Stochastic Navier-Stokes equations: ideas and results using nonstandard analysis

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(Joint with Marek Capiński, Jerry Keisler, Kasia Grzesiak, Brendan Enright)

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The STOCHASTIC NAVIER STOKES EQUATIONS (sNSe) in a bounded domain $D \subset \mathbb{R}^d$ ($d = 2, 3$) with multiplicative noise:

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\begin{cases}
    du = [\nu \Delta u - \langle u, \nabla \rangle u + f(t, u) - \nabla p] dt + g(t, u)dw_t \\
    \text{div } u = 0
\end{cases}
\]

$u(t, x, \omega) =$ (random) velocity of the fluid at the location $x \in D$ at time $t$:

$u : [0, \infty) \times D \times \Omega \to \mathbb{R}^d$

$\Omega =$ domain of an underlying probability space.
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- attractors for sNSe
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\end{array} \right.
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**Aim of the talk:** to sketch informally the Loeb space approach and what can be achieved in these areas.
Mathematical Formulation - Hilbert space setting
Set $\mathcal{H} = \{ u \in C_0^\infty(D, \mathbb{R}^d) : \text{div } u = 0 \}$ with norms $|u|$ and $\|u\|$ derived from

$$(u, v) = \sum_{j=1}^{d} \int_{D} u^j(x)v^j(x)dx, \quad \langle (u, v) \rangle = \sum_{j=1}^{d} \left( \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right)$$

$H$ = closure of $\mathcal{H}$ in the norm $|u|$ and $V$ is the closure in norm $|u| + \|u\|$. 
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$\mathcal{H}$ = closure of $\mathcal{H}$ in the norm $|u|$ and $\mathcal{V}$ is the closure in norm $|u| + \|u\|$. $\mathcal{H}$ and $\mathcal{V}$ are Hilbert spaces with scalar products $(\cdot, \cdot)$ and $((\cdot, \cdot))$ resp.
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$\mathcal{H}$ = closure of $\mathcal{H}$ in the norm $|u|$ and $\mathbf{V}$ is the closure in norm $|u| + \|u\|$. $\mathcal{H}$ and $\mathbf{V}$ are Hilbert spaces with scalar products $(\cdot, \cdot)$ and $((\cdot, \cdot))$ resp. $A$ = self adjoint extension of the projection of $-\Delta$ in $\mathcal{H}$; $A$ has an orthonormal basis $\{e_k\}$ of eigenfunctions with eigenvalues $0 < \lambda_k \uparrow \infty$. 
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$\mathcal{H}_m = \text{span}\{e_1, \ldots, e_m\} \subset \mathcal{V}$. 

Mathematical Formulation - Hilbert space setting
Set $\mathcal{H} = \{ u \in C^\infty_0(D, \mathbb{R}^d): \text{div } u = 0 \}$ with norms $|u|$ and $\|u\|$ derived from

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The operator $B(u)$ is defined by $B(u)w = (\langle u, \nabla \rangle u, w)$.
Mathematical Formulation - Hilbert space setting

Set \( \mathcal{H} = \{ u \in C_0^\infty(D, \mathbb{R}^d) : \text{div} \ u = 0 \} \) with norms \(|u|\) and \(|u|\) derived from

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(u, v) = \sum_{j=1}^d \int_D u^j(x)v^j(x)dx,
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The operator \( B(u) \) is defined by \( B(u)w = (\langle u, \nabla \rangle u, w) \).

The sNSe are now formulated as a stochastic differential equation in \( \mathcal{H} \):

\[
du = [-\nu Au - B(u) + f(t, u)]dt + g(t, u)dw_t
\]

Initially regard this as an equation in \( \mathbf{V}' \) (the dual of \( \mathbf{V} \)) although it turns out that solutions live in \( \mathcal{H} \) (and in fact in \( \mathbf{V} \) for almost all times).
\[ du = [-\nu Au - B(u) + f(t, u)]dt + g(t, u)dw_t \] (1)

