Overview

1. What is a numeraire?

2. Arrow-Debreu model
   - Change of numeraire $\leftrightarrow$ change of measure

3. Continuous time
   - Self-financing portfolios
   - Pricing via change of numeraire
   - Exchange rates
   - Swap options
A numéraire (or numeraire) is a chosen standard by which value is computed.

Example (currencies are numeraires)

We may compute values w.r.t to USD 1$ or EUR 1 € or JPY (1 ¥).

Of course, others might prefer use commodities:

1 OZ of gold could be a numéraire.

Clearly, once we choose a numéraire e.g. 1 USD, we determine the value of other assets:
The change of numeraire problem

**Problem**

In *theory*, does it really matter which numeraire we choose?

Of course, in *practice* there are reasons to prefer *gold* to other commodities e.g. *corn*, *live cattle*, or one currency with respect to another (political reasons)...

But *intuitively* there should be no theoretical reason (at least at the scale of investors)

to *measure* value in *gold* or *USD* (there used to be also the “gold standard”)

⇒ Can we deduce/exploit this fact in our financial models?

The strong underlying *principle* will be always

*absence* of arbitrage opportunities.
Recall (from Tuesday) the simple model with $N \geq 1$ securities (i.e. bonds, stocks or derivatives)

\[
\vec{a} = (a_1, a_2, \ldots, a_N)
\]

can be held long or short by any investor.

Two times: $t = 0$ and a fixed future $t = 1$.
At $t = 0$ we have the observed spot prices of the $N$ securities

\[
\vec{p} = (p_1, p_2, \ldots, p_N) = (p_i)_{i=1}^N \in \mathbb{R}^N
\]

At $t = 1$, the market attains one state among $M$ possible “scenarios”

\[
s \in \{1, \ldots, M\}.
\]

If $s$ is attained ⇒ “dividends” (prices) of the securities at $t = 1$

\[
\vec{D}^s = (D_1^s, D_2^s, \ldots, D_N^s) \in \mathbb{R}^N
\]
Theorem

No arbitrage $\Rightarrow$ existence of positive weights $(\pi_s)_{s=1}^{M}$ such that

$$\bar{p} = \sum_{s=1}^{M} \bar{D}^s \pi_s, \quad \text{i.e. } p_i = \sum_{s=1}^{M} D_i^s \pi_s \text{ for every } i \in \{1, \ldots, N\}.$$

Next we assume that there is a risk-free security $a_1$ (e.g. a bond) such that in any scenario $s$ we have $D_i^s = 1 \iff a_1$ is our numeraire

Define $R$ and $\hat{\pi}$ (the interest rate and risk-neutral probability) by the relation

$$1 + R = \frac{1}{\sum_{s=1}^{M} \pi_s}, \quad \hat{\pi}_s = \frac{\pi_s}{\sum_{r=1}^{M} \pi_r}.$$

Then

$$p_1 = \frac{1}{1 + R}, \quad p_i = E_{\hat{\pi}} \left[ \frac{D_i}{1 + R} \right] = \sum_{i=1}^{M} D_i^s \pi_s.$$
Change of numeraire

What happens if instead of $a_1$ we fix another asset, e.g. $a_2$ as numeraire?

Assume $D_2^s > 0$ for every state of the market.

⇒ the value of the asset $a_i$, measured in units of $a_2$, at $t = 1$ is

$$v_i^s := \frac{D_i^s}{D_2^s}, \quad \text{if the market is in the state } s.$$ 

We rewrite the value at $t = 0$ of $a_i$ as

$$p_i = \sum_{s=1}^{M} D_i^s \pi_s = \sum_{s=1}^{M} \frac{D_i^s}{D_2^s} D_2^s \pi_s$$

hence if we measure the value in units of $a_2$, since

$$p_2 = \sum_{r=1}^{M} (D_2^r \pi_r)$$

we have

$$\frac{p_i}{p_2} = \sum_{s=1}^{M} v_i^s \frac{(D_2^s \pi_s)}{\sum_{r=1}^{M} (D_2^r \pi_r)} = E_2 \left[ v^s \right],$$

where $E_2$ is a different probability measure than $E_{\hat{\pi}}$. 
We found an instance of the general “mechanisms”:

