leftrightarrow (a_1, \ldots, a_n) \in R^{\mathcal{M}} \subseteq M^n$.

The definition for other formulas is given by induction:

- $\mathcal{M} \models \phi \wedge \psi \Leftrightarrow \mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$;
- $\mathcal{M} \models \neg\phi \Leftrightarrow \mathcal{M} \not\models \phi$;
- $\mathcal{M} \models \exists x \phi(x) \Leftrightarrow$ there is $a \in M$ such that $\mathcal{M} \models \phi(a)$.

Remark

A structure $\mathcal{M}$ associates to each formula ϕ the value 1 (if $\mathcal{M} \models \phi$) or the value 0 (if $\mathcal{M} \not\models \phi$)

An Example

Let $\mathcal{L} = \{\cdot, c_1\}$ where $\cdot$ is a binary function symbol and c_1 is a constant symbol.

An Example

Let $\mathcal{L} = \{\cdot, c_1\}$ where $\cdot$ is a binary function symbol and c_1 is a constant symbol.

Example

$$\langle \mathbb{Q}, \cdot^{\mathbb{Q}}, 1 \rangle$$

An Example

Let $\mathcal{L} = \{\cdot, c_1\}$ where $\cdot$ is a binary function symbol and c_1 is a constant symbol.

Example

$$\langle \mathbb{Q}, \cdot^{\mathbb{Q}}, 1 \rangle$$

- $\phi_1 \equiv \forall xyz[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$
- $\phi_2 \equiv \forall x \exists y(x \cdot y = y \cdot x = c_1)$
- $\phi_3 \equiv \forall x(x \cdot c_1 = c_1 \cdot x = c_1)$

An Example

Let $\mathcal{L} = \{\cdot, c_1\}$ where $\cdot$ is a binary function symbol and c_1 is a constant symbol.

Example

$$\langle \mathbb{Q}, \cdot^{\mathbb{Q}}, 1 \rangle$$

- $\phi_1 \equiv \forall xyz[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$
- $\phi_2 \equiv \forall x \exists y (x \cdot y = y \cdot x = c_1)$
- $\phi_3 \equiv \forall x (x \cdot c_1 = c_1 \cdot x = c_1)$

Let $T = \{\phi_1, \phi_2, \phi_3\}$ be the **theory of groups**.

An Example

Let $\mathcal{L} = \{\cdot, c_1\}$ where $\cdot$ is a binary function symbol and c_1 is a constant symbol.

Example

$$\langle \mathbb{Q}, \cdot^{\mathbb{Q}}, 1 \rangle$$

- $\phi_1 \equiv \forall xyz[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$
- $\phi_2 \equiv \forall x \exists y(x \cdot y = y \cdot x = c_1)$
- $\phi_3 \equiv \forall x(x \cdot c_1 = c_1 \cdot x = c_1)$

Let $T = \{\phi_1, \phi_2, \phi_3\}$ be the **theory of groups**. We write

$$\langle \mathbb{Q}, \cdot^{\mathbb{Q}}, 1 \rangle \models T$$

meaning that $\langle \mathbb{Q}, \cdot^{\mathbb{Q}}, 1 \rangle \models \phi_i$ for $1 \leq i \leq 3$.

ZFC

We work in the first order axiomatization of set theory ZFC, the language is $\mathcal{L} = \{\in\}$.

ZFC

We work in the first order axiomatization of set theory ZFC, the language is $\mathcal{L} = \{\in\}$.

Definition (ZFC)

- Axiom of Extensionality
- Axiom of Pairing
- Axiom of Union
- Axiom of Power Set
- Axiom of Infinity
- Axiom Schema of Separation
- Axiom Schema of Replacement
- Axiom of Regularity
- Axiom of Choice

The Universe of Sets

The universe of sets V

The Universe of Sets

The universe of sets V is obtained iteratively as follows:

$$V_0 = \emptyset$$

The Universe of Sets

The universe of sets V is obtained iteratively as follows:

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

The Universe of Sets

The universe of sets V is obtained iteratively as follows:

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\beta = \bigcup_{\alpha<\beta} V_\alpha \text{ if } \beta \text{ is a limit ordinal}$$

The Universe of Sets

The universe of sets V is obtained iteratively as follows:

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\beta = \bigcup_{\alpha<\beta} V_\alpha \text{ if } \beta \text{ is a limit ordinal}$$

$$V = \bigcup_{\alpha\in \mathsf{ON}} V_\alpha$$

The Universe of Sets

The universe of sets V is obtained iteratively as follows:

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\beta = \bigcup_{\alpha<\beta} V_\alpha \text{ if } \beta \text{ is a limit ordinal}$$

$$V = \bigcup_{\alpha \in \mathrm{ON}} V_\alpha$$

$\langle V, \in \rangle$ is a first order model of ZFC in the language $\{\in\}$.

