AN INTRODUCTION TO HOPF ALGEBRAS

BIBLIOGRAPHY:

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0. MOTIVATION

L/K Galois extension of number fields or p-adic fields, $G = \mathrm{GAL}(L/K)$. Denote by $\mathcal{O}_L$, $\mathcal{O}_K$ the ring of integers - valuation rings.

If L/K is <u>not</u> tame, then $\mathcal{O}_L \not\cong \mathcal{O}_K[G]$, no NIB.

LEOPOLDT $\rightsquigarrow$ $\mathfrak{U}_{L/K} = \{\lambda \in K[G] \mid \lambda \mathcal{O}_L \subseteq \mathcal{O}_L\}$

If $L/\mathbb{Q}_p$ is abelian, then $\mathcal{O}_L \cong \mathfrak{U}_{L/K}$ as $\mathfrak{U}_{L/K}$-module.

But if $K \neq \mathbb{Q}_p$, or $L/\mathbb{Q}_p$ not abelian, it may be not the case.

Since $\mathfrak{U}_{L/K}$ is the only $\mathcal{O}_K$-order in $K[G]$ over $\mathcal{O}_L$ can be free, what can we do if it is not?

IDEA: Look in a different K-algebra!

1. Introduce K-HOPF ALGEBRAS, and HOPF GALOIS EXTENSIONS
2. Generalise the associated order
3. Study $\mathcal{O}_L$ as module over it.

Byott: example with freeness in a "nonclassical" setting, but not over $\mathfrak{A}_{L/K}$.

1. HOPF ALGEBRAS

We fix R, a commutative ring (with unity) for all this lecture.

NOTATION:

- $\otimes = \otimes_R$
- If A, B are R-modules, then $A \otimes B$ is an R-module, and we denote by $\tau : A\otimes B \longrightarrow B \otimes A$ the SWITCH MAP.
$$a \otimes b \longmapsto b \otimes a$$

In the first part we worked widely with R-algebras:

- a ring A, with an R-module structure s.t. $\forall r \in R, a, a' \in A$,
$$r(aa') = (ra)a' = a(ra')$$

- a ring A, with a ring homomorphism
$$f : R \longrightarrow A$$
s.t. $f(R)$ is contained in the center of A.

DEF: an R-algebra is (A, μ, ι)

where:

- A is an R-module
- $\mu : A \otimes A \to A$ R-linear

  MULTIPLICATION
- $\iota : R \to A$ R-linear

  UNITY

s.t.

ASSOCIATIVITY:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

UNITARY:

$$\begin{tikzcd}
R \otimes A \arrow[r, "\iota \otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes R \arrow[l, "id \otimes \iota"'] \arrow[dl, "\cong"] \\
& A \arrow[r, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

DEF: R-COALGEBRA (C, Δ, ε)

where:

- C is an R-module
- $\Delta : C \to C \otimes C$ R-linear

  COMULTIPLICATION
- $\varepsilon : C \to R$ R-linear

  COUNITY

s.t.

COASSOCIATIVITY

$$\begin{tikzcd}
C \otimes C \otimes C \arrow[dr, phantom, "\circlearrowleft"] & C \otimes C \arrow[l, "\Delta \otimes id"'] \\
C \otimes C \arrow[u, "id \otimes \Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

COUNITARY

$$\begin{tikzcd}
C \otimes R & C \otimes C \arrow[l, "id \otimes \varepsilon"'] & R \otimes C \arrow[l, "\varepsilon \otimes id"'] \\
& C \arrow[ul, "\cong^{-1}"] \arrow[u, "\Delta"'] \arrow[ur, "\cong^{-1}"'] \arrow[r, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$c \otimes 1_R \mapsfrom c$

NOTATION:

- $aa' = \mu(a\otimes a')$
- $\iota(1_R) = 1_A$

$$\begin{tikzcd} 1_R\otimes a \arrow[r, mapsto] \arrow[dr, mapsto] & \iota(1_R)\otimes a \arrow[d, mapsto] \\ & a = \iota(1_R)a \end{tikzcd}$$

EX: · R

· if A,B are R-algebras, then $A\otimes B$ R-algebra:

$$\begin{tikzcd} A\otimes B\otimes A\otimes B \arrow[r, "id\otimes\tau\otimes id"] \arrow[dr] & A\otimes A\otimes B\otimes B \arrow[d, "\mu_A\otimes\mu_B"] \\ & A\otimes B \end{tikzcd}$$

