Parabolic sheaves, root stacks and the Kato-Nakayama space

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Outline

Parabolic sheaves as sheaves on “stacks of roots”, and log geometry.
Partly joint with A. Vistoli, and Carchedi-Scherotzke-Sibilla.
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Parabolic sheaves

Log schemes and (in)finite root stacks

Kato-Nakayama space and real roots
Parabolic sheaves (on a curve)

Let $X$ be a compact Riemann surface.

Narasimhan-Seshadri correspondence: there is a bijection

$$\{\text{unitary irreducible representations of } \pi_1(X)\} \leftrightarrow \{\text{degree 0 stable (holomorphic) vector bundles on } X\}. $$

(via local systems)
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What about the non-compact case?
Let $x_1, \ldots, x_k \in X$, and consider $X \setminus \{x_1, \ldots, x_k\}$.

There is a bijection

$$\{\text{unitary irreducible representations of } \pi_1(X \setminus \{x_1, \ldots, x_k\})\} \leftrightarrow \{\text{degree 0 stable parabolic vector bundles on } (X, \{x_1, \ldots, x_k\})\}$$

(Mehta-Seshadri, Deligne)
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(Mehta-Seshadri, Deligne)

The “parabolic” structure is meant to encode the action of the small loops around the punctures.
Definition
A parabolic bundle on \((X, D = \{x_1, \ldots, x_k\})\) is a holomorphic vector bundle \(E\) on \(X\) with additional data:
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for every point $x_i$ there is a filtration

$$0 \subset F_{i,h_i} \subset \cdots \subset F_{i,2} \subset F_{i,1} = E_{x_i}$$

of the fiber of $E$ over $x_i$, and weights $0 \leq a_{i,1} < \cdots < a_{i,h_i} < 1$. 
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\sim \text{ eigenspaces and eigenvalues of the matrix corresponding to a small loop } \gamma \in \pi_1(X \setminus \{x_1, \ldots, x_k\}) \text{ around the puncture } x_i.\]
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Parabolic bundles arising from representations of the algebraic fundamental group $\hat{\pi}_1(X \setminus \{x_1, \ldots, x_k\})$ always have rational weights.

If $D = x_1 + \cdots + x_k$ (divisor on $X$), by taking inverse images along $E \to E|_D$, a parabolic bundle can be seen as

$$E(-D) \subset F_h \subset \cdots \subset F_1 = E$$

with weights $0 \leq a_1 < \cdots < a_h < 1$. 
We can generalize and allow sheaves and maps

\[ E \otimes \mathcal{O}(-D) \to F_h \to \cdots \to F_1 = E \]

whose composition \( E(-D) \to E \) is multiplication by the section \( 1_D \) of the line bundle \( \mathcal{O}(D) \), and weights \( 0 \leq a_1 < \cdots < a_h < 1 \).

This definition makes sense for any variety \( X \) with an effective Cartier divisor \( D \subseteq X \) (Maruyama-Yokogawa).
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One defines morphisms, subsheaves, kernels, cokernels, etc. \( \rightsquigarrow \) a nice category of parabolic sheaves.

Parabolic sheaves are “best” defined on an arbitrary logarithmic scheme.
Log schemes (K. Kato, Fontaine-Illusie)

**Definition (Borne-Vistoli)**

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A log scheme is a scheme $X$ with
a sheaf of monoids $A$ and
a symmetric monoidal functor $A \to \text{Div}_X$.
$\text{Div}_X = (\text{symmetric monoidal fibered})$ category over $X_{\text{ét}}$ of line bundles with a global section $(L, s)$. 
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Definition (Borne-Vistoli)

A log scheme is a scheme $X$ with a sheaf of monoids $A$ and a symmetric monoidal functor $A \rightarrow \text{Div}_X$. \text{Div}_X = \text{(symmetric monoidal fibered) category over } X_{\text{ét}} \text{ of line bundles with a global section } (L, s).