Passing from the numeraire $a$ to $b$ corresponds to a change of probability, from

$$\pi^a_s := \frac{(D^s_{a} \pi_s)}{\sum_{r=1}^{M} (D^r_{a} \pi_r)}$$

to

$$\pi^b_s = \frac{(D^s_{b} \pi_s)}{\sum_{r=1}^{M} (D^r_{b} \pi_r)}.$$

The value of the asset $a_i$ (w.r.t. the numeraire $b$) at $t = 0$ is given by the expectation w.r.t. $\pi^b$ of the values at time $t = 1$

$$\frac{p_i}{p_b} = E^b \left[ \frac{D^s_{i}}{D^s_{b}} \right] = \sum_{s=1}^{M} \frac{D^s_{i}}{D^s_{b}} \pi^b_s.$$
Let us model the market with
- a probability space \((\Omega, \mathcal{A}, P)\),
- a filtration \((\mathcal{F}_t)_{t \in [0,T]}\)
- Itô processes \(\tilde{S}_t = (S^i_t)_{i=1,...,N}\)

A portfolio \(\tilde{H}_t = (H^1_t, \ldots, H^N_t)\) has value

\[
V_t := \tilde{H}_t \cdot \tilde{S}_t = \sum_{i=1}^{N} H^i_t S^i_t
\]

A numeraire is a strictly positive Itô process \(D_t\).

The prices actualized with respect to \(D\) become

\[
\frac{S^i_t}{D_t}.
\]
Change of numeraire and self-financing strategies

Proposition

The self-financing condition is invariant with respect to any chosen numeraire, i.e.

\[ dV_t = \sum_{i=1}^{N} H^i_t dS^i_t, \quad t \in (0, T) \]

if and only if

\[ d \left( \frac{V_t}{D_t} \right) = \sum_{i=1}^{N} H^i_t d \left( \frac{S^i_t}{D_t} \right), \quad t \in (0, T) \]
We use Itô formula for product

\[ d(VG) = GdV + VdG + dGdV \]

with \( G = 1/D \).

Since

\[ d \left( \vec{H} \cdot \vec{S} \right) = \vec{H} d \left( \vec{S} \right) \]

we have

\[
\begin{align*}
   d(VG) &= G\vec{H}d\vec{S} + \left( \vec{H} \cdot \vec{S} \right) dG + dG d \left( \vec{H} \cdot \vec{S} \right) \\
   &= G\vec{H}d\vec{S} + \left( \vec{H} \cdot \vec{S} \right) dG + dG\vec{H}d\vec{S} \\
   &= \vec{H} \cdot \left( G\vec{S} + \vec{S}dG + dGd\vec{S} \right) \\
   &= \vec{H} \cdot d \left( \vec{S}G \right)
\end{align*}
\]
Theorem

Let $P^0$ be a probability (equivalent to $P$) such that every

$$\frac{S_t^i}{S_t^0}, \quad i \in \{1, \ldots, N\}$$

is a martingale, and also

$$\frac{D_t}{S_t^0}.$$

Consider the new probability

$$P^D = \frac{1}{D_0} \cdot \frac{D_T}{S_T^0} P^0.$$

Then each

$$\frac{S_t^i}{D_t} \quad i \in \{1, \ldots, N\}$$

is a $P^D$-martingale (as also $\frac{D_t}{D_t} = 1$).
Proof

\[ E^0 [\cdot | \mathcal{F}_t] \Rightarrow \text{the conditional expectation w.r.t. } P^0 \text{ and} \]
\[ E^D [\cdot | \mathcal{F}_t] \Rightarrow \text{the conditional expectation w.r.t. } P^D. \]

\[ \frac{S^i_t}{S^0_t} \text{ is a } P^0 \text{ martingale} \Rightarrow \frac{S^i_t}{S^0_t} = E^0 \left[ \frac{S^i_T}{S^0_T} | \mathcal{F}_t \right]. \]

