Analyzing Asymptotics of Stochastic Processes through Optimal Transport

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SMAQ seminar

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[1] joint work with M. Mariani (arXiv:2307.10325)

Plan

The empirical measure of a process

- Consider a (stochastic) process $(X_t)_{t\geq 0}$ on E (metric Polish).
- Its occupation measure (at time T) is the measure on E

$$\mu_T^X = \int_0^T \delta_{X_s} ds$$

explicitly:

$$\mu_T^X(A) = \int_0^T I_{\{X_s \in A\}} ds \quad \forall A \in \mathcal{E}.$$

- Discrete time: $(Y_n)_{n=0}^{\infty} \Rightarrow X_t = Y_{\lfloor t \rfloor}$:

$$\mu_n^X = \sum_{i=0}^{n-1} \delta_{Y_i}.$$

- Applications:
  - Non-parametric statistics
  - Monte Carlo methods
  - Mean field theory

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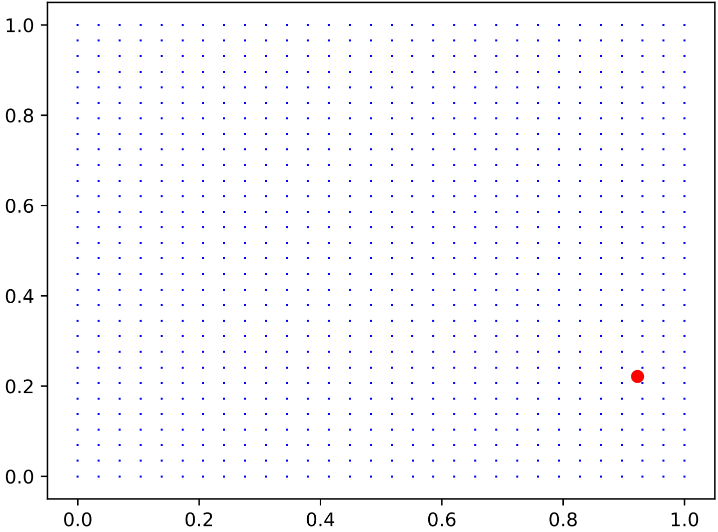
Consider $(Y_n)_{n=0}^{\infty}$ i.i.d. points uniformly distributed on $[0,1]^2$.



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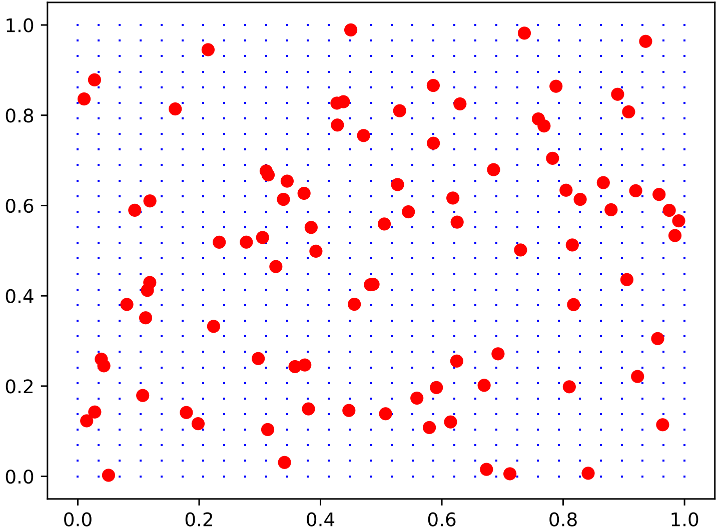

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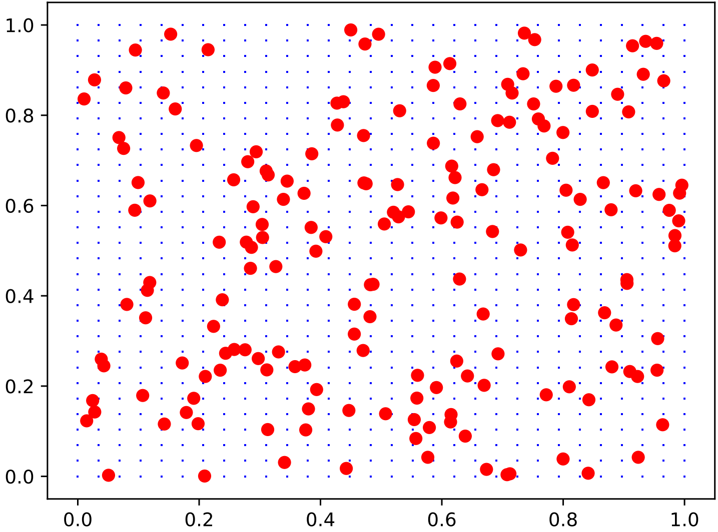

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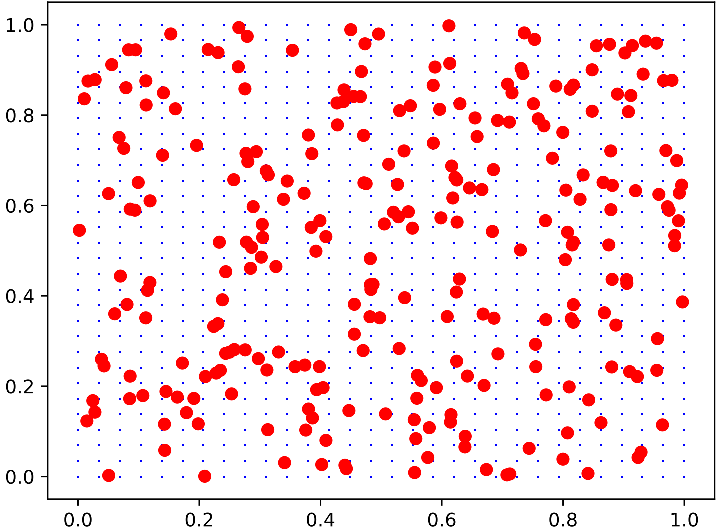

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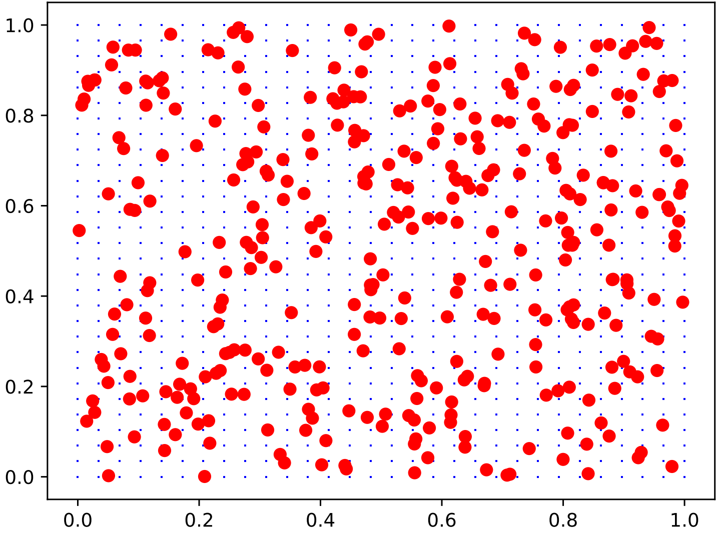