The Universe of Sets

The universe of sets V is obtained iteratively as follows:

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\beta = \bigcup_{\alpha<\beta} V_\alpha \text{ if } \beta \text{ is a limit ordinal}$$

$$V = \bigcup_{\alpha \in \mathrm{ON}} V_\alpha$$

$\langle V, \in \rangle$ is a first order model of ZFC in the language $\{\in\}$.

Remark

V is built step by step iterating the power-set operation.

Independence

Definition

We write $T \models \phi$ if every model $\mathcal{M}$ of T verifies

$$\mathcal{M} \models \phi$$

Independence

Definition

We write $T \models \phi$ if every model $\mathcal{M}$ of T verifies

$$\mathcal{M} \models \phi$$

Theorem (Completeness)

$$T \models \phi \iff T \vdash \phi$$

Independence

Definition

We write $T \models \phi$ if every model $\mathcal{M}$ of T verifies

$$\mathcal{M} \models \phi$$

Theorem (Completeness)

$$T \models \phi \iff T \vdash \phi$$

Let T be the theory of groups. Consider

$$\psi \equiv \forall xy(x \cdot y = y \cdot x)$$

Independence

Definition

We write $T \models \phi$ if every model $\mathcal{M}$ of T verifies

$$\mathcal{M} \models \phi$$

Theorem (Completeness)

$$T \models \phi \iff T \vdash \phi$$

Let T be the theory of groups. Consider

$$\psi \equiv \forall xy(x \cdot y = y \cdot x)$$

$$T \not\models \neg\psi$$

Independence

Definition

We write $T \models \phi$ if every model $\mathcal{M}$ of T verifies

$$\mathcal{M} \models \phi$$

Theorem (Completeness)

$$T \models \phi \iff T \vdash \phi$$

Let T be the theory of groups. Consider

$$\psi \equiv \forall xy(x \cdot y = y \cdot x)$$

$$T \not\models \neg\psi$$

$$T \not\models \psi$$

Independence

Definition

We write $T \models \phi$ if every model $\mathcal{M}$ of T verifies

$$\mathcal{M} \models \phi$$

Theorem (Completeness)

$$T \models \phi \iff T \vdash \phi$$

Definition

A formula ϕ is **independent** from a theory T if

$$T \not\models \phi \wedge T \not\models \neg\phi$$

Independence in Set Theory

In order to prove that a certain formula ϕ is independent from ZFC:

Independence in Set Theory

In order to prove that a certain formula ϕ is independent from ZFC:

- Consider $\langle V_1, E_1 \rangle, \langle V_2, E_2 \rangle$ two $\{\in\}$-structures which are models of ZFC;

Independence in Set Theory

In order to prove that a certain formula ϕ is independent from ZFC:

- Consider $\langle V_1, E_1\rangle, \langle V_2, E_2\rangle$ two $\{\in\}$-structures which are models of ZFC;
- Show that

$$\langle V_1, E_1\rangle \models \phi$$

$$\langle V_2, E_2\rangle \models \neg\phi$$

Independence in Set Theory

In order to prove that a certain formula ϕ is independent from ZFC:

- Consider $\langle V_1, E_1\rangle, \langle V_2, E_2\rangle$ two $\{\in\}$-structures which are models of ZFC;
- Show that

$$\langle V_1, E_1\rangle \models \phi$$

$$\langle V_2, E_2\rangle \models \neg\phi$$

Example

- *Continuum Hypothesis (Gödel* 1940*, Cohen* 1963*);*
- *Whitehead problem (Shelah* 1974*);*
- *Existence of outer automorhpisms in the Calkin Algebra (Phillips-Weaver* 2006*, Farah* 2010*).*

Boolean Algebras

A boolean algebra $\langle \mathsf{B}, \leq, \wedge, \vee, \neg, 0_\mathsf{B}, 1_\mathsf{B} \rangle$ is a complemented bounded distributive lattice.

Boolean Algebras

A boolean algebra $\langle \mathsf{B}, \leq, \wedge, \vee, \neg, 0_\mathsf{B}, 1_\mathsf{B} \rangle$ is a complemented bounded distributive lattice.

Example

$$\langle \mathcal{P}(X), \subseteq, \cap, \cup, A \to A^c, \emptyset, X \rangle$$

Boolean Algebras

A boolean algebra $\langle \mathrm{B}, \leq, \wedge, \vee, \neg, 0_{\mathrm{B}}, 1_{\mathrm{B}} \rangle$ is a complemented bounded distributive lattice.

