Ultraproducts in Functional Analysis

Marius Junge

Pisa, June 2008
General Remarks

Ultraproduct techniques are used in many branches of functional analysis (Banach spaces and operator algebras). More important than the ultrafilters are the spaces constructed with the help of ultrafilters. The new spaces look locally like the old one.
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- More important than the ultrafilters are the spaces constructed with the help of ultrafilters.
- The new spaces look locally like the old one.
The spaces

Let \((X_i)\) be a family of Banach spaces and
\[ B = \{ (x_i) : x_i \in X_i, \sup_i \|x_i\| < \infty \} \]
be the set of bounded sections.

Let \(U\) be an ultrafilter and
\[ N = \{ (x_i) : \lim_i, U \|x_i\|_{X_i} = 0 \} \]
Then \(\prod_i, U X_i / N\) equipped with the norm
\[ \| (x_i) + N \| = \lim_i, U \|x_i\|_{X_i} \]
is again a Banach space.
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is again a Banach space.
Examples

$X_i = \mathcal{L}(\Omega, \mu_i)$. Then $\prod_i X_i = \mathcal{L}(\Omega, \mu)$ for some large measure space $\Omega, \mu$. $X_i$ lattices, then the ultraproduct is also a lattice. $X_i$ Banach algebras, then the ultraproduct is a Banach algebra.
Examples

- $X_i = L_p(\Omega, \mu_i)$. 
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Factorization theory and ultra products

Theorem (Kwapien)

Let $X$ be a Banach space and $C > 0$ a constant such that

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\left( \sum_i \left\| \sum_j a_{ij} x_j \right\|_X^2 \right)^{1/2} \leq C \left\| a \right\|_\ell^2 \rightarrow _\ell^2 \left( \sum_i \left\| x_i \right\|_X^2 \right)^{1/2}.
$$

Then there is scalar product $(\cdot, \cdot)$ such that

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\left\| x \right\| \leq (x, x)^{1/2} \leq C \left\| x \right\|.
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Hernandez: Similar results for (quotient of subspaces) of $L^p$ spaces, even in the vector-valued setting.

Tools:
1) Use Grothendieck’s theory of tensor norms (trace duality) to show the result first for finite dimensional spaces.
2) Use that Hilbert spaces are stable under ultraproducts.
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More local theory

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More local theory

Remark: More results in this direction, due to Maurey, Pisier, Krivine (72-74):

Let $\ell_n^p$ be $\mathbb{R}^n$ equipped with the norm $\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$.

Let $X$ be an infinite dimensional Banach space and $p \geq 2$ be the infimum over all $q$ such that

$$\left(\sum_{k} \|x_k\|_q^q\right)^{1/q} \leq C \sup_{\epsilon_k = \pm 1} \|\sum_{k} \epsilon_k x_k\|_X.$$  

for some constant $C_q$. Then $X$ contains copies of $\ell_n^p$ of arbitrary dimension.
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Local properties

Let $X_i = X$ for all $i$. Then $Y = \prod U X$ is called a ultrapower.

Let $E \subset X$ be a finite dimensional subspace and $\varepsilon > 0$. Then there exist a finite dimensional subspace $E_\varepsilon \subset X$ and a linear isomorphism such that $\|u\|\|u - 1\| \leq (1 + \varepsilon)$.

Here $\|u\| = \sup_{x \neq 0} \|u(x)\|/\|x\|$.

Definition: If the above is satisfied for $Y$ and $X$ we say that $Y$ is finitely represented in $X$.

Major open problem in operator algebras: Is the predual of a von Neumann algebra finitely represented in the predual in $B(\ell^2)$?
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**Major open problem in operator algebras:** Is the predual of a von Neumann algebra finitely represented in the predual in $B(\ell_2)$?
A C*-algebra is a Banach algebra with involution * such that
\[ \|x\|^2 = \|xx^*\|. \]
Examples:
- \( A = \text{C}(K), K \) compact.
- \( A = \text{C}_0(K), K \) locally compact.
- \( B(\mathcal{H}) \), the bounded operators on Hilbert space, in particular \( M_n = B(\ell_2^n) \).

Finite dimensional C*-algebras are direct sums of matrix algebras.

Every C*-algebra is contained in some \( B(\mathcal{H}) \).

\( \text{C}\ast(\mathcal{F}_\infty) \), the universal algebra of infinitely many unitaries, \( \mathcal{F}_\infty \) free group in countably many generators.
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Von Neumann algebras

A von Neumann algebra is a unital subalgebra of $B(H)$ closed in the weak operator topology:

$$T\xrightarrow{\lambda} \text{WOT} T\text{iff } (h, T\lambda k) \xrightarrow{\lambda} (h, Tk).$$

Motivation: Functional calculus with measurable functions, spectral theory of unbounded operators.

Examples: $B(H)$, $L_\infty(\Omega, \mu)$, $L_\infty(\Omega, \mu; B(H))$ (random matrices).

$X \subset B(H)$ such that $X^* \subset X$, then $X' = \{T : Tx = xT, \forall x \in X\}$ is a vNa.

