A variant of the Hales-Jewett theorem

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van der Waerden / Hales-Jewett

van der Waerden's Theorem (1927)

Let $k, r \in \mathbb{N}$, $C_1 \cup \ldots \cup C_r = \mathbb{N}$. $\Rightarrow$ $\exists s$ and $a, d \in \mathbb{N}$ s.t.

$$a + d \cdot i \in C_s \quad \text{for } i = 0, \ldots, k$$

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Hales-Jewett Theorem (1963)

Let $k, r \in \mathbb{N}$, $C_1 \cup \ldots \cup C_r = \mathrm{Fin}(\mathbb{N} \times \{0, \ldots, k\})$. $\Rightarrow$ $\exists s$ and $\alpha \subseteq \mathbb{N} \times \{0, \ldots, k\}, \gamma \subseteq \mathbb{N}$ s.t.

$$\alpha \uplus \gamma \times \{i\} \in C_s \quad \text{for } i = 0, \ldots, k$$

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Szemerédi's Theorem	$\longleftrightarrow$	density Hales-Jewett
Polynomial van der Waerden	$\longleftrightarrow$	Polynomial Hales-Jewett

A combined additive and multiplicative van der Waerden theorem

Bergelson 2005

Let $k, r \in \mathbb{N}$ and $\mathbb{N} = C_1 \cup \ldots \cup C_r$. There exist a, b, d, s s.t.

$$b(a + id)^j \in C_s$$

for all $i, j \in \{0, \ldots, k\}$.

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In fact: Every set $C \subseteq \mathbb{N}$ of *positive upper multiplicative density* contains such configurations.
Idea: Uniform IP-Szemeredi implies that every such C contains a large set G of geometric progressions. Then Szemerédi's Theorem yields that G contains arithmetic progressions.

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$\mathcal{F} \dots$ family of finite subsets of $\mathbb{N}$.
$\mathcal{F}$ is *partition regular* iff one cell of any finite partition contains an element of $\mathcal{F}$.
(E.g. $\mathcal{F} = \{\{a, a+d, \dots, a+kd\} : a, d \in \mathbb{N}\}$.)

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Theorem

Let $k, r \in \mathbb{N}$, $C_1 \cup \ldots \cup C_r = \mathrm{Fin}(\mathbb{N} \times \{0, \ldots, k\})$ and let $\mathcal{F}$ be a partition regular family of finite sets.
$\Rightarrow \ \exists s, \alpha, \gamma$ and $F \in \mathcal{F}$ s.t.

$$\alpha \uplus \left(\gamma \uplus \{t\}\right) \times \{j\} \in C_s \quad \text{for all } j \in \{0, \ldots, k\} \text{ and } t \in F$$

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simpler: $C \subseteq \mathbb{Z}$ large $\Rightarrow \{(a, d) \in \mathbb{Z}^2 : a + id \in C, i = 0, \ldots, k\}$ large.

preservation of largeness

Furstenberg & Glasner 1998

Let $k \in \mathbb{N}$, assume that $C \subseteq \mathbb{Z}$ is piecewise syndetic.

$$\Rightarrow \{(a, d) : a, a + d, \ldots, a + kd \in C\} \text{ is piecewise syndetic in } \mathbb{Z}^2.$$

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$\Leftrightarrow$ $\exists p \in K(\beta G)$ s.t. $C \in p$.

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fact: One cell of any finite partition of G is piecewise syndetic.

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$C \subseteq G$ is *central* $\Leftrightarrow$ there is a minimal idempotent $p \in \beta G$ s.t. $S \in p$.
$p \in \beta G$ is idempotent if $p + p = p$, the idempotents are ordered by

$$p \leq q \ \Leftrightarrow \ p + q = q + p = p.$$

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sketch of proof: Set $\phi_i(a, d) = a + id$, let $\hat{\phi}_i : \beta(\mathbb{Z}^2) \to \beta\mathbb{Z}$ be its continuous extension. Goal:

$$\{(a, d) : \phi_0(a, d), \ldots, \phi_k(a, d) \in C\} \text{ is central}$$

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Pick a minimal idempotent p s.t. $C \in p$. $\Rightarrow \exists q \in \beta(G^2)$, minimal idempotent s.t.

$$\hat{\phi}_0(q) = \ldots = \hat{\phi}_k(q) = p.$$

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modest version

Assume that $C_1 \cup C_2 = \mathbb{Z}$. There exists $s \in \{1, 2\}$ s.t.

$$\{(a, d) : a, a + d, a + 2d \in C_s\}$$

is piecewise syndetic in $\mathbb{Z}^2$.