Example

$$\langle \mathcal{P}(X), \subseteq, \cap, \cup, A \to A^c, \emptyset, X \rangle$$

Example

$$\mathrm{MALG} = \langle \mathcal{M}/\mathcal{N}, \subseteq /_{\mathcal{N}}, \cap/_{\mathcal{N}}, \cup/_{\mathcal{N}}, [A] \to [I \setminus A]_{\mathcal{N}}, [\emptyset]_{\mathcal{N}}, [I]_{\mathcal{N}} \rangle$$

- $\mathcal{M}$ *is the family Lebesgue measurable subset of* $I = [0, 1]$
- $\mathcal{N}$ *is the family of null measure sets of* I

Boolean Algebras

A boolean algebra $\langle \mathrm{B}, \leq, \wedge, \vee, \neg, 0_{\mathrm{B}}, 1_{\mathrm{B}} \rangle$ is a complemented bounded distributive lattice.

Example

$$\langle \mathcal{P}(X), \subseteq, \cap, \cup, A \to A^c, \emptyset, X \rangle$$

Example

$$\mathrm{MALG} = \langle \mathcal{M}/\mathcal{N}, \subseteq /_{\mathcal{N}}, \cap/_{\mathcal{N}}, \cup/_{\mathcal{N}}, [A] \to [I \setminus A]_{\mathcal{N}}, [\emptyset]_{\mathcal{N}}, [I]_{\mathcal{N}} \rangle$$

- *$\mathcal{M}$ is the family Lebesgue measurable subset of $I = [0, 1]$*
- *$\mathcal{N}$ is the family of null measure sets of I*

B is **complete** if any $\{x_i : i \in \mathcal{I}\} \subseteq \mathrm{B}$ admits supremum $\bigvee_{i \in \mathcal{I}} x_i$ and infimum $\bigwedge_{i \in \mathcal{I}} x_i$.

Boolean Valued Models

Let B be a complete boolean algebra and fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

Boolean Valued Models

Let B be a complete boolean algebra and fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

A B-valued model is a tuple

$$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K \rangle$$

Boolean Valued Models

Let B be a complete boolean algebra and fix a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

A B-valued model is a tuple

$$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K \rangle$$

where:

- M is a non-empty set;
- $R_i^{\mathcal{M}}$ is a function:

$$R_i^{\mathcal{M}} : M^{n_i} \to \mathsf{B}$$
$$(\tau_1, \ldots, \tau_{n_i}) \mapsto [\![R_i(\tau_1, \ldots, \tau_{n_i})]\!]_{\mathsf{B}}^{\mathcal{M}}$$

- $f_j^{\mathcal{M}}$ is a function:

$$f_j^{\mathcal{M}} : M^{m_j+1} \to \mathsf{B}$$
$$(\tau_1, \ldots, \tau_{m_j}, \sigma) \mapsto \left[\!\left[f_j(\tau_1, \ldots, \tau_{m_j}) = \sigma\right]\!\right]_{\mathsf{B}}^{\mathcal{M}}$$

- $c_k^{\mathcal{M}}$ is a element of M.

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

- $[\![\phi \wedge \psi]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![\phi]\!]_{\mathsf{B}}^{\mathcal{M}} \wedge [\![\psi]\!]_{\mathsf{B}}^{\mathcal{M}}$
- $[\![\neg\phi]\!]_{\mathsf{B}}^{\mathcal{M}} = \neg\, [\![\phi]\!]_{\mathsf{B}}^{\mathcal{M}}$
- $[\![\exists x \phi(x)]\!]_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]_{\mathsf{B}}^{\mathcal{M}}$

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

- $[\![\phi \wedge \psi]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![\psi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\neg\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = \neg\, [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\exists x \phi(x)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]^{\mathcal{M}}_{\mathsf{B}}$

Remark

A B*-valued model* $\mathcal{M}$ *associates to each formula* ϕ *a value in* B*.*
First order models are B*-valued model for* $\mathsf{B} = \{0, 1\}$*.*

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

- $[\![\phi \wedge \psi]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![\psi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\neg\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = \neg\, [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\exists x \phi(x)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]^{\mathcal{M}}_{\mathsf{B}}$

Remark

A B*-valued model* $\mathcal{M}$ *associates to each formula* ϕ *a value in* B*.*
First order models are B*-valued model for* $\mathsf{B} = \{0, 1\}$*.*

Definition

ϕ is **valid** in $\mathcal{M}$ if $[\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}$.

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

- $[\![\phi \wedge \psi]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![\psi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\neg\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = \neg\, [\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}}$
- $[\![\exists x \phi(x)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]^{\mathcal{M}}_{\mathsf{B}}$

Remark

A B*-valued model* $\mathcal{M}$ *associates to each formula* ϕ *a value in* B*.*
First order models are B*-valued model for* $\mathsf{B} = \{0, 1\}$*.*

Definition

ϕ is **valid** in $\mathcal{M}$ if $[\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}$.