Let $G$ be a discrete group and $\lambda(g) e_h = e_{gh}$.

Then $VN(G) = \lambda(G)''$ is a von Neumann algebra.
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A von Neumann algebra is a unital subalgebra of $B(H)$ closed in the weak operator topology: $T_\lambda \to_{WOT} T$ if

$$(h, T_\lambda k) \to_\lambda (h, Tk).$$

**Motivation:** Functional calculus with measurable functions, spectral theory of unbounded operators.

**Examples:**

- $B(H)$. $L_\infty(\Omega, \mu)$, $L_\infty(\Omega, \mu; B(H)))$ (random matrices).
- $X \subset B(H)$ such that $X^* \subset X$, then $X' = \{T : Tx - xT = 0, \forall x \in X\}$ is a vNa.
- Let $G$ be a discrete group and $\lambda(g)e_h = e_{gh}$. Then $VN(G) = \lambda(G)^{''}$ is a von Neumann algebra.
Von Neumann algebra ultraprowers

Let $N$ be a von Neumann algebra and $\tau$ be a trace, i.e. a positive, normal functional with $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$. Then ultraproduct $N_\omega$ in $\text{vNa}$-lit is the quotient of $\ell_\infty(I, N)$ with respect to $I = \{ (x_i) : \lim_{i} \tau(x_i^*x_i) = 0 \}$.

Warning/Remark:
1) $I$ is much larger than $\{ (x_i) : \lim_{i} \| x_i \| = 0 \}$.
2) However, $(N_\omega)^*$ is a two-sided ideal in $\prod U N^*$.
3) The Chang-Keisler theorem for ultraproducts in the vNa-sense is missing.
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Property $\Gamma$

$N$ has property $\Gamma$ if $N' \cap N \neq C$.

**Example:**

1) Let $R = \bigotimes_{n \in \mathbb{N}} M_2$ the infinite tensor product of $2 \times 2$ matrices. Then $R$ has property $\Gamma$. Indeed, every von Neumann algebra which is the WOT closure of finite dimensional $C^*$-algebras, has this property (hyperfinite).

$VN(G)$ is hyperfinite iff $G$ is amenable.

2) Let $F_n$ be the free group in $n$ generators. Then $VN(F_n)$ does not have property $\Gamma$ (Murray/von Neumann). Hence, $VN(F_n)$ is not hyperfinite.
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More recent results

Recently (03) Christensen, Smith, Sinclair and Pop showed that for factors with property $\Gamma$ the bounded cohomology groups vanish. Popa showed that for $Q \subset N$, $Q \text{ contains no hyperfinite summand if and only if } Q' \cap (N^* N) \cup U \subset (N^* 1) U$ holds for the free product. This can be used to show that for every sub von Neumann algebra $Q$ of $VN(F_n)$ such that $Q' \cap L(VN(F_n))$ has no atoms, then $Q$ is hyperfinite (due to Ozawa). Popa has very successfully studied deformation/rigidity result in von Neumann algebras.
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Embedding in $R^U$

Problem 1: Let $N$ be a von Neumann algebra with a nice trace. Is there a trace preserving embedding of $N$ in $R^U$?

Remark: Then the range is automatically complemented with a conditional expectation $E: R^U \to N$, $E(axb) = aE(x)b$, $a, b \in N$, $x \in R^U$.

A good way to understand this is to ask whether for a finite set $x_1, \ldots, x_m \subset N$ there are matrices $y_1, \ldots, y_m \in M_n$ of $n \times n$ matrices such that $|\tau(x_1 \cdot \cdots \cdot x_k) - tr_n(y_1 \cdot \cdots \cdot y_k)| < \varepsilon$?
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Is there only one norm on $C^*(F_\infty \otimes C^*(F_\infty)$ which makes the tensor product a $C^*$-algebra?

Theorem (94) The four problems are all equivalent.
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**Theorem**

(94) *The four problems are all equivalent.*
Let $N$ be a von Neumann algebra with trace $\tau$. The $L_p$ spaces is defined by
$$\|x\|_p = \left[\tau(|x|^p)\right]^{1/p},$$
where $|x| = \sqrt{x^*x}$.

Theorem (J. Parcet–NC Rosenthal theorem) Let $X \subset L_1(N)$ be a reflexive subspace, then $X$ is isomorphic to a subspace of $L_p(N)$ for some $p > 1$. Indeed, there exists a positive $d \in L_1(N)$ and $u : X \to L_p$ such that
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\[(N_i)\text{ and } (M_j)\] be von Neumann algebras and 
\[z = \sum k x_k(i) \otimes y_k(j)\] a finite tensor.

Then
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**Exercise:**
Proof this for commutative \(N\) and \(M\).

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Connes used ultraproduct arguments in the classification of factors, and later in noncommutative geometry to relate singular values and integrals on manifolds. Matrix models and ultraproduct techniques are combined with Speicher's central limit approach to prove Khintchine type inequalities (inequalities for finite dimensional matrices!). Ultraproduct techniques are key for noncommutative stochastic integrals.

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