$(a\otimes b)(a'\otimes b') = aa'\otimes bb'$

$$R \xrightarrow{\cong} R\otimes R \xrightarrow{\iota_A\otimes\iota_B} A\otimes B$$

$$1_R \longmapsto 1_A\otimes 1_B$$

NOTATION:

- SWEEDLER NOTATION

$c\in C$, $\underbrace{\Delta(c)}_{\in\, C\otimes C} = \sum_{(c)} c_{(1)}\otimes c_{(2)}$

COUNITARY PROP:

$$c = \sum_{(c)} c_{(1)}\varepsilon(c_{(2)}) = \sum_{(c)} \varepsilon(c_{(1)})\, c_{(2)}$$

EX: · R

· if C,D are R-coalgebras, then $C\otimes D$ //

$$\begin{tikzcd} C\otimes D \arrow[r, "\Delta_C\otimes\Delta_D"] \arrow[dr] & C\otimes C\otimes D\otimes D \arrow[d, "id\otimes\tau\otimes id"] \\ & C\otimes D\otimes C\otimes D \end{tikzcd}$$

$$C\otimes D \xrightarrow{\varepsilon_A\otimes\varepsilon_B} R\otimes R \xrightarrow{\cong} R$$

A, B R-algebras, $f : A \to B$ R-linear.

DEF: f is an

- R-algebra homom. if it respects μ, ι

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \\
A \otimes A \arrow[u, "\mu_A"] \arrow[r, "f \otimes f"'] \arrow[ur, phantom, "\circlearrowleft"] & B \otimes B \arrow[u, "\mu_B"']
\end{tikzcd}
$$

$f(aa') = f(a) f(a')$

$$
\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
& R \arrow[ul, "\iota_A"] \arrow[ur, "\iota_B"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}
$$

- R-algebra anti. homom. if it respects ι,

$$
\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
A \otimes A \arrow[u, "\mu_A"] \arrow[r, "\tau"'] & A \otimes A \arrow[r, "f \otimes f"'] & B \otimes B \arrow[u, "\mu_B"']
\end{tikzcd}
$$

$f(aa') = f(a') f(a)$

C, D R-coalgebras, $f : C \to D$ R-linear.

DEF. f is an

- R-coalg. homomorphism if it respects Δ, ε

$$
\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta_C"'] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[d, "\Delta_D"] \\
C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\
& R &
\end{tikzcd}
$$

- R-coalg. anti-homomorphism, if it respects ε, and

$$
\begin{tikzcd}
C \arrow[rr, "f"] \arrow[d, "\Delta_C"'] & \circlearrowleft & D \arrow[dr, "\Delta_D"] & \\
C \otimes C \arrow[r, "\tau"'] & C \otimes C \arrow[rr, "f \otimes f"'] & & D \otimes D
\end{tikzcd}
$$

DEF: an R-BIALGEBRA is $(H, \mu, \iota, \Delta, \varepsilon)$ where:

1) (H, μ, ι) R-algebra.
2) (H, Δ, ε) R-coalgebra.
3) Δ, ε are R-algebra homomorphisms.

$$\begin{tikzcd}
H \arrow[r, "\Delta"] & H \otimes H \\
H \otimes H \arrow[u, "\mu"] \arrow[r, "\Delta \otimes \Delta"'] \arrow[ur, phantom, "\circlearrowright"] & H \otimes H \otimes H \otimes H \arrow[u, "\mu_{H \otimes H}"']
\end{tikzcd}$$

(RK: Δ, ε R-alg. homom $\Leftrightarrow$ μ, ι are R-coalg. homom).

DEF: An R- Hopf algebra is $(H, \mu, \iota, \Delta, \varepsilon, \lambda)$ where

1) $(H, \mu, \iota, \Delta, \varepsilon)$ bialgebra.