Theorem (Soundness)

Let $\mathcal{L}$ *be a language, if* ϕ *is a* $\mathcal{L}$*-formula which is syntactically provable by a* $\mathcal{L}$*-theory* T*, and* T *is valid in a* B*-valued model* $\mathcal{M}$*, then* $[\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}$*.*

Starting from the interpretation of the symbols in $\mathcal{L}$, we define the boolean value of each formula ϕ inductively:

- $[\![\phi \wedge \psi]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![\phi]\!]_{\mathsf{B}}^{\mathcal{M}} \wedge [\![\psi]\!]_{\mathsf{B}}^{\mathcal{M}}$
- $[\![\neg\phi]\!]_{\mathsf{B}}^{\mathcal{M}} = \neg\, [\![\phi]\!]_{\mathsf{B}}^{\mathcal{M}}$
- $[\![\exists x \phi(x)]\!]_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]_{\mathsf{B}}^{\mathcal{M}}$

Remark

A B*-valued model* $\mathcal{M}$ *associates to each formula* ϕ *a value in* B*. First order models are* B*-valued model for* $\mathsf{B} = \{0, 1\}$*.*

Definition

ϕ is **valid** in $\mathcal{M}$ if $[\![\phi]\!]_{\mathsf{B}}^{\mathcal{M}} = 1_{\mathsf{B}}$.

Theorem (Soundness)

Let $\mathcal{L}$ *be a language, if* ϕ *is a* $\mathcal{L}$*-formula which is syntactically provable by a* $\mathcal{L}$*-theory* T*, and* T *is valid in a* B*-valued model* $\mathcal{M}$*, then* $[\![\phi]\!]_{\mathsf{B}}^{\mathcal{M}} = 1_{\mathsf{B}}$*. More in general* $\phi \vdash \psi \Rightarrow [\![\phi]\!]_{\mathsf{B}}^{\mathcal{M}} \leq [\![\psi]\!]_{\mathsf{B}}^{\mathcal{M}}$*.*

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra and $\mathcal{L} = \{\in\}$. V^{B}, the **boolean valued universe**, is a boolean valued version of V.

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra and $\mathcal{L} = \{\in\}$. V^{B}, the **boolean valued universe**, is a boolean valued version of V.
$\mathcal{P}(X) \equiv 2^X$ identifying a subset with its characteristic function.

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra and $\mathcal{L} = \{\in\}$. V^{B}, the **boolean valued universe**, is a boolean valued version of V.
$\mathcal{P}(X) \equiv 2^X$ identifying a subset with its characteristic function.
$\mathcal{P}_{\mathsf{B}}(X) \equiv \mathsf{B}^X$ where for $f : X \to \mathsf{B}$, $f(a)$ is the boolean value of the concept *a belongs to* X.

Boolean Valued Universe

Let $\mathsf{B} \in V$ a complete boolean algebra and $\mathcal{L} = \{\in\}$. V^{B}, the **boolean valued universe**, is a boolean valued version of V.
$\mathcal{P}(X) \equiv 2^X$ identifying a subset with its characteristic function.
$\mathcal{P}_{\mathsf{B}}(X) \equiv \mathsf{B}^X$ where for $f : X \to \mathsf{B}$, $f(a)$ is the boolean value of the concept *a belongs to* X.
Therefore:

$$V_0^{\mathsf{B}} = \emptyset$$

$$V_{\alpha+1}^{\mathsf{B}} = \left\{ f : X \to \mathsf{B} \mid X \subset V_\alpha^{\mathsf{B}} \right\}$$

$$V_\beta^{\mathsf{B}} = \bigcup_{\alpha<\beta} V_\alpha^{\mathsf{B}} \text{ if } \beta \text{ is a limit ordinal}$$

$$V^{\mathsf{B}} = \bigcup_{\alpha \in ON} V_\alpha^{\mathsf{B}}$$

Boolean Valued Universe

V^{B} is a B-valued model with the following definitions, given by induction on the rank of the elements:

- $[\![\tau \in \sigma]\!] = \bigvee_{\chi \in \mathsf{dom}(\sigma)} ([\![\tau = \chi]\!] \wedge \sigma(\chi))$;
- $[\![\tau \subseteq \sigma]\!] = \bigwedge_{\chi \in \mathsf{dom}(\tau)} (\tau(\chi) \to [\![\chi \in \sigma]\!])$;
- $[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$.

Boolean Valued Universe

V^{B} is a B-valued model with the following definitions, given by induction on the rank of the elements:

- $[\![\tau \in \sigma]\!] = \bigvee_{\chi \in \mathsf{dom}(\sigma)}([\![\tau = \chi]\!] \wedge \sigma(\chi))$;
- $[\![\tau \subseteq \sigma]\!] = \bigwedge_{\chi \in \mathsf{dom}(\tau)}(\tau(\chi) \to [\![\chi \in \sigma]\!])$;
- $[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$.