The equation is understood as a weak integral equation:

\[ u(t) = u_0 + \int_0^t [\nu Au(s) - B(u(s)) + f(s, u(s))]ds + \int_0^t g(s, u(s))dw_s \]

the first \( \int \) = Bochner integral; the second \( \int \) = Ichikawa’s extension of the Itô integral to Hilbert spaces; evaluated by testing against functions in \( \mathbf{V} \).
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the first \( \int \) = Bochner integral; the second \( \int \) = Ichikawa's extension of the Itô integral to Hilbert spaces; evaluated by testing against functions in \( V \). The noise \( w: [0, \infty) \times \Omega \to H \) is a Wiener process with trace class covariance.
\[ du = \left[ -\nu Au - B(u) + f(t, u) \right] dt + g(t, u) dw_t \]  

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The coefficients

\[ g : [0, \infty) \times V \to L(H, H) \quad \text{and} \quad f : [0, \infty) \times V \to V'. \]

can be quite general - we only need appropriate continuity and growth conditions. (The restriction to \( V \) in the domains is sufficient because solutions will lie in \( V \) for almost all times.)
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The coefficients

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can be quite general - we only need appropriate continuity and growth conditions. (The restriction to \( V \) in the domains is sufficient because solutions will lie in \( V \) for almost all times.)

\textbf{Note} The pressure has disappeared, because \( \nabla p = 0 \) in \( V' \).
Basic Existence Theorem

Theorem
For any $u_0 \in H$ and given $f, g$ there is an adapted probability space $\Omega$ carrying an $H$-valued Wiener process $w$ and a (weak) solution of the stochastic Navier–Stokes equations.
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For any $u_0 \in H$ and given $f, g$ there is an adapted probability space $\Omega$ carrying an $H$-valued Wiener process $w$ and a (weak) solution of the stochastic Navier–Stokes equations. That is, an adapted stochastic process $u : [0, \infty) \times \Omega \to H$ such that for a.a. $\omega$
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(i) $u(\cdot, \omega) \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap C(0, T; H_{weak})$ for all $T < \infty$, 

(ii) $u(t) = u_0 + \int_0^t [\nu A u(s) - B(u(s)) + f(s, u(s))] \, ds + \int_0^t g(s, u(s)) \, dw_s$.
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Theorem
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(ii) for all \( t \geq 0 \)
\[
    u(t) = u_0 + \int_0^t \left[ \nu A u(s) - B(u(s)) + f(s, u(s)) \right] ds + \int_0^t g(s, u(s)) dw_s
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The classical approach to solving

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(1) solve an approximate version (the Galerkin approximation) in each finite dimensional space \( H_n \) on a probability space \( \Omega_n \) with Wiener process \( w_n \).
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This needs specialized compactness theorems and ways to enlarge the spaces \( \Omega_n \) to a “limit” probability space (which may depend on the solution).
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(1) solve an approximate version (the Galerkin approximation) in each finite dimensional space $H_n$ on a probability space $\Omega_n$ with Wiener process $w_n$
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This needs specialized compactness theorems and ways to enlarge the spaces $\Omega_n$ to a “limit” probability space (which may depend on the solution).

Loeb space methods provide a single space $\Omega$ (a Loeb space) and a Wiener process $w$ carrying solutions for all (random) initial conditions and all $f, g$.

This makes them powerful for discussing attractors and optimal control theory for sNSe. Loeb spaces are saturated and homogeneous.
LOEB SPACE METHODS FOR sNS\(_e\)
LOEB SPACE METHODS FOR sNSe

NONSTANDARD ANALYSIS The hyperreals or nonstandard reals $\mathbb{R} \supsetneq \mathbb{R}$ is a field such that $\mathbb{R}$ contains non-zero infinitesimal numbers; and positive and negative infinite numbers using the following definitions:

Let $x \in \mathbb{R}$. We say that

(i) $x$ is infinitesimal if $|x| < \varepsilon$ for all $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$;

(ii) $x$ is finite if $|x| < r$ for some $r \in \mathbb{R}$;

(iii) $x$ is infinite if $|x| > r$ for all $r \in \mathbb{R}$.

(iv) $x$ and $y$ are infinitely close, denoted by $x \approx y$, if $x - y$ is infinitesimal. (So $x \approx 0$ means that $x$ is infinitesimal.)

