# Ultrafilters and Set Theory

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- measure theory (David Fremlin)

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- Ultrafilters and determinacy,
- Cofinality of ultrapowers, pcf theory.

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Less: Suffices to preserve operations of $\leq 2$ arguments.

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**Open:** Do maximal linked families follow from the assumption that every set can be linearly ordered?

Any map $3^X \to 3$ that respects all **unary** operations on 3 (as canonically extended to $3^X$) is given by an ultrafilter. (Lawvere)

Among all the weak forms of AC in “Consequences of the Axiom of Choice” (Howard and Rubin), BPI has the most equivalent forms listed.

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Such ultrafilters can be proved to exist if we assume CH (or certain weaker assumptions), but not in ZFC alone.

Selective ultrafilters have the stronger, Ramsey property that every partition of $[\omega]^n$ into finitely many pieces has a homogeneous set in $\mathcal{U}$. (Kunen)

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If $\mathcal{U}$ is merely a P-point, then you get $H \in \mathcal{U}$ with a weaker homogeneity property: There exists $f : \omega \to \omega$ such that one piece of the partition contains all those infinite subsets $\{x_0 < x_1 < \dots\}$ for which $f(x_n) \leq x_{n+1}$ for all $n$.

Mixed partition theorems:

Let $\mathcal{U}$ and $\mathcal{V}$ be non-isomorphic selective ultrafilters, and let $[\omega]^\omega$ be partitioned into an analytic piece and a co-analytic piece. Then there exist $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that one piece of the partition contains all the sets chosen alternately from $A$ and $B$, i.e., all $\{a_0 < b_0 < a_1 < b_1 < \dots\}$ with all $a_i \in A$ and all $b_i \in B$.

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The same goes for non-nearly-coherent P-points.

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This is an equivalence relation on the non-principal ultrafilters on $\omega$.

The number of equivalence classes can be 1, can probably be 2, can be $2^{2^{\aleph_0}}$, and cannot be any other infinite cardinal.

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$\mathfrak{d}$ is the minimum number of functions $\omega \to \omega$ to dominate all such functions.

Any non-principal ultrafilter on $\omega$ generated by $< \mathfrak{d}$ sets is a P-point. (Ketonen)

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- The ultrapowers of $\omega$ by non-principal ultrafilters on $\omega$ all have cofinality $> \mathfrak{u}$. (Mildenberger)
- The ideal of compact operators on Hilbert space is not the sum of two properly smaller ideals.

## Ultrafilters are Bad Sets

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It follows that the existence of such $\mathcal{U}$ contradicts the axiom of determinacy. But there's a more direct contradiction.

## An Undetermined Game

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Neither player has a winning strategy in this game.

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So are the restrictions of the club filter on $\aleph_2$ to the sets

$$\{\alpha : \mathrm{cf}(\alpha) = \aleph_0\} \quad \text{and} \quad \{\alpha : \mathrm{cf}(\alpha) = \aleph_1\}.$$

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Ultrapowers with respect to these ultrafilters are essential in the combinatorial theory of cardinals under AD, and even in descriptive set theory.

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- They amount to $V$-complete homomorphisms $\mathcal{B} \to 2$ and thus let us convert $\mathcal{B}$-valued models to 2-valued ones.
- They play a key role in the formalization of what is true in $V^{\mathcal{B}}$.

## Non-generic Ultrafilters and Forcing

When forcing over models of ZFC, genericity is not needed to turn $V^{\mathcal{B}}$ into a 2-valued model. Any ultrafilter in $\mathcal{B}$ will do — even one in the ground model. (Vopěnka)

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Any statement with truth-value 1 in $V^{\mathcal{B}}$ will be true in the 2-valued quotient.

But there may be new ordinals in the 2-valued model produced by this process.

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So (in ZFC) every set is obtainable from ordinals and ultrafilters (in Boolean algebras).

**Intuition:** Ultrafilters provide a second fundamental building block, after ordinals, for the universe of sets.