More concretely: if $P$ is a monoid, a symmetric monoidal functor $L : P \rightarrow \text{Div}(X)$ sends

$$p \mapsto (L_p, s_p)$$

with isomorphisms

$$L_p \otimes L_q \cong L_{p+q}$$

carrying $s_p \otimes s_q$ to $s_{p+q}$.
If $D \subseteq X$ is an eff. Cartier divisor we get a log scheme $(X, D)$: take the symmetric monoidal functor $\mathbb{N} \to \text{Div}(X)$ sending $1$ to $(\mathcal{O}(D), 1_D)$.

If $D$ has $r$ irreducible components $D_1, \ldots, D_r$ and is simple normal crossings you might want to “separate the components” with the functor $\mathbb{N}^r \to \text{Div}(X)$ sending $e_i$ to $(\mathcal{O}(D_i), 1_{D_i})$ (Iyer-Simpson, Borne).
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Example: start with $X$ non-proper, compactify to $X \subseteq \overline{X}$ with SNC complement $D = \overline{X} \setminus X = D_1 \cup \ldots \cup D_r$, and take $(\overline{X}, (D_1, \ldots, D_r))$. 
How to think about this

To visualize the log scheme \((X, L: A \to \text{Div}_X)\), think about the stalks of the sheaf \(A\).

There is a largest open subset \(U \subseteq X\) where \(A_p = 0\) (might be empty).
In the “divisorial” case, \(U = X \setminus D\).
How to think about this

To visualize the log scheme \((X, L: A \to \text{Div}_X)\), think about the stalks of the sheaf \(A\).

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More generally \(A\) is locally constant on a stratification (\(\sim\) discrete data).

Example: \(X = \mathbb{A}^2\), \(D = \{xy = 0\}\). The stalks of the sheaf \(A\) are

\[
\begin{align*}
0 & \quad \text{on} \quad \mathbb{A}^2 \setminus \{xy = 0\} \\
\mathbb{N} & \quad \text{on} \quad \{xy = 0\} \setminus \{(0,0)\} \\
\mathbb{N}^2 & \quad \text{on} \quad \{(0,0)\}.
\end{align*}
\]
Parabolic sheaves (on log schemes)

A parabolic sheaf on $X$ with respect to $D$

$$E \otimes \mathcal{O}(-D) \rightarrow F_h \rightarrow \cdots \rightarrow F_1 = E$$

with rational weights $0 \leq a_1 < \cdots < a_h < 1$ with common denominator $n$
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$$E \otimes \mathcal{O}(-D) \to F_h \to \cdots \to F_1 = E$$

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can be seen as a diagram

$$-1 \quad -a_h \quad \cdots \quad -a_2 \quad -a_1 \quad 0$$

$$E \otimes \mathcal{O}(-D) \longrightarrow F_h \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow E$$

of sheaves placed in the interval $[-1, 0]$, with maps going in the positive direction.
We can fill the (possible) “gaps” in $[-1, 0] \cap \frac{1}{n}\mathbb{Z}$ by “looking at the sheaf on the left”, and extend out of $[-1, 0]$ by tensoring with powers of $\mathcal{O}(D)$.

(so that $E_{q+1} \cong E_q \otimes \mathcal{O}(D)$ for every $q \in \frac{1}{n}\mathbb{Z}$)
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We obtain a functor

$$\frac{1}{n}\mathbb{Z} \to \text{Qcoh}(X)$$

where there is one arrow $a \to b$ in $\frac{1}{n}\mathbb{Z}$ if and only if $a \leq b$ (i.e. there is $p \in \frac{1}{n}\mathbb{N}$ such that $a + p = b$).
Let $X$ be a log scheme with log structure $L: P \to \text{Div}(X)$, and choose an index $n \in \mathbb{N}$ (\sim common denominator of the weights).