One way to construct $\mathbb{R}$ is as an ultrapower of the reals $\mathbb{R} = \mathbb{R}^N$ where $U$ is a nonprincipal ultralfilter (or maximal filter) on $N$. An example of a non-zero infinitesimal is given by $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$. 
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One way to construct $\mathbb{R}^\ast$ is as an ultrapower of the reals $\mathbb{R}^\ast = \mathbb{R}^N \cup U$ where $U$ is a nonprincipal ultrafilter (or maximal filter) on $\mathbb{N}$. An example of a non-zero infinitesimal is given by $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$. 
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$$\mathbb{R} = \mathbb{R}^\mathbb{N} \mathcal{U}$$

where $\mathcal{U}$ is a nonprincipal ultrafilter (or maximal filter) on $\mathbb{N}$.
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(i) \( x \) is **infinitesimal** if \( |x| < \varepsilon \) for all \( \varepsilon > 0, \varepsilon \in \mathbb{R} \);
(ii) \( x \) is **finite** if \( |x| < r \) for some \( r \in \mathbb{R} \);
(iii) \( x \) is **infinite** if \( |x| > r \) for all \( r \in \mathbb{R} \).
(iv) \( x \) and \( y \) are **infinitely close**, denoted by \( x \approx y \), if \( x - y \) is infinitesimal. (So \( x \approx 0 \) means that \( x \) is infinitesimal)

One way to construct \( \mathbb{R} \) is as an **ultrapower** of the reals

\[
\mathbb{R} = \mathbb{R}^N \mathcal{U}
\]

where \( \mathcal{U} \) is a nonprincipal ultrafilter (or maximal filter) on \( \mathbb{N} \).

An example of a non-zero infinitesimal is given by \( (1, \frac{1}{2}, \frac{1}{3}, \ldots) \mathcal{U} \).
Define addition and multiplication on \(*\mathbb{R}\) pointwise (this is safe) and it is then easy to see that

\((\mathbb{R}, +, \times, <)\) is an ordered field.
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\[(*\mathbb{R}, +, \times, <)\] is an ordered field.

A good way to picture \(*\mathbb{R}\) is as follows (note that some features in the diagram are yet to be explained).
\[ \text{monad}(r) = \{ x \in {}^\ast\mathbb{R} : x \approx r \} \]

The Hyperreals
Now extend all sets $A$, functions $f$ and relations $R$ on $\mathbb{R}$ to $\ast \mathbb{R}$ pointwise – with the extensions denoted by $\ast A$, $\ast f$ and $\ast R$.

**Examples**: $\ast \mathbb{N}$, $\ast \mathbb{Z}$ and $\ast \mathbb{Q}$, the sets of *hypernatural numbers*, *hyperintegers* and *hyperrationals* respectively. We can talk about an infinite (hyper)natural number $N$. 
Properties of $\mathbb{R}^*$ are given systematically by the following:

**Theorem (Transfer Principle)**

*Let $\varphi$ be any first order statement. Then*

$$\varphi \text{ holds in } \mathbb{R} \iff \mathbb{R}^* \text{ holds in } \mathbb{R}^*$$
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A **first order statement** $\varphi$ (respectively $*\varphi$): refers to elements of $\mathbb{R}$ (respectively $*\mathbb{R}$), both fixed and variable, and to fixed relations and functions $f, R$ (respectively $*f, *R$), with quantification ($\forall x, \exists y$) only for elements.
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To get back to \(\mathbb{R}\) from \(*\mathbb{R}\)*:

**Theorem (Standard Part Theorem)**

*If* \(x \in *\mathbb{R}\) *is finite, then there is a unique* \(r \in \mathbb{R}\) *such that* \(x \approx r\); *i.e. any finite hyperreal* \(x\) *is uniquely expressible as* \(x = r + \delta\) *with* \(r\) *a standard real and* \(\delta\) *infinitesimal.*
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**Definition (Standard Part)**

If $x$ is a finite hyperreal the unique real $r \approx x$ is called the **standard part** of $x$, written $r = \circ x = \text{st}(x)$. 
A NONSTANDARD UNIVERSE
Repeat the above construction to give \( *A \) for any mathematical object or structure \( A \); e.g. \( *M \) for a metric space with \( *d : *M \times *M \rightarrow *\mathbb{R} \).
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The most economical way to do this is for a whole mathematical universe \( \mathbb{V} \) with \( A \in \mathbb{V} \) for each object \( A \) that might be needed.
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\textbf{Theorem (The Transfer Principle)}

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Elements (objects) belonging to the nonstandard universe $^*\mathbb{V}$ are called \textit{internal}.
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Repeat the above construction to give *A for any mathematical object or structure A; e.g. *M for a metric space with *d : *M × *M → *R.
The most economical way to do this is for a whole mathematical universe V with $A \in V$ for each object A that might be needed. Information about the resulting nonstandard universe *V is given by:

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Elements (objects) belonging to the nonstandard universe \(*\mathbb{V}\) are called *internal*.