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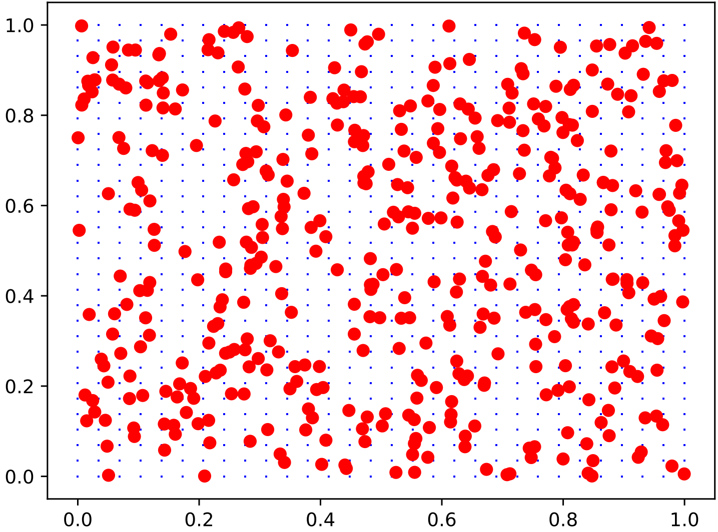

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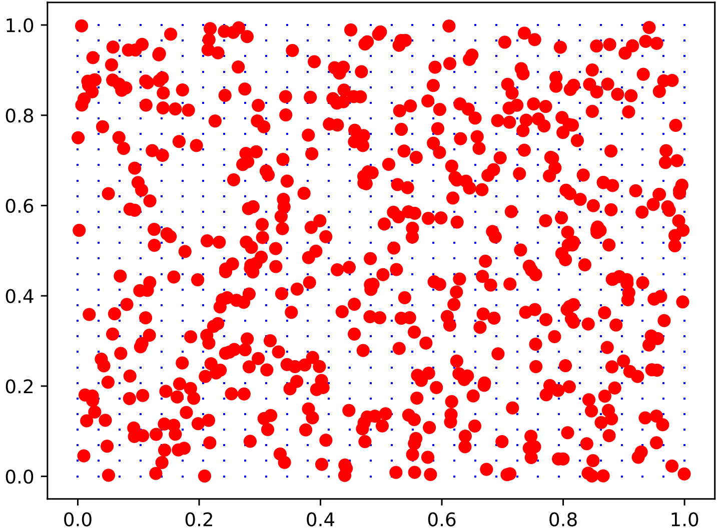

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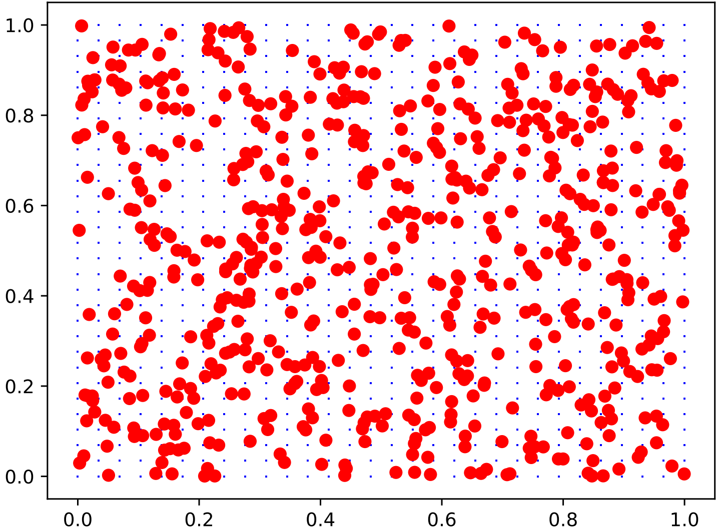

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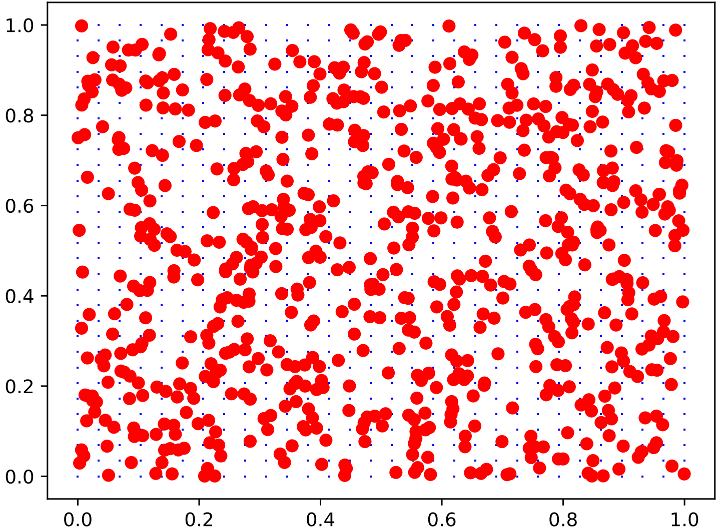

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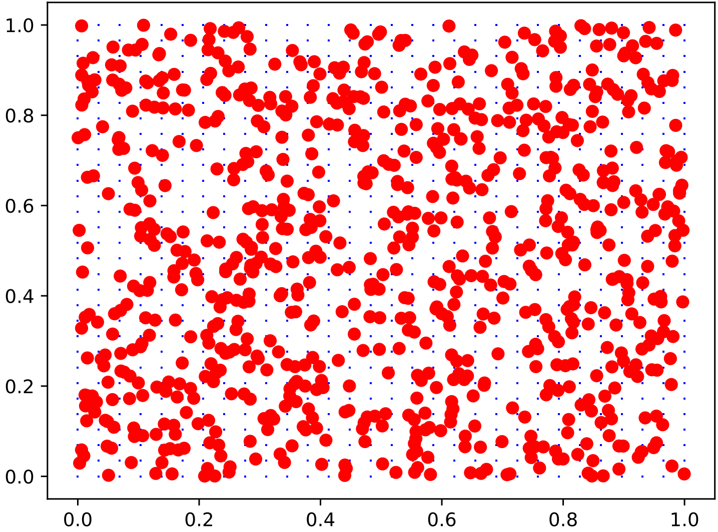

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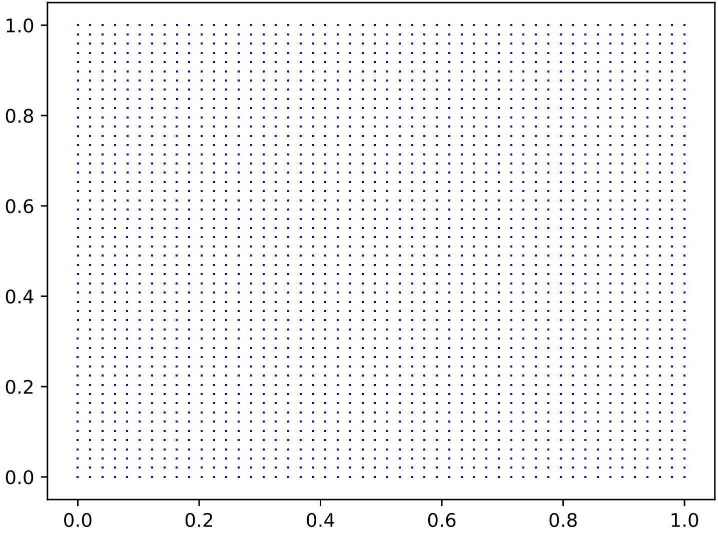
$(X_t) = (B_t)_{t\geq 0}$ is Brownian motion on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.