We have a formula for conditional expectation w.r.t. different probabilities:

\[ E^D [X | A] = \frac{E^0 [X f I_A]}{E^0 [f I_A]} = \frac{E^0 [X f | A]}{E^0 [f | A]} \quad \text{with } f = \frac{1}{D_0} \cdot \frac{D_T}{S^0_T}. \]

Theorem

For any \((P^D \text{ integrable})\) random variable \(X\),

\[ E^D [X | \mathcal{F}_t] = \frac{E^0 [X f | \mathcal{F}_t]}{E^0 [f | \mathcal{F}_t]}, \quad \text{with } f = \frac{dP^D}{P^0} = \frac{1}{D_0} \cdot \frac{D_T}{S^0_T}. \]

\[ \Rightarrow \quad E^D \left[ \frac{S^i_T}{D_T} | \mathcal{F}_t \right] = \frac{E^0 \left[ \frac{S^i_T D_T}{D_T D_0 S^0_T} | \mathcal{F}_t \right]}{E^0 \left[ \frac{1}{D_0} \cdot \frac{D_T}{S^0_T} | \mathcal{F}_t \right]} = \frac{D_0}{D_0} \cdot \frac{E^0 \left[ \frac{S^i_T}{S^0_T} | \mathcal{F}_t \right]}{E^0 \left[ \frac{D_T}{S^0_T} | \mathcal{F}_t \right]} = \frac{S^i_T}{S^0_T} = \frac{S^i_t}{S^0_t} = \frac{S^i_t}{D_t}. \]
A consequence of the previous theorem is the possibility to compute prices w.r.t. $P^D$ instead of $P^0$.

**Corollary**

The value at time $t \in [0, T]$ of an asset $X$ can be computed as

$$V_t = S_t^0 E^0 \left[ \frac{X}{S_t^0} \big| \mathcal{F}_t \right] = D_t E^D \left[ \frac{X}{D_t} \big| \mathcal{F}_t \right].$$

Also the self-financing **hedging** strategy can be computed w.r.t. $D_t$.

If $D_t = S_t^i$ for some $i$, this has the advantage of reducing the number of parameters by one.

Let us consider some examples of applications.
A remark

Assume that an Itô process $S_t > 0$ is in the form

$$(dS)_t = S_t(\mu_t dt + \sigma_t dW_t),$$

hence the quadratic variation is

$$d[S]_t = \langle dSdS \rangle = S_t^2 \sigma_t^2 dt.$$

The process $S_t^{-1}$ is an Itô process, with and by Itô formula with

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f''(x) = +2 \frac{1}{x^3},$$

we have

$$dS^{-1} = f'(S)dS + \frac{1}{2}f''(S)d[S]$$

$$= -\frac{1}{S^2}dS + \frac{1}{S^3}S^2 \sigma^2 dt$$

$$= -\frac{1}{S_t^2} (S_t(\mu_t dt + \sigma_t dW_t)) + S_t^{-1} \sigma_t^2 dt$$

$$= S^{-1} \left( (-\mu_t + \sigma_t^2) dt - \sigma_t dW_t \right)$$
Siegel paradox on exchange rates

Assume that we have

- two different currencies, 1 and 2,
- and corresponding (deterministic) bonds

\[ B_t^1 = e^{r_1 t}, \quad B_t^2 = e^{r_2 t} \]

with interest rates \( r^1, r^2 \).

- \( B^i \) is expressed in currency \( i \in \{1, 2\} \)
- a stochastic exchange rate \( R_t \)

\[ dR_t = R_t(\mu dt + \sigma dW_t) \]

so that a unit of currency 2 equals \( R_t \) units of currency 1.