**Remark** The standard part mapping extends to the “nearstandard” elements of any extension metric (or topological) space \(*M\) - in particular the space \(*\mathbb{H}\). It is easy to show that elements \(U\) in \(*\mathbb{H}\) with \(|U|\) finite are nearstandard in the weak topology.
LOEB MEASURES

A *Loeb measure space* is a measure constructed from a nonstandard (i.e. *internal*) measure (essentially it is an *ultraproduct of measures*).

Suppose that an internal set $\Omega$ and an internal algebra $\mathcal{A}$ of subsets of $\Omega$, are given, $\mu$ is a finite internal finitely additive measure on $\mathcal{A}$;

\begin{align*}
\mu(\mathcal{A} \cup \mathcal{B}) &= \mu(\mathcal{A}) + \mu(\mathcal{B}) \quad \text{for disjoint } \mathcal{A}, \mathcal{B} \in \mathcal{A}, \quad \mu(\Omega) \text{ is finite.}
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by $\circ \mu(A) = \ast(\mu(A))$. Then $(\Omega, \mathcal{A}, \circ \mu)$ is a *standard finitely additive measure space* but $\mathcal{A}$ is not $\sigma$-additive in general.
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Theorem (Loeb 1975)

There is a unique $\sigma$-additive extension of $^\circ \mu$ to the $\sigma$-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. The completion of this measure is the Loeb measure corresponding to $\mu$, denoted $\mu_L$ and the completion of $\sigma(\mathcal{A})$ is the Loeb $\sigma$-algebra, denoted by $L(\mathcal{A})$. 
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Let $F : \Omega \to \ast \mathbb{R}$ be $S$-integrable. Then $\circ F : \Omega \to \mathbb{R}$ and

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Similar relationships connect internal (i.e. nonstandard) stochastic integrals to standard stochastic integrals on the Loeb space.
LOEB SPACE SOLUTIONS TO STOCHASTIC NSe

(1) Use standard SDE methods + Transfer to solve the Galerkin approximation to the sNSe in dimension $N$ ($N \in ^{\ast}\mathbb{N}$ infinite)

$$dU(\tau) = [-\nu^* AU(\tau) + ^* B_N(U) + ^* f_N(\tau, U(\tau))]d\tau + ^* g_N(\tau, U(\tau))dW_\tau$$

$U$ is an internal stochastic processes $U : ^*[0, T] \times \Omega \to H_N \subset ^*H$ on an internal space $\Omega_0 = (\Omega, \mathcal{A}, \mathcal{P})$ with internal Wiener process $W$ in $H_N$
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(2) Establish an energy estimate. There is a finite constant $E$ (independent of $N$) such that

$$\mathbb{E} \left( \sup_{\tau \leq T} |U(\tau)|^2 + \nu \int_0^T \|U(\sigma)\|^2 d\sigma \right) < E$$  \hspace{1cm} \text{(Energy)}

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(3) The energy estimate means that for a.a. $\omega$, $|U(\tau)|$ is finite for all $\tau \leq T$ and hence weakly nearstandard. The integral equation for $U$ gives that for a.a. $\omega$, if $\sigma \approx \tau$ then $U(\sigma) \approx U(\tau)$. Hence
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(4) Define a **standard weakly continuous process** $u : [0, T] \times \Omega \to H$ by

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(5) Show that this $u$ solves the sNSe on the Loeb space corresponding to $\Omega_0$ i.e. $\Omega = (\Omega, L(A), \mathcal{P}_L)$ with filtration derived from that on $\Omega_0$
Hence

**Theorem (Capiński & NJC (1991))**

*There is an adapted probability space $\Omega$ carrying an $H$-valued Wiener process $w$ such that for any ($L^2$-random) $u_0 \in H$ and $f, g$ (continuous with linear growth) there is a (weak) solution of the stochastic Navier–Stokes equations.*
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**Theorem (Capiński & NJC (1991))**