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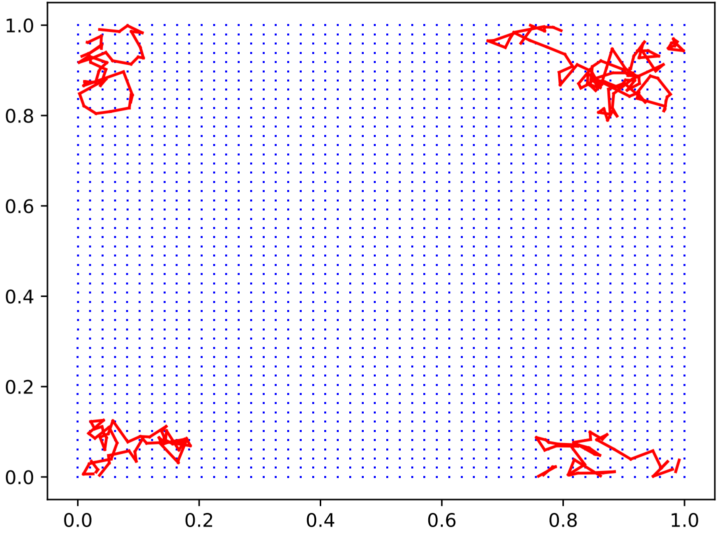

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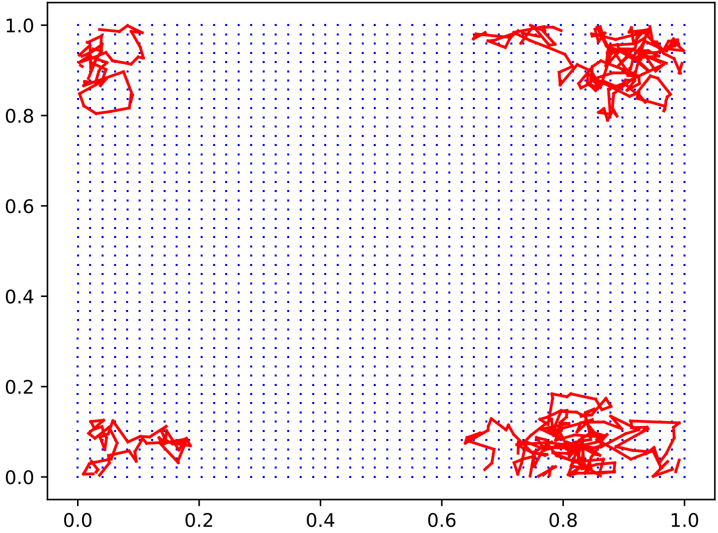

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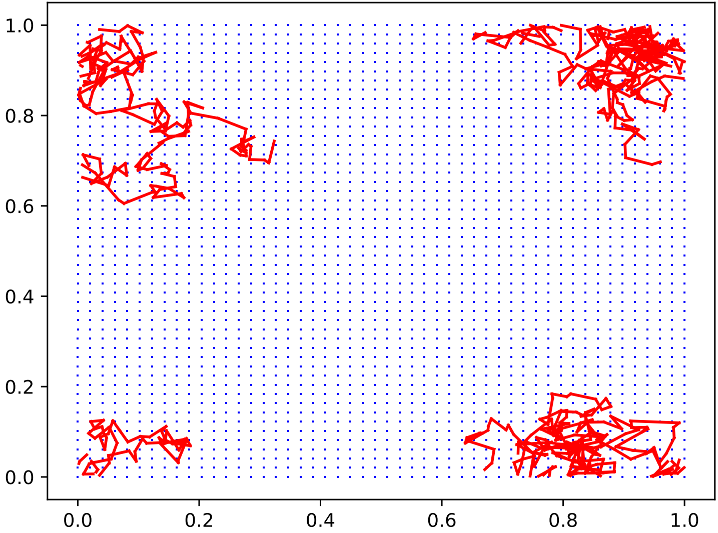

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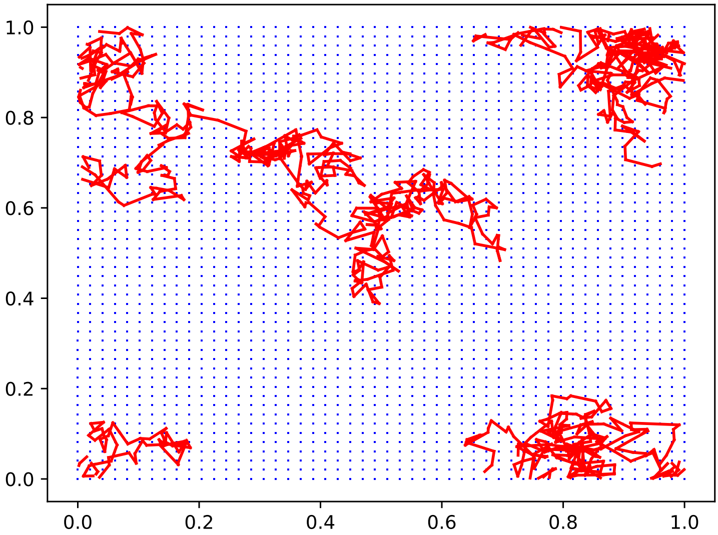

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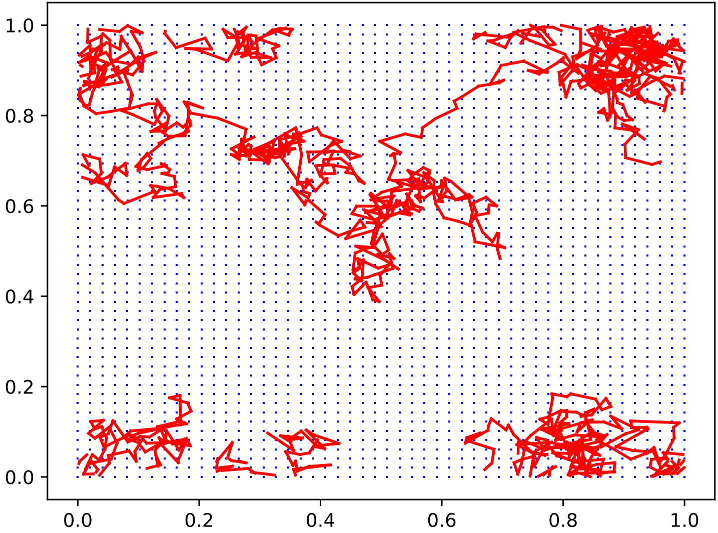

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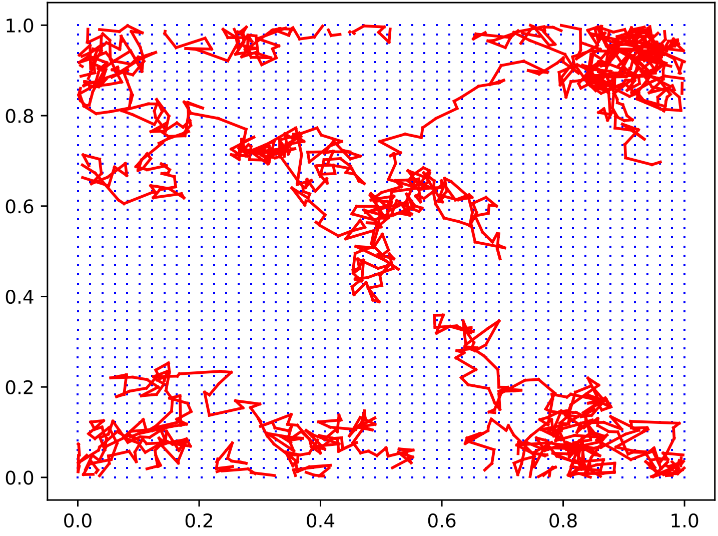