In currency 1, the market is given by two assets:

\[ (B_t^1, B_t^2 R_t). \]

In currency 2, the market is given by two assets:

\[ (B_t^1 R_t^{-1}, B_t^2). \]
In the equivalent martingale measure $P^0$ (risk neutral measure) we have

$$\frac{B_t^2 R_t}{B_t^1} = e^{(r^2-r^1)t} R_t$$

must be a martingale. By Ito formula (with respect to the measure $P^0$)

$$d \left( \frac{B_t^2 R_t}{B_t^1} \right) = e^{(r^2-r^1)t} R_t \left( (r^2 - r^1) dt + dR \right) \Rightarrow dR = R \left( (r^1 - r^2) dt + \sigma dW^0_t \right).$$

Also with respect to $P^0$, the equation for the inverse exchange rate is

$$d \left( R^{-1} \right) = R^{-1} \left( \left( -(r^1 - r^2) + \sigma^2 \right) dt - \sigma dW^0_t \right)$$

there is an extra term $\sigma^2$ which gives no symmetry.

⇒ The choice of one numeraire “transfers” all the risk to the others assets.

Choose currency 2 as a numeraire, i.e. $D_t = B_t^2 R_t$. Then in the probability $P^D$, we have that

$$\frac{B_t^1}{B_t^2 R_t}$$

is a $P^D$-martingale.

this leads to the “natural” equation

$$d \left( R^{-1} \right)_t = R_t^{-1} \left( -(r^1 - r^2) dt - \sigma dW^D_t \right)$$
Consider a market with three assets $S^0$, $S^1$, $S^2$,

$$dS^0_t = S^0_t \, r \, dt, \quad dS^1_t = S^1_t \left( \mu_1 dt + \sigma_1 dW^1_t \right), \quad dS^2_t = S^2_t \left( \mu_2 dt + \sigma_2 dW^2_t \right)$$

with $W^1_t$, $W^2_t$ independent.

We want to price the swap option that gives the possibility to the holder to exchange $S^2_T$ with $S^1_T$ without additional costs. Its value at time $T$ is

$$(S^1_T - S^2_T)^+ = \left( \frac{S^1_T}{S^2_T} - 1 \right)^+ S^2_T$$

Idea: swap option is a call option with strike price 1, if numeraire is $S^2$. 
\[
\frac{d \left( \frac{S^1}{S^2} \right)}{S^2} = \frac{S^1}{S^2} \left( \ldots \right) dt - \sigma_2 dW_t^2 + \sigma_1 dW_t^1 \\
= \frac{S^1}{S^2} \left( \ldots \right) dt + \sqrt{\sigma_1^2 + \sigma_2^2} \frac{\sigma_1 dW_t^1 - \sigma_2 dW_t^2}{\sqrt{\sigma_1^2 + \sigma_2^2}} \\
= \frac{S^1}{S^2} \left( \ldots \right) dt + \sqrt{\sigma_1^2 + \sigma_2^2} dW^* \\
\]

Where \( W^* \) is a Brownian motion w.r.t. \( P \).

If we use the risk-neutral probability \( P^2 \), corresponding to the numeraire \( S^2 \),

\[
d \left( \frac{S^1}{S^2} \right) = \frac{S^1}{S^2} \left( \sqrt{\sigma_1^2 + \sigma_2^2} \right) dB^* \\
\]

where \( B^* \) is a \( P^2 \)-Brownian motion and

\[
V_t = S_t^2 E^2 \left[ \frac{(S_T^1 - S_T^2)^+}{S_T^2} \right] | F_t = S_t^2 E^2 \left[ \left( \frac{S_T^1}{S_T^2} - 1 \right)^+ \right] | F_t \\
= S_t^2 C \left( t, T, \frac{S_t^1}{S_t^2}, 1, 0, \sqrt{\sigma_1^2 + \sigma_2^2} \right) \\
\]

where \( C (t, T, x, K, r, \sigma) \) Black-Scholes formula for price of call option.