There is an adapted probability space $\Omega$ carrying an $H$-valued Wiener process $w$ such that for any ($L^2$-random) $u_0 \in H$ and $f, g$ (continuous with linear growth) there is a (weak) solution of the stochastic Navier–Stokes equations. That is, an adapted stochastic process $u : [0, \infty) \times \Omega \to H$ such that for a.a. $\omega$

(i) $u \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap C(0, T; H_{\text{weak}})$ for all $T < \infty$,

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Application 1: ATTRACTORS FOR STOCHASTIC NAVIER–STOKES EQUATIONS
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For a *deterministic* dynamical system with uniqueness write $S_t v = \text{value at time } t \text{ of the solution with } u(0) = v$.

An *attractor* is a compact set $A \subseteq H$ such that $S_t A = A$ and for any open set $G \supset A$ and bounded set $B \subset H$, eventually we have $S_t B \subseteq G$.

Intuitively an attractor is given by $A = \{S_\tau V : V \in B \text{ and } \tau \text{ an infinite time} \}$ where $B \subseteq H$ is a chosen bounded set (an absorbing set). This can be made precise using the ideas of NSA.

For stochastic systems there is a variety of notions including

1. *measure attractors* - limiting behaviour of the measure induced on path space (Schmalfuß and others).
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Loeb space methods give new results for each of (2)–(4) for $S_N$ for drift and noise of the form $f(u)$ and $g(u)$. 
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2. *stochastic attractors* (Crauel & Flandoli)
3. *process attractors* (NJC & Keisler)
4. *neo-attractors* (NJC & Keisler)

Loeb space methods give new results for each of (2) - (4) for sNSe for drift and noise of the form $f(u)$ and $g(u)$.
Stochastic attractors for sNSe

For this only makes sense for $d = 2$ (where we have uniqueness).

Crauel & Flandoli's idea: a stochastic attractor is a random compact set $A(\omega)$ that, at time 0, attracts trajectories starting at $-\infty$ (compared to the usual idea of an attractor being a set at time $\infty$ that attracts trajectories starting at time 0).

The Loeb space approach to solving the sNSe can be modeled by starting the solutions at any given negative time - including infinite negative time; then intuitively a random attractor $A(\omega)$ = points in $H$ that can be reached at time $t = 0$ starting at some infinite negative time.

Making this precise gives:

Theorem (Capiùski & NJC 1999) For special forms of the noise term $g(u)$ in the 2D sNSe there is a stochastic attractor $A(\omega)$ (compact in the strong topology of $H$).

Precise definition and proof - too long and complicated!
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Process attractors for sNSe

For \( d = 3 \) uniqueness is unknown. To overcome this for the deterministic NSe, Sell replaced the phase space \( H \) by \( \mathcal{W} = \) all solution paths for the deterministic NavierStokes equations.

The semigroup action \( \mathcal{S}_t \) on \( \mathcal{W} \) is time translation: if \( u = u(\cdot) \in \mathcal{W} \) then \( \mathcal{S}_t u = v \in \mathcal{W} \) is given by \((\mathcal{S}_t u)(s) = u(t+s)\).

This has the crucial semi-group property \( \mathcal{S}_{t_1} \circ \mathcal{S}_{t_2} = \mathcal{S}_{t_1 + t_2} \) along with \( \mathcal{S}_0 u = u \).

Theorem (Sell (1996))

There is global attractor \( \mathcal{A} \subseteq \mathcal{W} \) for the 3-dimensional (deterministic) NavierStokes equations.
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**Theorem (Sell (1996))**

*There is global attractor $A \subseteq \mathbf{W}$ for the 3-dimensional (deterministic) Navier–Stokes equations.*
Extension of Sell’s idea to the stochastic NS-equations.

Basic idea. Let $X$ be a set of solutions to the sNSe on a space $\Omega$ with Wiener process $w$. 
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\begin{align*}
\theta_1 \circ \theta_0 &= \text{id} \\
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Suppose now that $X$ is closed under $S_t$. Then a natural definition of a process attractor for the class $X$ is $A \subseteq X$ such that

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(a) A set of laws \( \mathcal{A} \subset \text{Law}(X) \) is a \textit{law-attractor} if

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(ii) \textit{(Attraction)} For any open set \( O \supset \mathcal{A} \) and bounded \( Z \subset \text{Law}(X) \), \( \hat{S}t Z \subseteq O \) eventually (i.e. this holds for all \( t \geq t_0 \)).