Denote by $\frac{1}{n} P^{\text{wt}}$ the category with objects the elements of $\frac{1}{n} P^{gp}$ and arrows $a \to b$ elements $p \in \frac{1}{n} P$ such that $a + p = b$. 
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**Definition (Borne-Vistoli)**

A parabolic sheaf on $X$ with weights in $\frac{1}{n}P$ is a functor $E: \frac{1}{n}P^{\text{wt}} \to \text{Qcoh}(X)$ together with isomorphisms

$E_{a+p} \cong E_a \otimes L_p$ for any $a \in \frac{1}{n}P^{\text{gp}}$ and $p \in P$

(that satisfy some compatibility properties).
Example
If the log structure on $X$ is given by a snc divisor $D = D_1 + D_2$ with 2 irreducible components and we take $n = 2$, then a parabolic sheaf can be seen as $E(0,0) \otimes O(-D_1) \rightarrow E(-1,0) \rightarrow E(0,0)$ in the "negative unit square", and extended outside by tensoring with powers of $O(D_1)$ and $O(D_2)$. 
Example

If the log structure on $X$ is given by a snc divisor $D = D_1 + D_2$ with 2 irreducible components and we take $n = 2$, then a parabolic sheaf can be seen as

$$
E_{(0,0)} \otimes \mathcal{O}(-D_1) \rightarrow E_{(-\frac{1}{2},0)} \rightarrow E_{(0,0)}
$$

$$
E_{(0,-\frac{1}{2})} \otimes \mathcal{O}(-D_1) \rightarrow E_{(-\frac{1}{2},-\frac{1}{2})} \rightarrow E_{(0,-\frac{1}{2})}
$$

$$
E_{(0,0)} \otimes \mathcal{O}(-D) \rightarrow E_{(-\frac{1}{2},0)} \otimes \mathcal{O}(-D_2) \rightarrow E_{(0,0)} \otimes \mathcal{O}(-D_2)
$$

in the “negative unit square”, and extended outside by tensoring with powers of $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$. 
Take a log scheme $X$ with log structure $L: A \to \text{Div}_X$, and $n \in \mathbb{N}$.

The $n$-th root stack $\sqrt[n]{X}$ parametrizes liftings

$$A \longrightarrow \text{Div}_X \quad \text{with} \quad \text{lifting} \quad \nabla^n: \text{Div}_X \rightarrow \text{Div}_X$$

where $\nabla^n: \text{Div}_X \rightarrow \text{Div}_X$ is given by

$$(L, s) \mapsto (L \otimes^n, s \otimes^n).$$
If the log structure is induced by an irreducible Cartier divisor $D \subseteq X$, the stack $\sqrt[n]{X}$ parametrizes $n$-th roots of the divisor $D$.

That is, pairs $(L, s)$ such that $(L, s)^{\otimes n} \cong (\mathcal{O}(D), 1_D)$.

If $X$ is a compact Riemann surface and $D = x_1 + \ldots + x_k$, then $\sqrt[n]{X}$ is an orbifold with coarse moduli space $X$, and stabilizer $\mathbb{Z}/n\mathbb{Z}$ over the punctures $x_i$. 
Root stacks are tame Artin stacks, Deligne–Mumford in good cases (for example if char($k) = 0$).

**Theorem (Borne-Vistoli)**

Let $X$ be a log scheme with log structure $A \to \text{Div}_X$. There is an equivalence between

parabolic sheaves on $X$ with weights in $\frac{1}{n}A$, and

quasi-coherent sheaves on the root stack $\sqrt[n]{X}$. 
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*Let $X$ be a log scheme with log structure $A \to \text{Div}_X$. There is an equivalence between*

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*quasi-coherent sheaves on the root stack $\sqrt[n]{X}$.*

The “pieces” $E_a$ of the parabolic sheaves are obtained (roughly) as eigensheaves for the action of the stabilizers of $\sqrt[n]{X}$. 
The infinite root stack
(with A. Vistoli)

If $n \mid m$, there is a projection morphism

$$\sqrt[m]{X} \to \sqrt[n]{X}$$

that corresponds to raising to the $\frac{m}{n}$-th power.
This gives a projective system of algebraic stacks.
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**Definition**