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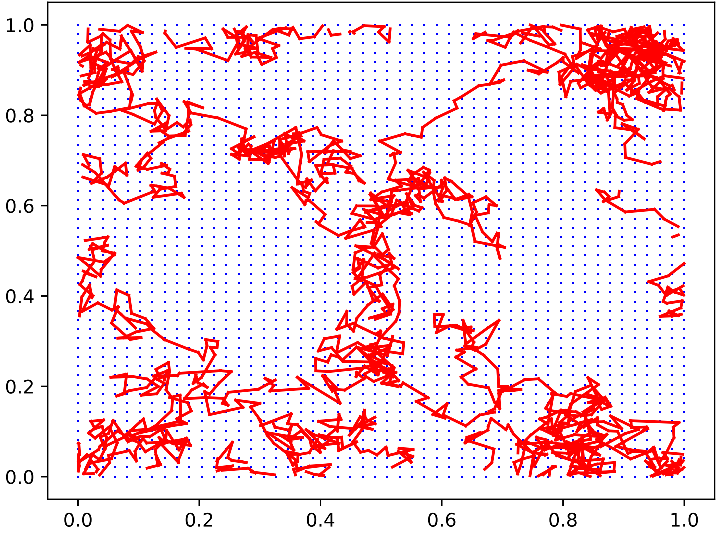

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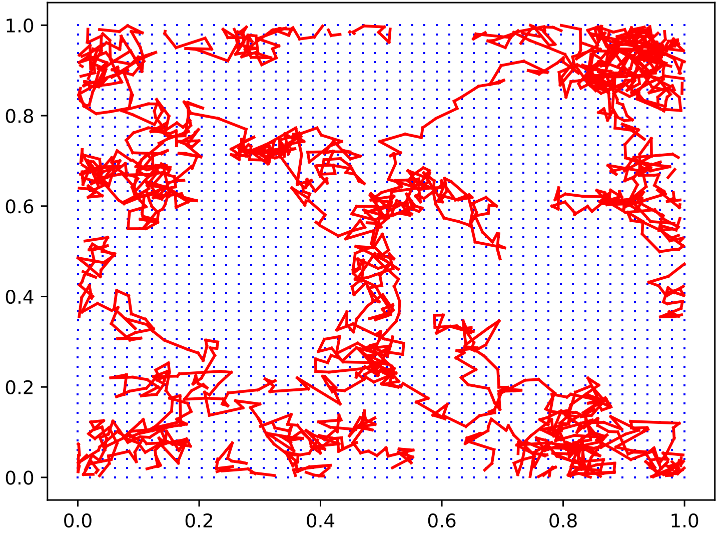

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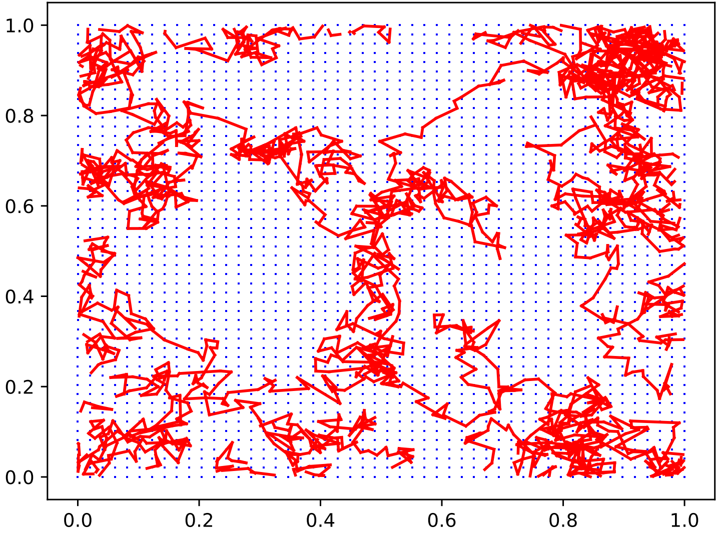

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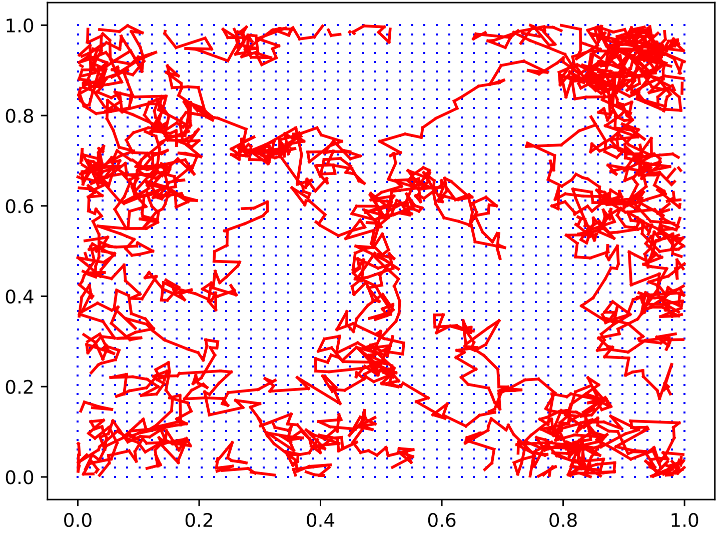

Convergence of Empirical Measure

- Assuming e.g. stationarity and ergodicity of X limit theorems are known:

$$\lim_{T\to\infty} \frac{1}{T}\mu_T^X = \mathfrak{m},$$

where $\mathfrak{m}$ is (the) invariant measure of the process:

$$\forall f \in C_b(E) \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T f(X_t)dt \to \int_E f d\mathfrak{m}.$$

- Problem: quantify convergence.
- Need a metric on the measures over the state space E.

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Metric induced by a family of functions

Weak convergence $\mu_n \to \nu$:

$$\lim_{n\to\infty} \int_E f d\mu_n \to \int_E f d\nu \quad \forall f \in C_b(E).$$

Idea: fix a $\mathcal{F} \subseteq C_b(E)$ and set

$$\mathrm{d}_{\mathcal{F}}(\mu, \nu) = \sup_{f \in \mathcal{F}} \left| \int_E f d\mu - \int_E f d\nu \right|$$

1. $\mathcal{F} = \{\|f\|_\infty \leq 1/2\} \Rightarrow \mathrm{d}_{TV}(\mu, \nu)$ total variation
2. $\mathcal{F} = \{\mathrm{Lip}(f) \leq 1\} \Rightarrow W^1(\mu, \nu)$ Wasserstein-Kantorovich
3. $\mathcal{F} = \{\|f\|_\infty \leq 1/2, \mathrm{Lip}(f) \leq 1\} \Rightarrow \mathrm{d}_{BL}(\mu, \nu)$ Bounded-Lipschitz (Dudley)
4. $E = \mathbb{R}$, $\mathcal{F} = \{I_{(-\infty, x]}\}_{x \in \mathbb{R}} \Rightarrow \mathrm{d}_{KS}(\mu, \nu)$ Kolmogorov

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Optimal transport of mass

Idea: find the cheapest way to move the mass μ into ν.

- Fix a cost $c(x, y) : E \times E \to [0, \infty)$ to transport a unit of mass from x to y:

$$\mathrm{d}(x, y), \quad \mathrm{d}(x, y)^p \quad (p > 0), \quad I_{\{x \neq y\}}$$

- A map $T : E \to E$ pushes μ into ν if $\mu(T^{-1}(A)) = \nu(A) \quad \forall A \subseteq E$ Borel.

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Optimal transport of mass

Idea: find the cheapest way to move the mass μ into ν.