Theorem

Every axiom of ZFC is valid in V^{B}.

Boolean Valued Universe

V^{B} is a B-valued model with the following definitions, given by induction on the rank of the elements:

- $[\![\tau \in \sigma]\!] = \bigvee_{\chi \in \mathsf{dom}(\sigma)}([\![\tau = \chi]\!] \wedge \sigma(\chi))$;
- $[\![\tau \subseteq \sigma]\!] = \bigwedge_{\chi \in \mathsf{dom}(\tau)}(\tau(\chi) \rightarrow [\![\chi \in \sigma]\!])$;
- $[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$.

Theorem

Every axiom of ZFC is valid in V^{B}.

Remark

Suppose we can find B *complete boolean algebra such that*

$$0_{\mathsf{B}} < [\![CH]\!]^{V^{\mathsf{B}}} < 1_{\mathsf{B}}$$

Boolean Valued Universe

V^{B} is a B-valued model with the following definitions, given by induction on the rank of the elements:

- $[\![\tau \in \sigma]\!] = \bigvee_{\chi \in \mathrm{dom}(\sigma)}([\![\tau = \chi]\!] \wedge \sigma(\chi))$;
- $[\![\tau \subseteq \sigma]\!] = \bigwedge_{\chi \in \mathrm{dom}(\tau)}(\tau(\chi) \to [\![\chi \in \sigma]\!])$;
- $[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$.

Theorem

Every axiom of ZFC is valid in V^{B}.

Remark

Suppose we can find B *complete boolean algebra such that*

$$0_{\mathsf{B}} < [\![CH]\!]^{V^{\mathsf{B}}} < 1_{\mathsf{B}}$$

Then CH is independent from ZFC because

$$ZFC \vdash CH \Rightarrow [\![CH]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}}$$

$$ZFC \vdash \neg CH \Rightarrow [\![\neg CH]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}} \Rightarrow [\![CH]\!]^{V^{\mathsf{B}}} = 0_{\mathsf{B}}$$

Ultrafilters

Definition

Given B a boolean algebra, a **ultrafilter** G is a subset of B such that

- $1_B \in G$, $0_B \notin G$
- if $x \in G$ and $y \in G$ then $x \wedge y \in G$
- if $x \in G$ and $x \leq y$ then $y \in G$
- if $x \notin G$ then $\neg x \in G$

Ultrafilters

Definition

Given B a boolean algebra, a **ultrafilter** G is a subset of B such that

- $1_{\mathsf{B}} \in G$, $0_{\mathsf{B}} \notin G$
- if $x \in G$ and $y \in G$ then $x \wedge y \in G$
- if $x \in G$ and $x \leq y$ then $y \in G$
- if $x \notin G$ then $\neg x \in G$

Definition (Stone Space)

$$St(\mathsf{B}) = \{G \subseteq \mathsf{B} : G \text{ is a ultrafilter}\}$$

Ultrafilters

Definition

Given B a boolean algebra, a **ultrafilter** G is a subset of B such that

- $1_{\mathrm{B}} \in G$, $0_{\mathrm{B}} \notin G$
- if $x \in G$ and $y \in G$ then $x \wedge y \in G$
- if $x \in G$ and $x \leq y$ then $y \in G$
- if $x \notin G$ then $\neg x \in G$

Definition (Stone Space)

$$St(\mathrm{B}) = \{G \subseteq \mathrm{B} : G \text{ is a ultrafilter}\}$$

In V^{B}, given $G \in St(\mathrm{B})$, define the following equivalence relation:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

Ultrafilters

Definition

Given B a boolean algebra, a **ultrafilter** G is a subset of B such that

- $1_B \in G$, $0_B \notin G$
- if $x \in G$ and $y \in G$ then $x \wedge y \in G$
- if $x \in G$ and $x \leq y$ then $y \in G$
- if $x \notin G$ then $\neg x \in G$

Definition (Stone Space)

$$St(B) = \{G \subseteq B : G \text{ is a ultrafilter}\}$$

In V^B, given $G \in St(B)$, define the following equivalence relation:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

The quotient V^B/G has a natural structure of first order model.

