KATO-NAKAYAMA SPACES VS INFINITE ROOT STACKS

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Abstract. I will talk about a comparison result between two objects that one can associate to a fine saturated log scheme over the complex numbers, namely the Kato-Nakayama space and the infinite root stack (joint work with D. Carchedi, S. Scherotzke and N. Sibilla). I will start by giving an introduction to log geometry through motivations and examples, then I will briefly introduce these two objects, that are different incarnations of the "log part" of the geometry of a log scheme. Towards the end I will state our result, and (time permitting) give an idea of what it boils down to.

Contents

1. Introduction

I want to talk about a comparison result (joint work with D. Carchedi, S. Scherotzke and N. Sibilla) between two objects that you can associate to a log scheme, so I will start by talking about log schemes. Towards the second half I will introduce these two objects, the Kato-Nakayama space and the infinite root stack, and talk about the comparison.

Here is the plan of the talk.

§1. Log geometry
§2. Kato-Nakayama space and infinite root stack
§3. The comparison

2. Log geometry

Log geometry was originally born in arithmetic context in the late 80's (Fontaine-Illusie, Deligne-Faltings, K. Kato), and later spread to touch various other areas, in particular moduli theory. In short, it is an enhanced version of algebraic geometry, where you have an additional structure to take along for the ride. This extra structure often (not always) gives a way to keep track of "boundaries" (and in some sense talk about "manifolds with boundary") in algebraic geometry.

It can also be seen as a generalization of the theory of (normal) toric varieties. Toric varieties are in some sense completely combinatorial, in the sense that the combinatorics determines the geometry. A general log scheme is a more flexible blend of combinatorics and geometry.

Every normal affine toric variety is of the form $X = \operatorname{Spec} \mathbb{C}[P]$, with P an integral, saturated, finitely generated monoid, and the combinatorics (the monoid P) completely controls the geometry. How do you recover P from the toric variety X? You need some additional datum. Let's say you are given the torus $T \subseteq X$ and the action on T on X, that extends the action by multiplication on itself.

Then you can take the group of characters of the torus $M = \operatorname{Hom}(T, \mathbb{C}^\times)$ (which is naturally $\mathbb{Z}^k$, once you've chosen an isomorphism $T \cong (\mathbb{C}^\times)^k$), and look at the subset $P = \{m \in M \mid t^m \text{ extends as a regular function to the whole } X\}$. Then $X = \operatorname{Spec} \mathbb{C}[P]$. This shows you how to think of the elements of P, as characters of the torus that extend as regular functions across the boundary.

For example let's look at $\mathbb{G}_m^2 \subseteq \mathbb{A}^2$. Then a character is a pair (m, n) of integers, and the regular function $x^n \cdot y^m$ extends to the boundary exactly if $n, m \geq 0$, so we get back $\operatorname{Spec} \mathbb{C}[\mathbb{N}^2]$.

Now let's generalize this picture. Say that you have a dense open embedding $U \subseteq X$ of smooth varieties (this will work best if the complement $D = X \setminus U$ is a normal crossings, or more generally "toric", divisor). One example in which this happens is if you have to do stuff on a non-proper U, but you really wish it were proper. So you compactify it to a smooth proper X, and you can choose this so that the complement D is a normal crossings (even SNC) divisor, and then you work on X but you want to remember that you care about U. Log geometry gives you a systematic way to do that.

Let's consider again invertible regular functions on U that extend to the boundary. This time, since we're not aiming to get something completely combinatorial, we'll get a sheaf. Precisely, take

$$M_U = \{f \in \mathcal{O}_X \mid f|_U \in \mathcal{O}_U^\times, \text{ i.e. } f \text{ is invertible outside of } D\}$$

where you should think of local sections f doing that. This gives a subsheaf $M_U \subseteq \mathcal{O}_X$ of monoids, and $\mathcal{O}_X^\times \subseteq M_U$. This sheaf remembers the "boundary" D. Its stalks, if you disregard the units, encode the combinatorics of how the components of D meet.

Note that you can't pullback the open embedding $U \subseteq X$ via an arbitrary map $Y \to X$, but by using (a generalization of) the sheaf of monoids above, you get something more functorial, that can be pulled back along arbitrary morphisms.