- Fix a cost $c(x, y) : E \times E \to [0, \infty)$ to transport a unit of mass from x to y:

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Duality

Linear (convex) duality links the two viewpoints:

$$\mathrm{d}_{TV}(\mu,\nu) = \min_K \int_{E\times E} I_{\{x\neq y\}} K(dy|x)\mu(dx)$$

$$W^1(\mu,\nu) = \min_K \int_{E\times E} \mathrm{d}(x,y) K(dy|x) d\mu(dx)$$

$$\mathrm{d}_{BL}(\mu,\nu) = \min_K \int_{E\times E} \min\{1, \mathrm{d}(x,y)\} K(dy|x) d\mu(dx)$$

$$\mathrm{d}_{Kol}(\mu,\nu) = (\textit{exercise}?)$$

Plan

1. Introduction
2. Main results
3. Some ideas from the proof
4. Conclusion

Notation

- Let
  1. $E = \mathbb{R}^d$ with Euclidean distance $\mathsf{d}_{\mathbb{R}^d}(x, y) = |x - y|$
  2. or $E = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (with flat quotient distance)

$$\mathsf{d}_{\mathbb{T}^d}(x, y) = \min_{k \in \mathbb{Z}^d} |x - y + k|$$

- Write $|A|$ for Lebesgue measure $\mathscr{L}^d$ for $A \subseteq E$ (also on $\mathbb{T}^d$)
- For $\Omega \subseteq E$,

$$W^1_\Omega(\mu, \nu) = W^1(\mu \llcorner \Omega, \nu \llcorner \Omega)$$

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Asymptotics for i.i.d. points

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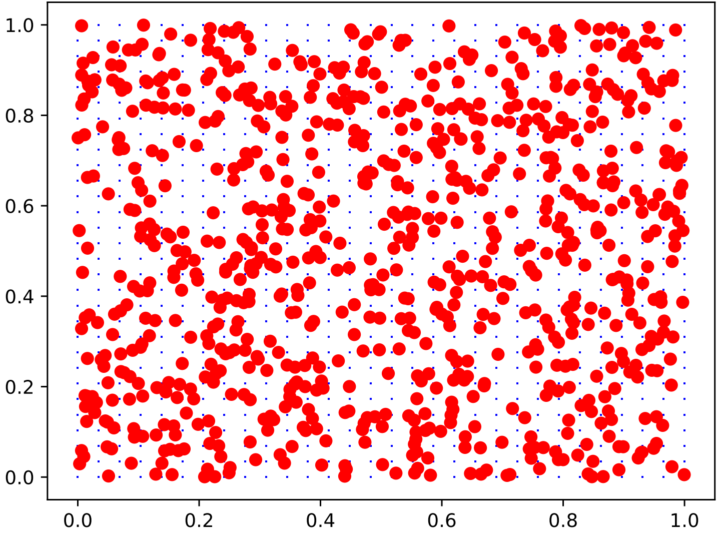

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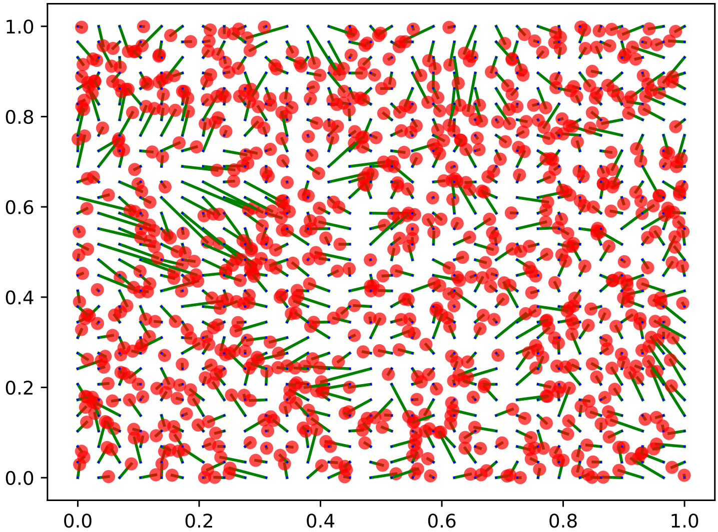

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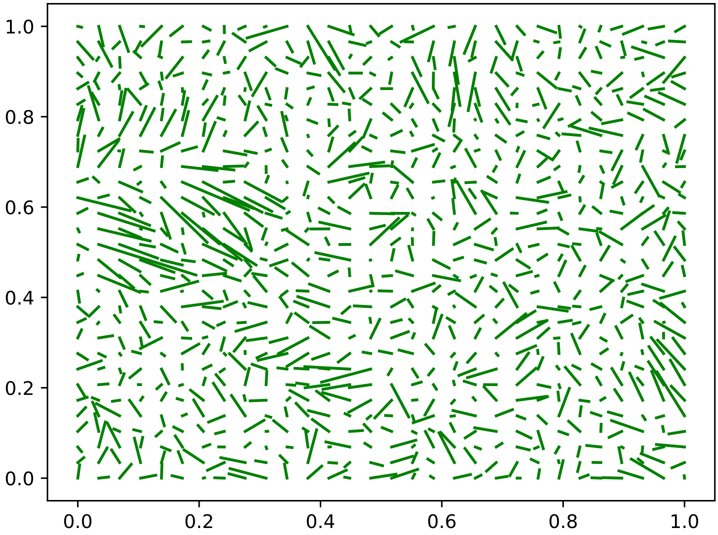

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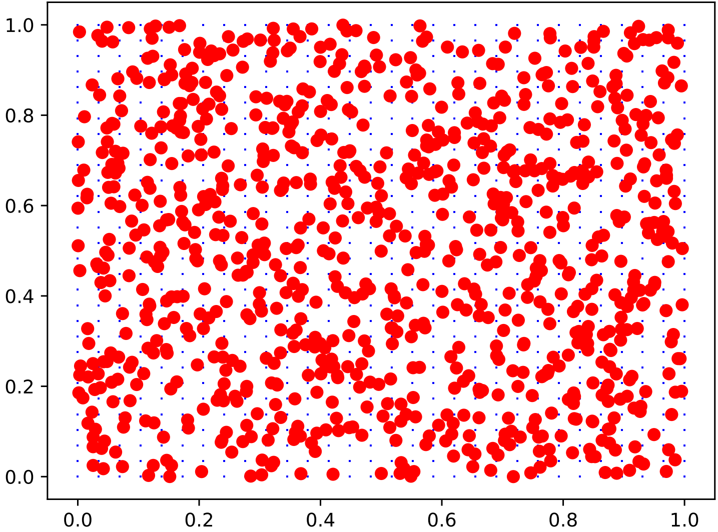

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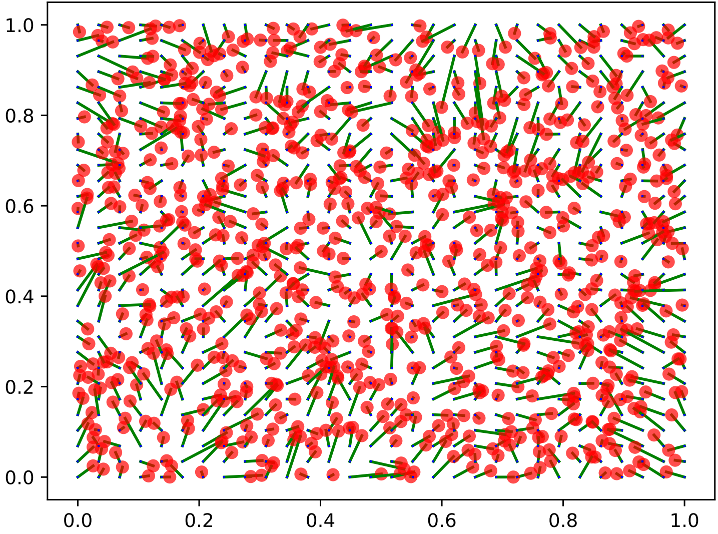