Cohen's Forcing Theorem

Theorem (Cohen's Forcing Theorem)

Assume $\mathsf{B} \in V$ *is a complete boolean algebra and* $G \in St(\mathsf{B})$. *Then*

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \mathsf{ZFC}$$

Moreover

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \phi([\tau_1]_G, \ldots, [\tau_n]_G) \Leftrightarrow [\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$$

Cohen's Forcing Theorem

Theorem (Cohen's Forcing Theorem)

Assume $\mathsf{B} \in V$ *is a complete boolean algebra and* $G \in St(\mathsf{B})$*. Then*

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \mathsf{ZFC}$$

Moreover

$$\langle V^{\mathsf{B}}/G, \in_G \rangle \models \phi([\tau_1]_G, \ldots, [\tau_n]_G) \Leftrightarrow [\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$$

Remark

Forcing is a machine which produces first order models of ZFC*. The truth value of independent formulas in these models depends on the combinatorial properties of* B *and on the choice of* G*.*

B-names for Complex Numbers

Definition

$\sigma \in V^{\mathsf{B}}$ is a B-**name for a complex number** if

$$[\![\sigma \text{ is a complex number}]\!] = 1_{\mathsf{B}}$$

Denote with $\mathbb{C}^{\mathsf{B}}$ the set of all B-names for complex numbers.

B-names for Complex Numbers

Definition

$\sigma \in V^{\mathsf{B}}$ is a B-**name for a complex number** if

$$[\![\sigma \text{ is a complex number}]\!] = 1_{\mathsf{B}}$$

Denote with $\mathbb{C}^{\mathsf{B}}$ the set of all B-names for complex numbers.

$\mathbb{C}^{\mathsf{B}}$ is the boolean valued version of the set of the complex numbers.

B-names for Complex Numbers

Definition

$\sigma \in V^{\mathsf{B}}$ is a B-**name for a complex number** if

$$[\![\sigma \text{ is a complex number}]\!] = 1_{\mathsf{B}}$$

Denote with $\mathbb{C}^{\mathsf{B}}$ the set of all B-names for complex numbers.

$\mathbb{C}^{\mathsf{B}}$ is the boolean valued version of the set of the complex numbers.

$$V \to V^{\mathsf{B}} \to V^{\mathsf{B}}/G$$
$$\mathbb{C} \to \mathbb{C}^{\mathsf{B}} \to \mathbb{C}^{\mathsf{B}}/G$$

Generic Absoluteness

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathsf{B})$.

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathsf{B})$*. Then:*

$$\langle \mathbb{C}, +, \cdot, \overline{}, 0, 1 \rangle \prec_{\Sigma_2} \langle \mathbb{C}^{\mathsf{B}}/G, +, \cdot, \overline{}, 0, 1 \rangle$$

Generic Absoluteness

Theorem

Let B *a complete boolean algebra and* $G \in St(\mathrm{B})$*. Then:*

$$\langle \mathbb{C}, +, \cdot, \bar{\ }, 0, 1 \rangle \prec_{\Sigma_2} \langle \mathbb{C}^{\mathrm{B}}/G, +, \cdot, \bar{\ }, 0, 1 \rangle$$

Remark

This means that forcing can be used as a tool to prove theorems within ZFC*. To prove that a* Σ^1_2*-formula* ϕ *is true in ZFC, it is not necessary to show that it holds in every model of ZFC. It is enough to find one model of a certain form in which* ϕ *holds.*

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, * \rangle$ is a $\mathbb{C}$-algebra such that:

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, * \rangle$ is a $\mathbb{C}$-algebra such that:

1. $\langle \mathcal{A}, +, \|.\| \rangle$ is a Banach space
2. $\|xz\| \leq \|x\| \|z\|$
3. $(x + y)^* = x^* + y^*$
4. $(xy)^* = y^* x^*$
5. $(\lambda x)^* = \overline{\lambda} x^*$
6. $x^{**} = x$
7. $\|x^* x\| = \|x\|^2$

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, *\rangle$ is a $\mathbb{C}$-algebra such that:

1. $\langle \mathcal{A}, +, \|.\|\rangle$ is a Banach space
2. $\|xz\| \leq \|x\|\|z\|$
3. $(x+y)^* = x^* + y^*$
4. $(xy)^* = y^*x^*$
5. $(\lambda x)^* = \overline{\lambda}x^*$
6. $x^{**} = x$
7. $\|x^*x\| = \|x\|^2$

Example

- Commutative: $L^\infty([0,1])$, $\mathcal{C}(X)$ for X compact Hausdorff

C^*-algebras

Definition

A C^*-**algebra** $\langle \mathcal{A}, +, \cdot, \|.\|, *\rangle$ is a $\mathbb{C}$-algebra such that:

1. $\langle \mathcal{A}, +, \|.\|\rangle$ is a Banach space
2. $\|xz\| \leq \|x\|\|z\|$
3. $(x+y)^* = x^* + y^*$
4. $(xy)^* = y^* x^*$
5. $(\lambda x)^* = \overline{\lambda} x^*$
6. $x^{**} = x$
7. $\|x^* x\| = \|x\|^2$

Example

- Commutative: $L^\infty([0,1])$, $\mathcal{C}(X)$ for X compact Hausdorff
- Non-commutative: $\mathcal{B}(H)$ for H Hilbert space