The *infinite root stack* of \( X \) is the inverse limit

\[
\sqrt[\infty]{X} = \lim_{\leftarrow n} \sqrt[n]{X}.
\]

The stack \( \sqrt[\infty]{X} \) parametrizes compatible systems of roots of all orders. It is not algebraic, but it has local presentations as a quotient stack.
If $X$ is a compact Riemann surface with the log structure induced by the divisor $D = x_1 + \cdots + x_k$, the infinite root stack $\sqrt[\infty]{X}$

- looks like $X$ outside of $D$, and
- there is a stabilizer group $\widehat{\mathbb{Z}}$ at each of the points $x_i$. 
If $X$ is a compact Riemann surface with the log structure induced by the divisor $D = x_1 + \cdots + x_k$, the infinite root stack $\sqrt[\infty]{X}$ looks like $X$ outside of $D$, and there is a stabilizer group $\hat{\mathbb{Z}}$ at each of the points $x_i$.

**Theorem (\text{-, Vistoli})**

*There is an equivalence between quasi-coherent sheaves on $\sqrt[\infty]{X}$ and parabolic sheaves on $X$ with arbitrary rational weights.*

($\rightsquigarrow$ *moduli spaces for parabolic sheaves*)
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($\cong$ moduli spaces for parabolic sheaves)

**Theorem (-, Vistoli)**

*Every isomorphism $\sqrt[\infty]{X} \cong \sqrt[\infty]{Y}$ of stacks comes from a unique isomorphism of log schemes $X \cong Y$.***
The Kato-Nakayama space

From now on consider schemes locally of finite type over $\mathbb{C}$. Let $X$ be a log scheme.

There is an “underlying topological space” $X_{\log}$ with a surjective map $\tau: X_{\log} \to X$. 

Note that $S^1 = B\mathbb{Z}$, and so $\hat{S}^1_k \cong \hat{B}\mathbb{Z}^k$. 

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The fiber of $\tau$ over $x \in X$ can be identified with $(S^1)^k$, where $k = \text{rank of the free abelian group } A_{x}^{gp}$.

(over the locus $U \subseteq X$ where the log structure is trivial, $\tau$ is an isomorphism)
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The fiber of \( \tau \) over \( x \in X \) can be identified with \( (S^1)^k \), where \( k = \text{rank of the free abelian group } A^\text{gp}_x \).

(over the locus \( U \subseteq X \) where the log structure is trivial, \( \tau \) is an isomorphism)

The (reduced) fiber of \( \sqrt[\infty]{X} \to X \) over \( x \) is \( \widehat{B\mathbb{Z}}^k \), where \( k \) is the same number.

Note that \( S^1 = B\mathbb{Z} \), and so \( (S^1)^k \cong \widehat{B\mathbb{Z}}^k \).
There is a functor $(−)_{top}$ that associates to a stack over schemes an underlying topological stack in the sense of Noohi (≈ analytification functor).
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**Theorem (Carchedi, Scherotzke, Sibilla, -)**

*There is a canonical map of topological stacks*

$$\Phi_X : X_{\text{log}} \to \sqrt[\infty]{X}_{\text{top}}$$

*that induces an equivalence upon profinite completion.*

The description of the map is easier if one interprets $X_{\text{log}}$ itself as parametrizing “roots” of a certain kind.
$X_{\text{log}}$ as a root stack

As the stack $\sqrt[n]{X}$ parametrizes

\[
A \to \text{Div}_X \leftarrow \text{Div}_X \uparrow \uparrow \wedge n
\]
$X_{\text{log}}$ as a root stack

As the stack $\sqrt[n]{X}$ parametrizes

\[
\begin{array}{ccc}
A & \rightarrow & \text{Div}_X \\
\downarrow & & \downarrow \\
& \text{Div}_X & \\
\end{array}
\]

it turns out $\sqrt[n]{X}_{\text{top}}$ parametrizes

\[
\begin{array}{ccc}
A & \rightarrow & [\mathbb{C}/\mathbb{C}^\times]_X \\
\downarrow & & \downarrow \\
& [\mathbb{C}/\mathbb{C}^\times]_X & \\
\end{array}
\]