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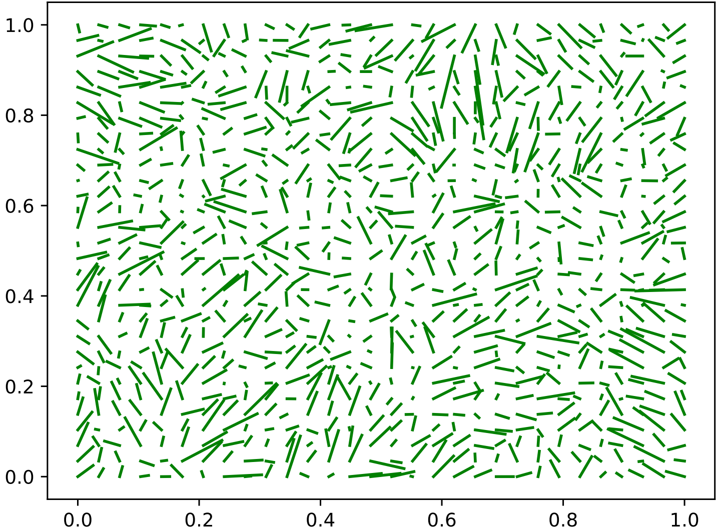

- Consider i.i.d. $(Y_n)_{n=1}^{\infty}$ uniform on $[0,1]^d$
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Moreover, if $d > 2$, $\mathbb{P}$-a.s.

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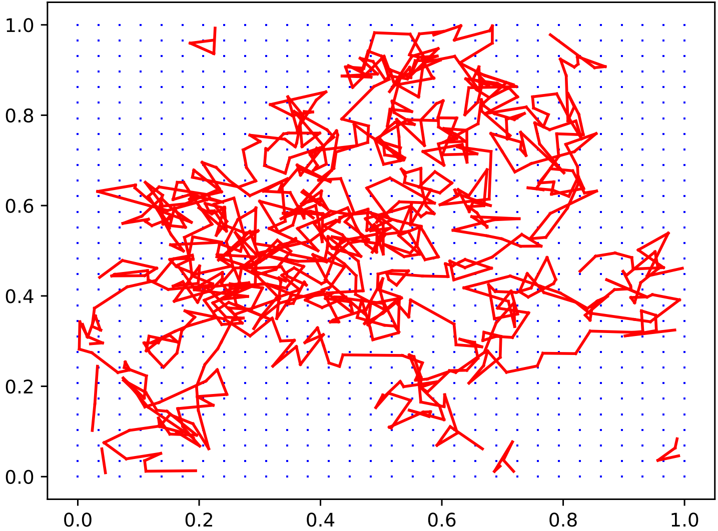

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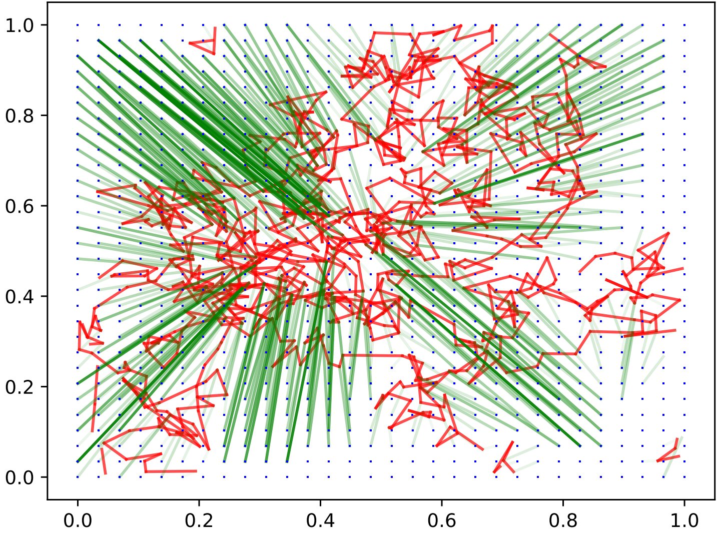

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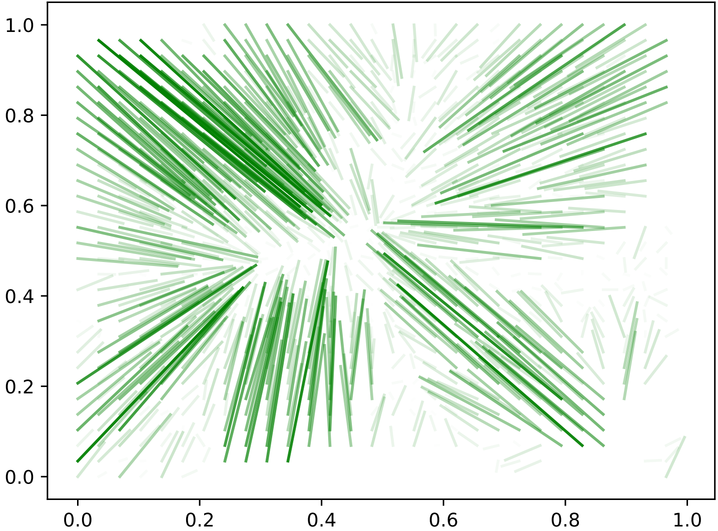

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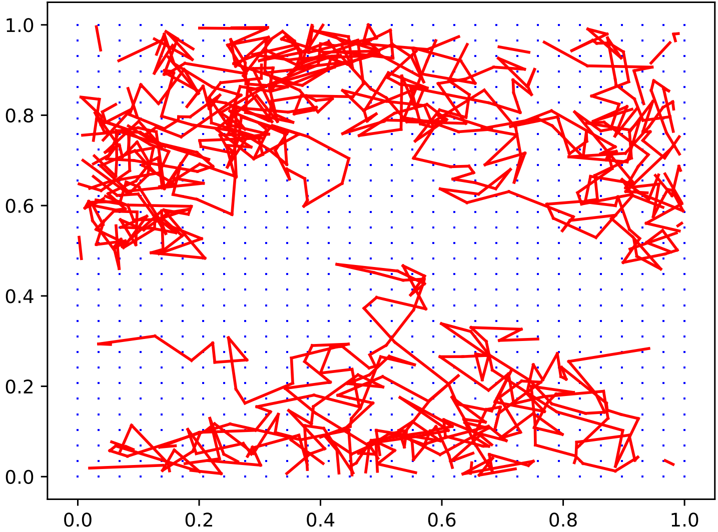

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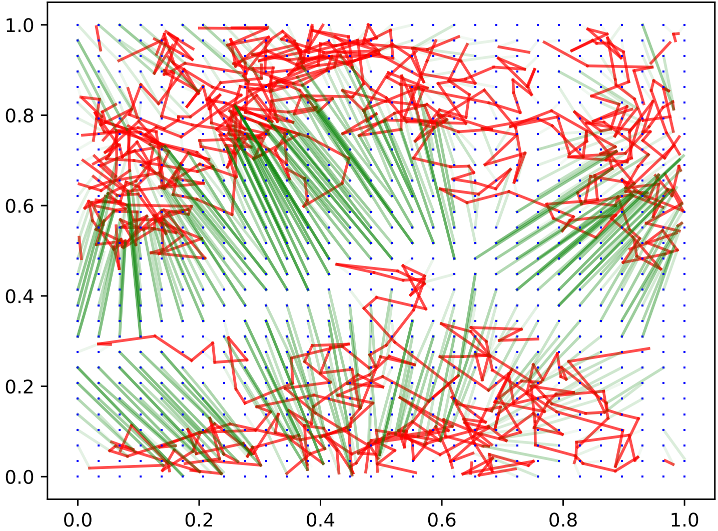