Definition 2.1 (K. Kato)**.** A log scheme is a triple (X, M, α) with X a scheme, M a sheaf of monoids (for the étale topology of X) and $\alpha\colon M \to \mathcal{O}_X$ a morphism of sheaves of monoids (where $\mathcal{O}_X$ is a monoid by multiplication), and α identifies the units, i.e. $\alpha|_{\alpha^{-1}\mathcal{O}_X^\times} \colon \alpha^{-1}\mathcal{O}_X^\times \to \mathcal{O}_X^\times$ is an isomorphism.

So in particular $\mathcal{O}_X^\times \subseteq M$. The quotient $\overline{M} = M/\mathcal{O}_X^\times$ contains most of the action.

Example 2.2. If $U \subseteq X$ is a dense open embedding, we get a log scheme (X, M_U, i). In particular if $\operatorname{Spec} \mathbb{C}[P]$ is an affine toric variety, the above construction applied to the embedding $T \subseteq \operatorname{Spec} \mathbb{C}[P]$ gives a canonical log structure on $\operatorname{Spec} \mathbb{C}[P]$.

Example 2.3. If X is any scheme, $(X, \mathcal{O}_X^\times, i)$ is a log scheme (with the *trivial*) log structure. This embeds schemes into log schemes.

This suggest that the "non-trivial" part of the log structure is encoded in $\overline{M}$, as remarked above.

Example 2.4. Let's look at $\mathbb{A}^2$ again: the stalks of the sheaf $\overline{M}$, that give a crude image of the log structure, are 0 outside of the coordinate axes, $\mathbb{N}$ on each one of the axes but outside the origin, and $\mathbb{N}^2$ on the origin. You can see that the log structure "encodes" the combinatorics of the intersections of the components of the boundary divisor.

I want to give an alternative definition, that is more useful in some situations. In the map $\alpha\colon M \to \mathcal{O}_X$, the units don't give anything non-trivial, so we might as well mod out by them. By doing that (in the stacky sense) we get $L = \overline{\alpha}\colon \overline{M} \to [\mathcal{O}_X/\mathcal{O}_X^\times]$. I am assuming that the action of $\mathcal{O}_X^\times$ on M is without stabilizers, something that is usually true.

The quotient $[\mathcal{O}_X/\mathcal{O}_X^\times]$ can be seen as $[\mathbb{A}^1/\mathbb{G}_m]_X$, and denoted also by Div_X, the stack of "generalized" Cartier divisors on X, consisting of pairs (L, s), where L is a line bundle and s is a global section, over some étale open $U \to X$.

The map L is still a map of monoids, i.e. there are given isomorphisms $(L_{a+b}, s_{a+b}) \cong (L_a, s_a)\otimes(L_b, s_b)$ that satisfy various compatibility conditions. The name for L is "symmetric monoidal functor", and it has "trivial kernel", in the sense that if a local section a maps to something isomorphic to $(\mathcal{O}_X, 1)$, then $a = 0$.

This alternative definition is completely equivalent to Kato's.

3. Kato-Nakayama space and infinite root stack

The geometry of the "monoid" part in a log scheme (X, M, α) is quite mysterious. There have been some attempts to capture it via different incarnations. I'm going to talk about two of them, and about the relationship between them.

The Kato-Nakayama space is a topological space $X_{\log}$ attached to a log analytic space X (that for example can arise as Y_{an} for some log scheme Y of finite type over $\mathbb{C}$). There is a surjective map $\tau\colon X_{\log} \to X$, and the fiber $\tau^{-1}(x) \cong \mathrm{Hom}(\overline{M}_x^{\mathrm{gp}}, S^1) \cong (S^1)^k$, where k is the rank of the free abelian group $\overline{M}_x^{\mathrm{gp}}$. This complicated fiber is somehow a "topological avatar" of the log structure.

Example 3.1. Take $\mathbb{A}^1 = \mathbb{C}$ with the divisorial log structure coming from the origin. Then $X_{\log} = \mathbb{R}_{\geq 0} \times S^1$, and the projection is $\mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C}$ given by $(r, a) \mapsto r \cdot a$. This is bijective (and a homeomorphism) outside of the origin, and the fiber over the origin is $\{0\} \times S^1$.

In good cases (like this one) $X_{\log}$ is a manifold with boundary. This explains how, in those cases, log structures can be though of as giving a notion of manifold with boundary in algebraic geometry.

For each $n \in \mathbb{N}$, any log scheme has an n-th root stack $\sqrt[n]{X} \to X$ that parametrizes n-th roots of the log structure $L\colon \overline{M} \to \mathrm{Div}_X$, i.e. extensions of L to a symmetric monoidal functor $\frac{1}{n}\overline{M} \to \mathrm{Div}_X$.