(note $\text{Div}_X \sim [\mathbb{A}^1/\mathbb{G}_m]_X$).
A way to map to something that dominates every morphism
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A way to map to something that dominates every morphism $\wedge n: [\mathbb{C}/\mathbb{C}^\times]_X \to [\mathbb{C}/\mathbb{C}^\times]_X$ is to (in some sense) **extract a logarithm**.

Consider the stack $X_H$ that parametrizes

\[
\begin{array}{c}
A \\
\downarrow \quad \rightarrow [\mathbb{C}/\mathbb{C}^\times]_X \\
\downarrow \\
[\mathbb{H}/\mathbb{C}^+]_X
\end{array}
\]

where $H = \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $\exp$ is induced by $H \to \mathbb{C}$ given by

\[(x, y) \mapsto x \cdot e^{iy}\]

and by the exponential $\mathbb{C}^+ \to \mathbb{C}^\times$. 
For every $n$ we have a factorization

\[
\begin{array}{c}
\left[ \mathbb{C}/\mathbb{C}^\times \right] \\
\downarrow \exp \\
\left[ \mathbb{H}/\mathbb{C}^+ \right]
\end{array} \xleftarrow{\exp} \left[ \mathbb{C}/\mathbb{C}^\times \right] \xrightarrow{\wedge n} \left[ \mathbb{C}/\mathbb{C}^\times \right] \\
\downarrow \phi_n
\]

where $\phi_n: \left[ \mathbb{H}/\mathbb{C}^+ \right] \to \left[ \mathbb{C}/\mathbb{C}^\times \right]$ is given by $\mathbb{H} \to \mathbb{C}$

\[
(x, y) \mapsto (\sqrt[n]{x}, y/n) \mapsto \sqrt[n]{x} \cdot e^{i\frac{y}{n}}
\]

and by $\mathbb{C}^+ \to \mathbb{C}^\times$ given by $z \mapsto e^{\frac{z}{n}}$. 
Now the diagram

\[
A \rightarrow [\mathbb{C}/\mathbb{C}^\times]_X \quad \xymatrix{ \left[\mathbb{H}/\mathbb{C}^+\right]_X \ar[rr]^-{\phi_n} \ar@{.>}[dr]^\exp \ar@{.>}[d] \ar@{.>}[ur]^\wedge n & & \left[\mathbb{C}/\mathbb{C}^\times\right]_X \ar[ll]^-{\exp} \ar[dl]^-{\phi_n} } \]

gives a natural transformation \( X_{\mathbb{H}} \rightarrow \sqrt[n]{X}_{\text{top}} \).

These are compatible and give \( X_{\mathbb{H}} \rightarrow \sqrt[\infty]{X}_{\text{top}} = \lim_{\leftarrow n} \sqrt[n]{X}_{\text{top}} \).
Theorem (in progress) -, Vistoli)

*The topological space* $X_{\log}$ *represents the stack* $X_{\mathbb{H}}$.

The morphism $X_{\log} \to \sqrt[\infty]{X_{\text{top}}}$ that we obtain is the one mentioned before.
Theorem ((in progress) -, Vistoli)

The topological space $X_{\log}$ represents the stack $X_H$.

The morphism $X_{\log} \to \sqrt[\infty]{X_{\text{top}}}$ that we obtain is the one mentioned before.

This is related to

- real roots of the log structure, i.e. diagrams

\[
\begin{array}{c}
A \\
\downarrow \\
A_{\mathbb{R}_{\geq 0}}
\end{array} \longrightarrow \begin{array}{c}
[\mathbb{C}/\mathbb{C}^\times]_X \\
\end{array}
\]

- parabolic sheaves with real weights.
Thank you for your attention!