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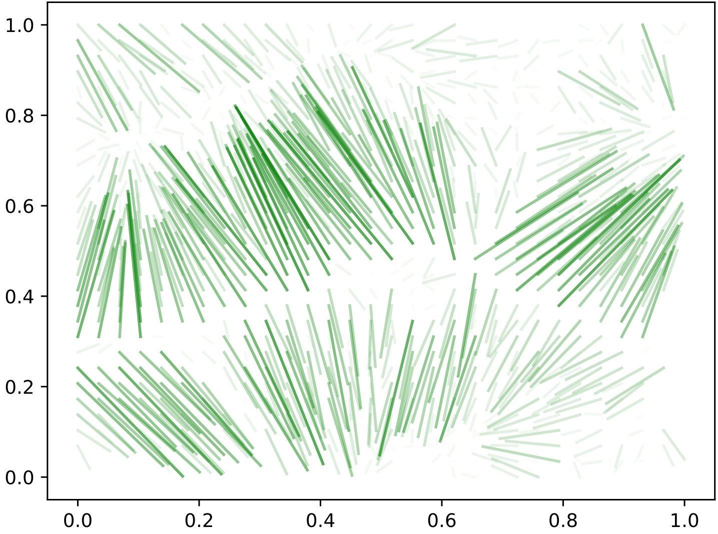

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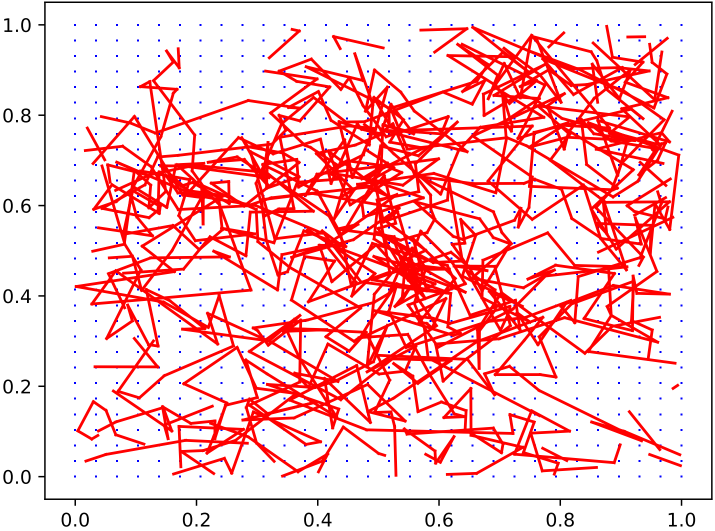

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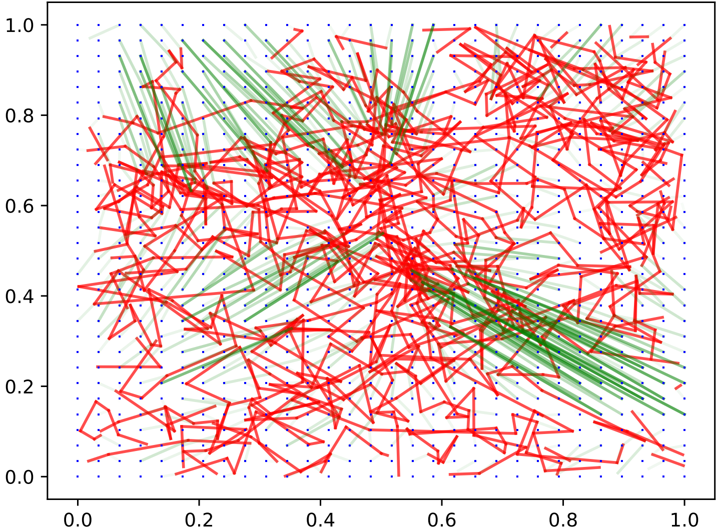

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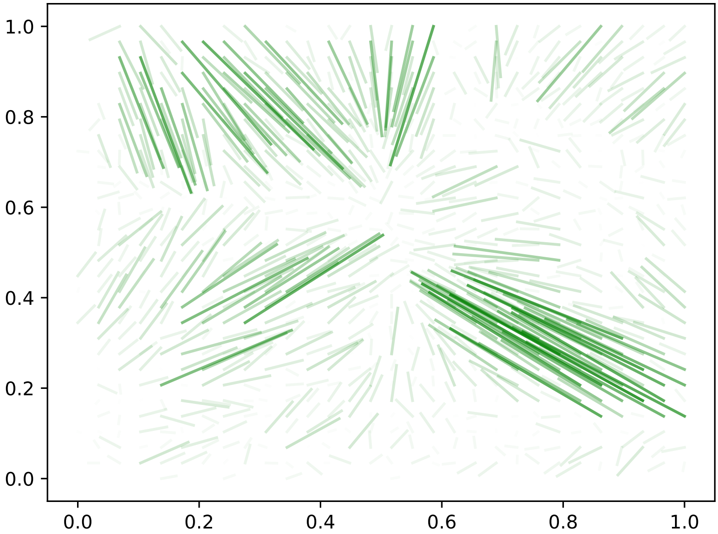

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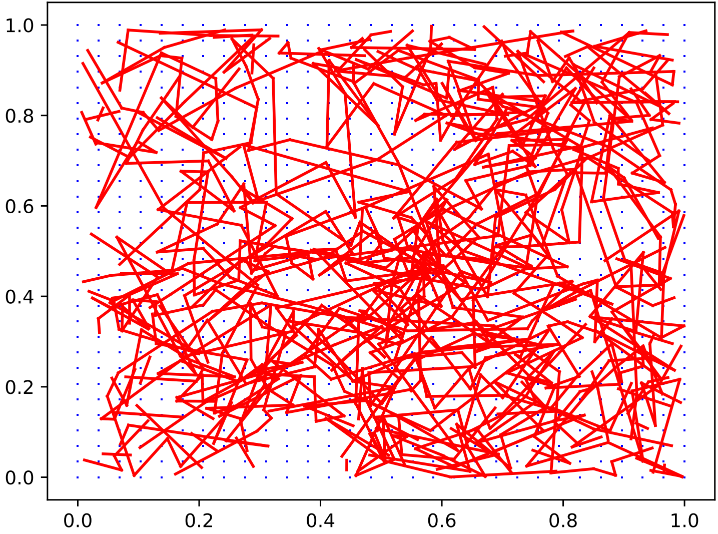

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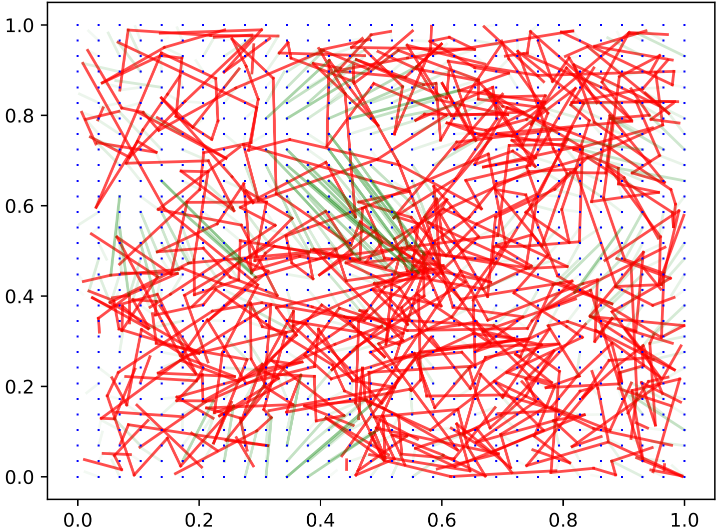