The space $\mathcal{C}(St(\mathrm{B}))$

On $St(\mathrm{B})$ define the topology generated by $\{\mathcal{O}_a\}_{a \in \mathrm{B}}$, where:

$$\mathcal{O}_a = \{G \in St(\mathrm{B}) : a \in G\}$$

The space $\mathcal{C}(St(\mathrm{B}))$

On $St(\mathrm{B})$ define the topology generated by $\{\mathcal{O}_a\}_{a \in \mathrm{B}}$, where:

$$\mathcal{O}_a = \{G \in St(\mathrm{B}) : a \in G\}$$

Proposition

$St(\mathrm{B})$ is a compact Hausdorff space and $\{\mathcal{O}_a\}_{a \in \mathrm{B}}$ is a basis of clopen sets.

The space $\mathcal{C}(St(\mathrm{B}))$

On $St(\mathrm{B})$ define the topology generated by $\{\mathcal{O}_a\}_{a\in\mathrm{B}}$, where:

$$\mathcal{O}_a = \{G \in St(\mathrm{B}) : a \in G\}$$

Proposition

$St(\mathrm{B})$ is a compact Hausdorff space and $\{\mathcal{O}_a\}_{a\in\mathrm{B}}$ is a basis of clopen sets.

We will work with the following commutative C^*-algebra:

$$\mathcal{C}(St(\mathrm{B})) = \{f : St(\mathrm{B}) \to \mathbb{C} : f \text{ is continuous}\}$$

The space $\mathcal{C}(St(\mathrm{B}))$

There are well understood examples of such spaces.

The space $\mathcal{C}(St(\mathrm{B}))$

There are well understood examples of such spaces.
Let
$$\mathrm{B} = \mathrm{MALG}$$

The space $\mathcal{C}(St(\mathsf{B}))$

There are well understood examples of such spaces.
Let
$$\mathsf{B} = \mathsf{MALG} = \mathcal{M}/\mathcal{N}$$

The space $\mathcal{C}(St(\mathsf{B}))$

There are well understood examples of such spaces.
Let
$$\mathsf{B} = \mathsf{MALG} = \mathcal{M}/\mathcal{N}$$

This is a complete boolean algebra.

The space $\mathcal{C}(St(\mathsf{B}))$

There are well understood examples of such spaces.
Let

$$\mathsf{B} = \mathsf{MALG} = \mathcal{M}/\mathcal{N}$$

This is a complete boolean algebra.
Using Gelfand Transform it can be shown that:

Proposition

$$\mathcal{C}(St(\mathsf{MALG})) \cong L^{\infty}([0,1])$$

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathsf{B}))$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. Define

$$[\![R(f,g)]\!] = \mathring{\overline{\{G \in St(\mathrm{B}) : f(G)Rg(G)\}}} = \mathring{\overline{(f \times g)^{-1}[R]}}$$

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. Define

$$[\![R(f,g)]\!] = \mathring{\overline{\{G \in St(\mathrm{B}) : f(G) R g(G)\}}} = \mathring{\overline{(f \times g)^{-1}[R]}}$$

This set univocally determines an element of B ($\mathrm{B} \cong \mathrm{RO}(St(\mathrm{B}))$).

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. Define

$$[\![R(f,g)]\!] = \overline{\{G \in St(\mathrm{B}) : f(G)Rg(G)\}}^{\circ} = \overline{(f \times g)^{-1}[R]}^{\circ}$$

This set univocally determines an element of B ($\mathrm{B} \cong \mathrm{RO}(St(\mathrm{B}))$).

- $\mathcal{C}(St(\mathrm{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. Define

$$[\![R(f,g)]\!] = \overset{\circ}{\overline{\{G \in St(\mathrm{B}) : f(G) R g(G)\}}} = \overset{\circ}{\overline{(f \times g)^{-1}[R]}}$$

This set univocally determines an element of B ($\mathrm{B} \cong \mathrm{RO}(St(\mathrm{B}))$).

- $\mathcal{C}(St(\mathrm{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.
- The set $\{c_x : x \in \mathbb{C}\}$, where c_x is the constant function with value x, is a copy of $\mathbb{C}$ in $\mathcal{C}(St(\mathrm{B}))$.

A Boolean Valued Extension of $\mathbb{C}$

Let B be a complete boolean algebra and $R \subseteq \mathbb{C} \times \mathbb{C}$ a binary Borel relation on $\mathbb{C}$. Consider $f, g \in \mathcal{C}(St(\mathrm{B}))$. Define

$$[\![R(f,g)]\!] = \overline{\{G \in St(\mathrm{B}) : f(G) R g(G)\}}^{\circ} = \overline{(f \times g)^{-1}[R]}^{\circ}$$

This set univocally determines an element of B ($\mathrm{B} \cong \mathrm{RO}(St(\mathrm{B}))$).