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Even for this weaker definition, existence requires a rather large probability space.
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(NJC & H.J. Keisler, 2004) There is a Loeb space $\Omega$ and a natural class of solutions $X$ that has a process attractor $A$. The class $X$ contains solutions to the sNSe for all $L^2$ random initial conditions.
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Remark It can be shown that if $\Omega$ is any sufficiently rich space (for example if $\Omega$ is a Loeb space) then any process attractor $A$ is not compact.
The definition of a process attractor is somewhat unsatisfactory and does not capture the full strength of what was proved above. This is fully captured by the notion of a neo-attractor.

Keisler's theory of neo-metric spaces involves notions of neo-open, neo-compact, and neo-continuous. These arise very naturally for metric spaces that are constructed using nonstandard analysis—such as the class of solutions $X$ in the above theorem.

Neo-compact is weaker than compact in general; neo-open is weaker than open, but neo-continuous is stronger than continuous.

Theorem (NJC & H.J. Keisler, 2005) The attractor $A$ of the above theorem is a neo-attractor; that is

1. (Invariance) $S t A = A$ for all $t \geq 0$;
2. $A$ is neo-compact;
3. for any neo-open set $G \supset A$ and bounded set $B \subset X$, eventually $S t B \subseteq G$. 

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Suppose we have a minimizing sequence of controls $\theta^*_n : [0, T] \rightarrow M$ (metric space) for a given optimal control problem, say. That is the cost of using control $\theta^*_n$ equals $J(\theta^*_n) \downarrow J_0$ where $J_0$ is the minimum of all costs for controls for the given system. Then NSA allows us to speak of the nonstandard control $\theta_N$ for any infinite $N$. We can usually make sense of $J(\theta_N)$ and we will have $J(\theta_N) \approx J_0$. In many circumstances we can then take standard parts to produce an optimal control $\theta = \circ \theta_N$.

This idea has been applied to the NSSe in a variety of settings, always involving a Loeb space so that solutions for all controls live on the same probability space. Results have been obtained for 2D systems of the form

$$u(t) = u_0 + \int_0^t \left\{ -\nu A u(s) - B(u(s)) + f(s, u, \theta(s, u)) \right\} ds + \int_0^t g(s, u) dw(s)$$

with $\theta$ Hölder continuous, or with $\theta$ having no feedback in $u$, or with the feedback consisting of cumulative digital observations of the solution at a fixed finite number of times.
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Suppose we have a minimizing sequence of controls $\theta_n : [0, T] \to M$ ($M$ a metric space) for a given optimal control problem, say. That is

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Application 2: OPTIMAL CONTROL THEORY (NJC & Katarzyna Grzesiak)

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This idea has been applied to the sNSe in a variety of settings, always involving a Loeb space so that solutions for all controls live on the same probability space. Results have been obtained for 2D systems of the form

$$u(t) = u_0 + \int_0^t \{-\nu Au(s) - B(u(s)) + f(s, u, \theta(s, u))\} \, ds + \int_0^t g(s, u) \, dw(s)$$

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For the **3D equations** results are only for systems with no feedback: i.e. \( \theta = \theta(t) \). The possible non-uniqueness of solutions requires a large space to work in - one containing all possible solutions for a given control to allow initially the existence of an optimal solution for a given control.

\[
\text{where } U: \mathbb{R} \rightarrow \mathbb{R}^n.
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Then we standardise the control to give \( \theta = \Theta \) and as in the basic existence proof show that it is possible to take \( u(t, \omega) = \theta(U(t, \omega)) \) It remains to prove that \( u \) is a solution for control \( \theta \) and \( J(\theta) = \theta(J(\Theta)) \) to give optimality.

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A typical situation in either 2D or 3D is that we have a nonstandard control \( \Theta \) (possibly \( \theta_N \)) and a nonstandard solution for it:

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U(\tau) = U_0 + \int_0^\tau \{-\nu^* A U(s) - ^* B(U(s)) + ^* f(s, U, \Theta(U))\} \, ds
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+ \int_0^\tau ^* g(s, U) \, dW(s)
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where \( U : ^*[0, T] \rightarrow ^*H \) or \( H_N \).
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**Details:** NJC & K.Grzesiak: Stochastics (2005) and AMO (2007).
Application 3: NON-HOMOGENEOUS (i.e. non-constant density)
STOCHASTIC NSe with multiplicative noise
These model the velocity $u$ and density $\rho$ of a mixture of viscous incompressible fluids of varying density in a bounded domain $D \subset \mathbb{R}^d \ (d = 2, 3)$
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(3) Loeb space methods (NJC & Brendan Enright): solve the stochastic equations with general multiplicative noise for $d = 2, 3$ assuming $M \geq \rho_0 \geq m > 0$. 