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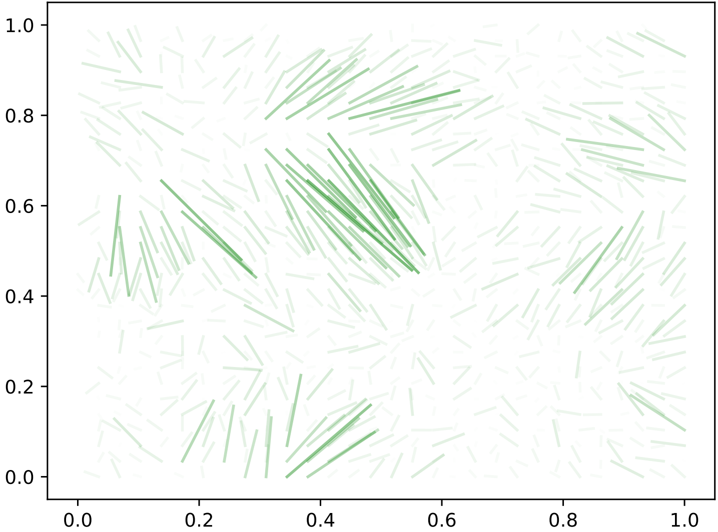

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Main result

Theorem (Mariani-T.)

If $d > 4$, then, $\mathbb{P}$-a.s.

$$\limsup_{T\to\infty} \frac{W^1_{\mathbb{T}^d}\left(\int_0^T \delta_{B_s} ds\right)}{T \cdot T^{-1/(d-2)}} \le c_{\mathcal{I}}(d) \in (0,\infty).$$

- $\mathcal{I}$ stands for "interlacement" (more on this later)
- Conjecture: the limit exists and $=$ holds.

Proposition

If $d > 4$ and $\ell = T^{-\gamma}$ with $\bar{\gamma}(d) < \gamma < 1/(d-2)$, then,

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Strategy in the i.i.d. case

Theorem (BdMonvel-Martin)

Let $d > 2$, $(Y_n)_{n=1}^{\infty}$ be i.i.d. uniform on $[0,1]^d$. Then,

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[W^1_{[0,1]^d}\left(\sum_{i=1}^n \delta_{Y_i}\right)\right]}{n \cdot n^{-1/d}} = c(d) \in (0,\infty).$$

Proof can be split into two steps:

1. Show that for a Poisson point process $\mathcal{P}$ on $\mathbb{R}^d$ (with intensity $\mathcal{L}^d$)

$$\lim_{L\to\infty} \frac{\mathbb{E}\left[W^1_{[0,L]^d}(\mathcal{P})\right]}{L^d} = c(d) \in (0,\infty)$$

   + existence follows by self-similarity and sub-additivity arguments
   - $c(d)$ is not explicit.

2. Perform a rescaling and de-Poissonization argument to reduce to n (deterministic) i.i.d. points on $[0,1]^d$.

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We need

1. an analogue of the Poisson point process for Brownian motion on $\mathbb{T}^d$:

   $\Rightarrow$ Brownian interlacement occupation measure

2. to perform a de-Poissonization (de-interlacing) argument from the interlacement to a single Brownian path.

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Brownian interlacement

- First introduced by Sznitman, we study its occupation measure.

Definition

Let $d \geq 3$, consider any $L \geq 1$ and

1. i.i.d. Brownian motions on $\mathbb{R}^d$ with initial law uniform on the sphere ∂D_L
2. an independent Poisson r.v. N with mean $\mathrm{Cap}(D_L) = L^{d-2}\,\mathrm{Cap}(D_1)$,

and define

Equality holds (in law)

$\Rightarrow$ the Brownian interlacement occupation measure is well-defined.

- $d \geq 3 \Rightarrow$ Brownian motion is transient.

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- $\mathbb{E}\left[\mathcal{I}(A)\right] = |A|$
- Concentration: if $\operatorname{diam}(A) \geq 1$, then

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Let $d \geq 5$. There exists

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Direct applications of tools from the i.i.d. literature

- de-Poissonization
- geometric decomposition

$\Rightarrow$ result for a deterministic number of Brownian paths.

Proposition

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A coupling argument allows for initial laws *close enough* to uniform on ∂D_1.

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- geometric decomposition

$\Rightarrow$ result for a deterministic number of Brownian paths.

Proposition

Let $d \geq 5$ and $(B^i)_{i=1}^{\infty}$ be i.i.d. Brownian motions on $\mathbb{R}^d$ with uniform initial law on the sphere ∂D_1. Then,

$$\lim_{m\to\infty} \frac{\mathbb{E}\left[W_{D_1}^p\left(\sum_{i=1}^m \int_0^\infty \delta_{B_s^i} ds\right)\right]}{(m/\operatorname{Cap}(D_1))^{1-1/(d-2)}} = c_{\mathcal{I}}(d)|D_1|.$$

A coupling argument allows for initial laws *close enough* to uniform on ∂D_1.

De-interlacing

Proposition

If $d > 4$ and $\ell = T^{-\gamma}$ with $\bar{\gamma}(d) < \gamma < 1/(d-2)$, then,

$$\lim_{T\to\infty} \frac{\mathbb{E}\left[W^1_{D_\ell}\left(\int_0^T \delta_{B_s} ds\right)\right]}{T \cdot T^{-1/(d-2)}|D_\ell|} = c_{\mathcal{I}}(d).$$

Sketch of the argument:

1. Split $[0, T]$ into $n \gg 1$ time intervals of length $\rho \sim T/n$ and replace a single Brownian path with n i.i.d. stationary $(B^i)_{i=1}^n$ on $\mathbb{T}^d$.
2. Consider the rare events A_i that B^i hits D_ℓ before ρ:
$$P(A_i) \sim \rho \ell^{d-2} \operatorname{Cap}(D_1) \ll 1 \quad \text{if } \rho \ll \ell^{d-2}.$$
3. By law of large numbers, the contribution of $\sum_{i=1}^n \mu_\rho^{B^i} \llcorner D_\ell$ comes from
$$m \sim n \cdot P(A_i) = n \cdot \rho \ell^{d-2} \operatorname{Cap}(D_1) \quad \text{terms.}$$

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Step 2 uses a result on hitting probabilities. Notation:

$$\tau_{D_\ell} = \text{first hitting time of } D_\ell \text{ for process } B$$

$$\nu_\rho(A) := \mathbb{P}(B_{\tau_{D_\ell}} \in A | 0 < \tau_{D_\ell} \leq \rho).$$

Proposition

Let $d \geq 3$, $0 < \gamma < d - 2$, and

$$\rho \sim \ell^{-\gamma} \quad \text{as } \ell \to 0.$$

For stationary Brownian motion B on $\mathbb{T}^d$,

$$\left|\mathbb{P}(0 < \tau_{D_\ell}(B) \leq \rho) - \rho \ell^{d-2} \operatorname{Cap}(D_1)\right| \lesssim (\rho \ell^{(d-2)})^2 |\log \ell| + \ell^d.$$

Moreover,

$$W^1_{D_\ell}(\nu_\rho, \mathrm{Unif}_{\partial D_\ell}) \lesssim \ell \cdot \left(\rho \ell^{d-2} |\log \ell| + \ell^2/\rho\right),$$

- The first bound is well-known in the literature (also used by Sznitman)
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- Brownian interlacement theory

are useful to analyze asymptotics of stochastic processes through optimal transport.

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