If the log structure of X is determined by a single smooth divisor $D \subseteq X$, i.e. by the map $\mathbb{N} \to \mathrm{Div}(X)$ that sends 1 to $(\mathcal{O}_X(D), 1_D)$, then the root stack $\sqrt[n]{X}$ universally parametrizes pairs (L, s) with an isomorphism $(L, s)^{\otimes n} \cong (\mathcal{O}_X(D), 1_D)$, which are "$n$-th roots" of D (in fact, the universal $(\mathcal{L}, s) \cong (\mathcal{O}(\mathcal{D}), 1_{\mathcal{D}})$ for a divisor $\mathcal{D}$ on $\sqrt[n]{X}$). In this case $\sqrt[n]{X} \to X$ is an isomorphism outside of D, and basically a μ_n-gerbe over D.

Example 3.2. If $X = \operatorname{Spec} A$ is affine and $f \in A$ is a local equation for D, the root stack is the quotient $[\operatorname{Spec}(A[x]/(x^n - f))/\mu_n]$.

By increasing n we get an inverse system (for $n \mid m$ there is a projection $\sqrt[m]{X} \to \sqrt[n]{X}$). Take this root process to the extreme: the infinite root stack is the inverse limit $\sqrt[\infty]{X} = \varprojlim_n \sqrt[n]{X}$. It parametrizes compatible systems of roots of every possible order.

The reduced fiber of $\pi\colon \sqrt[\infty]{X} \to X$ over x is $\pi^{-1}(x)_{\mathrm{red}} \cong B\widehat{\mathbb{Z}^k} = (B\widehat{\mathbb{Z}})^k$.

4. The comparison

There is a clear similarity between the Kato-Nakayama space and the infinite root stack, in the fact that $S^1 = B\mathbb{Z}$, and $\widehat{S^1} \cong B\widehat{\mathbb{Z}}$. This led us to prove:

Theorem 4.1 (CSS-). *There is a canonical morphism of topological stacks $X_{\log} \to \sqrt[\infty]{X}_{\mathrm{top}}$, that induces an equivalence on profinite completions.*

I'm just going to show you what happens in two examples.

Example 4.2. If X is the standard log point, then $X_{\log} \cong S^1$ and $\sqrt[\infty]{X}_{\mathrm{top}} = B\widehat{\mathbb{Z}}$. The map $S^1 \to B\widehat{\mathbb{Z}}$ is given by the sequence of $B\mathbb{Z}/n$-torsors $\{S^1 \to S^1, z \mapsto z^n\}_{n\in\mathbb{N}}$.

Example 4.3. Assume now $X = \mathbb{A}^1 = \mathbb{C}$ with the log structure coming from the origin. Then $X_{\log} \cong \mathbb{R}_{\geq 0} \times S^1$, and $\sqrt[n]{X}_{\mathrm{top}} = [\mathbb{C}/(\mathbb{Z}/n)]$, where the projection $[\mathbb{C}/(\mathbb{Z}/n)] \to \mathbb{C}$ is induced by $z \mapsto z^n$. For any n we have a map $\mathbb{R}_{\geq 0} \times S^1 \to [\mathbb{C}/(\mathbb{Z}/n)]$ that corresponds to a $\mathbb{Z}/n$-(topological)torsor $\phi\colon P \to \mathbb{R}_{\geq 0} \times S^1$, with an equivariant map $P \to \mathbb{C}$. There's a very natural one: $P = \mathbb{R}_{\geq 0} \times S^1$ itself, and $\phi\colon \mathbb{R}_{\geq 0} \times S^1 \to \mathbb{R}_{\geq 0} \times S^1$ is given in coordinates by $(r, a) \mapsto (r^n, a^n)$ (this is induced by $z \mapsto z^n$ from $\mathbb{A}^1$ to $\mathbb{A}^1$). The equivariant map $\mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C}$ is the usual $(r, a) \mapsto r \cdot a$. This system of torsors gives the map $X_{\log} \to \sqrt[\infty]{X}_{\mathrm{top}}$ in this case.

Note that ϕ is a torsor exactly because of the presence of the S^1s in the Kato-Nakayama construction (on the algebraic side the map $z \mapsto z^n$ from $\mathbb{A}^1$ to $\mathbb{A}^1$ is ramified!).