Definition of a weak solution  The velocity and the density will be stochastic processes living on an adapted probability space $\Omega$.

Definition
Given $u_0 \in H$, $\rho_0 \in L^\infty(D)$, $f : [0, T] \times H \to H$ and $g : [0, T] \times H \to L(H, H)$ a pair of stochastic processes $(\rho, u)$ is a weak solution to the stochastic nonhomogeneous Navier-Stokes equations if

(i) $u \in L^2([0, T] \times \Omega, V)$ and for a.a. $\omega$ $u(\cdot, \omega) \in L^\infty(0, T; H) \cap L^2(0, T; V)$

(ii) $\rho \in L^\infty([0, T] \times D \times \Omega)$

(iii) (Velocity) for almost all $T_0 \leq T$, for all $\Phi \in C^1(0, T; V)$

$$\rho(T_0) u(T_0) - (\rho_0 u_0, \Phi(0)) = \int_0^{T_0} \left[ (\rho u, \Phi') + \langle u, \nabla \rangle \Phi \right] dt + \int_0^{T_0} (\Phi, \rho g) dw$$

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Theorem (NJ C & Brendan Enright, JDE 2006) Suppose that \( u_0 \in H \) and \( \rho_0 \in L^\infty(D) \) with \( 0 < m \leq \rho_0(x) \leq M \), and \( f, g \) satisfy natural continuity and growth conditions. Then there is a weak solution \((\rho, u)\) to the stochastic nonhomogeneous Navier-Stokes equations with

\[
\mathbb{E} \left( \sup_{t \leq T} |u(t)|^2 + \nu \int_0^T ||u(t)||^2 \, dt \right) < \infty
\]

and for almost all \( \omega \), for all \( t \)

\[
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Main idea of the proof (Broadly similar to the homogeneous case.)
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3. Show that for almost all $(\tau, \omega)$ the density $R(\tau, \omega)$ is nearstandard.
Main idea of the proof (Broadly similar to the homogeneous case.)

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6. Show that the pair $(u, \rho)$ is a solution to the stochastic nonhomogeneous Navier-Stokes equations on the adapted Loeb space

$$\Omega = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$$

where $P = \Pi_L$, $\mathcal{F} = L(A)$ and $(\mathcal{F}_t)_{t \geq 0}$ is the filtration obtained from $(\mathcal{A}_\tau)_{\tau \geq 0}$. 
**Regularity in dimension 2**

In the 2D setting (i.e. a fluid moving in a bounded domain in the plane) there is more regularity to the solution, provided \( g \) has a little more regularity.
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Theorem
Suppose that \( d = 2 \) and the initial condition \( u_0 \in V \) and \((\rho, u)\) is the solution to the stochastic non-homogeneous Navier-Stokes equations constructed above. Suppose further that \( g : [0, t] \times V \to L(H, V) \) and \( |g(t, u)|_{H, V} \leq a(t)(1 + ||u||) \). Then almost surely:
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Then almost surely:

(a) \( \sup_{t \in [0, T]} ||u(t)|| + \int_T^T |Au(t)|^2 \, dt < \infty \) where \( A = -\Delta \);
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(c) the equation for $u(t, \omega)$ holds for all $T_0 \leq T$. 
Concluding remarks - what makes nonstandard methods useful in the study of Navier-Stokes equations?
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Concluding remarks - what makes nonstandard methods useful in the study of Navier-Stokes equations?
1. No need for limiting arguments and specialized compactness theorems to get a convergent subsequence from a sequence of finite dimensional Galerkin approximations. In fact the specialized compactness theorems (and the appropriate topology) are discovered as by-products.
2. The richness of Loeb spaces means that all activity can take place in a single underlying probability space - not only convenient but essential for formulating some ideas - eg process attractors and optimal controls in 3D.