- $\mathcal{C}(St(\mathrm{B}))$ is a B-valued model for $\mathcal{L} = \{R\}$.
- The set $\{c_x : x \in \mathbb{C}\}$, where c_x is the constant function with value x, is a copy of $\mathbb{C}$ in $\mathcal{C}(St(\mathrm{B}))$.

Proposition

$\mathcal{C}(St(\mathrm{B}))$ *is a* B*-valued extension of* $\mathbb{C}$*.*

$\mathcal{C}^+(St(\mathsf{B})) \cong \mathbb{C}^{\mathsf{B}}$

Theorem

Fix a set

$$\mathcal{L} = \{R_i : i \in I\} \cup \{F_j : j \in J\}$$

where:

- *for* $i \in I$, R_i *is a Borel subset of* $\mathbb{C}^{n_i}$;
- *for* $j \in J$, F_j *is a Borel function from* $\mathbb{C}^{m_j}$ *to* $\mathbb{C}$.

Then

$$\mathcal{C}^+(St(\mathsf{B})) \cong \mathbb{C}^{\mathsf{B}}$$

as B*-valued models in the language* $\mathcal{L}$.

$\mathcal{C}(St(\mathsf{B}))$ and generic absoluteness

Proposition

Assume G is a V-generic filter on B*. Then*

$$\mathcal{C}^+(St(\mathsf{B}))/G \cong \mathcal{C}(St(\mathsf{B}))/G$$

$\mathcal{C}(St(\mathsf{B}))$ and generic absoluteness

Proposition

Assume G is a V-generic filter on B. Then

$$\mathcal{C}^+(St(\mathsf{B}))/G \cong \mathcal{C}(St(\mathsf{B}))/G$$

Theorem

Let B be a complete boolean algebra and $G \in St(\mathsf{B})$ be V-generic.

$\mathcal{C}(St(\mathrm{B}))$ and generic absoluteness

Proposition

Assume G is a V-generic filter on B. Then

$$\mathcal{C}^+(St(\mathrm{B}))/G \cong \mathcal{C}(St(\mathrm{B}))/G$$

Theorem

Let B be a complete boolean algebra and $G \in St(\mathrm{B})$ be V-generic. Assume $R_1, \ldots, R_n$ are Borel relations on $\mathbb{C}$ and $F_1, \ldots, F_m$ are Borel functions on $\mathbb{C}$.

$\mathcal{C}(St(\mathsf{B}))$ and generic absoluteness

Proposition

Assume G is a V-generic filter on B. Then

$$\mathcal{C}^+(St(\mathsf{B}))/G \cong \mathcal{C}(St(\mathsf{B}))/G$$

Theorem

Let B be a complete boolean algebra and $G \in St(\mathsf{B})$ be V-generic. Assume $R_1, \ldots, R_n$ are Borel relations on $\mathbb{C}$ and $F_1, \ldots, F_m$ are Borel functions on $\mathbb{C}$. Then:

$$\langle \mathbb{C}, R_1, \ldots, F_m \rangle \prec_{\Sigma_2} \langle \mathbb{C}^{\mathsf{B}}/G, R_1/G, \ldots, F_m/G \rangle$$

$\mathcal{C}(St(\mathrm{B}))$ and generic absoluteness

Proposition

Assume G is a V-generic filter on B. Then

$$\mathcal{C}^+(St(\mathrm{B}))/G \cong \mathcal{C}(St(\mathrm{B}))/G$$

Theorem

Let B be a complete boolean algebra and $G \in St(\mathrm{B})$ be V-generic. Assume $R_1, \dots, R_n$ are Borel relations on $\mathbb{C}$ and $F_1, \dots, F_m$ are Borel functions on $\mathbb{C}$. Then:

$$\langle \mathbb{C}, R_1, \dots, F_m \rangle \prec_{\Sigma_2} \langle \mathbb{C}^{\mathrm{B}}/G, R_1/G, \dots, F_m/G \rangle$$

Therefore:

$$\langle \mathbb{C}, R_1, \dots, F_m \rangle \prec_{\Sigma_2} \langle \mathcal{C}(St(\mathrm{B}))/G, R_1/G, \dots, F_m/G \rangle$$

Grazie per l'attenzione!

Essential bibliography

[1] Andrea Vaccaro and Matteo Viale, *Generic absoluteness and boolean names for elements of a Polish space*, Boll. Unione Mat. Ital. **10** (2017), no. 3, 293–319. MR3691801

[2] Matteo Viale, *Forcing the truth of a weak form of Schanuel's conjecture*, Confluentes Math. **8** (2016), no. 2, 59–83. MR3633221