2) $\lambda : H \to H$ R-linear, ANTIPODE, s.t.

$$\begin{tikzcd}
& H\otimes H \arrow[rr, "\lambda\otimes id"] & & H\otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & R \arrow[rr, "\iota"] & & H \\
& H\otimes H \arrow[rr, "id\otimes\lambda"'] & & H\otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

RK: λ is the inverse of id_H WRT $*$ CONVOL PRODUCT

RK: In [CH100], it is required that λ is an anti homom. of R-algebras and R-coalgebras
In [UND15], it is shown that this fact is "free" from our definition

DEF: H, H' R-Hopf algebras, $f: H \to H'$ R-linear.
f is an R-Hopf algebra homom. if it respects
$\mu, \iota, \Delta, \varepsilon, \lambda$

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "\lambda_H"'] \arrow[dr, phantom, "\circlearrowleft"] & H' \arrow[d, "\lambda_{H'}"] \\ H \arrow[r, "f"'] & H' \end{tikzcd}$$

DEF: An R-Hopf algebra H is

1) COMMUTATIVE if $\underset{\overset{\shortparallel}{aa'}}{\mu(a\otimes a')} = \underset{\overset{\shortparallel}{a'a}}{\mu(a'\otimes a)} \Leftrightarrow \boxed{\mu\circ T = \mu}$

2) COCOMMUTATIVE if $T\circ\Delta = \Delta$, $\sum_{(h)} h_{(1)}\otimes h_{(2)} = \sum_{(h)} h_{(2)}\otimes h_{(1)}$

3) ABELIAN if it both.

Ex: Let G be a finite group. $R[G]$ free R-module with basis $\{\sigma : \sigma \in G\}$

① $R[G]$ R-algebra: $\sigma, \tau \in G$

$$\mu(\sigma \otimes \tau) = \sigma\tau \quad (\text{in } G)$$

$$\iota(1_R) = 1_G$$

② $R[G]$ R-coalgebra

$$\Delta(\sigma) = \sigma \otimes \sigma$$

$$\varepsilon(\sigma) = 1_R \quad \left(\varepsilon\left(\sum_{\sigma \in G} r_\sigma \sigma \right) = \sum_{\sigma \in G} r_\sigma \right)$$ AUGMENTATION MAP

③ $R[G]$ R-Hopf algebra

$$\lambda(\sigma) = \sigma^{-1}$$

$$(\iota \circ \varepsilon) = \mu \circ (\lambda \otimes \mathrm{id}) \circ \Delta$$

- $\iota(\varepsilon(\sigma)) = \iota(1_R) = 1_{R[G]}$

- $\Delta(\sigma) = \sigma \otimes \sigma \qquad \mu(\lambda(\sigma) \otimes \sigma) = \mu(\sigma^{-1} \otimes \sigma) = 1_{R[G]}$

Rk: 1) $R[G]$ is always cocommutative $\tau \circ \Delta = \Delta$

$\Delta(\sigma) = \sigma \otimes \sigma$

2) $R[G]$ commutative $\Leftrightarrow$ G abelian

3) $\lambda^2 = id$ $\left(\lambda(\sigma) = \sigma^{-1} \right)$

(H commutative or cocommutative $\Rightarrow$ $\lambda^2 = id$)

2. DUAL OF HOPF ALGEBRAS

Let H be an R-module, and $H^* = \mathrm{Hom}_R(H, R)$, also an R-module.

DEF: a FINITE R-module is a finitely generated projective R-module.

LEMMA: Let H be a finite R-module.

1) H is a direct summand of a free R-module of finite rank.
2) H^* is finite
3) H is REFLEXIVE: $H \longrightarrow H^{**}$ isomorphism of R-modules
$h \longmapsto (f \mapsto f(h))$

We write $\langle \, , \rangle : H^* \otimes H \longrightarrow R$
$f \otimes h \longmapsto \langle f, h \rangle = f(h)$

4) H admits a PROJECTIVE COORDINATE SYSTEM (PCS):
$\{h_i, f_i\}_{i=1}^n$, with $h_i \in H$, $f_i \in H^*$, s.t. $\forall h \in H$,
$$h = \sum_{i=1}^n \langle f_i, h \rangle h_i$$

RK: H finite R-module with PCS $\{h_i, f_i\}_{i=1}^n \Rightarrow$
H^* " " " " $\{f_i, h_i\}_{i=1}^n$

FACT: Let H be a finite R-Hopf algebra. Then H^* is again a finite R-Hopf algebra

Let $r \in R$, $h, h' \in H$, $f, f' \in H^*$.

- $(ff')(h) = (f \otimes f')(\Delta(h)) = \sum_{(h)} f(h_{(1)}) f'(h_{(2)})$
- $u_{H^*}(r) = r\,\varepsilon_H \qquad (\varepsilon_H : H \to R$ R-linear $\Rightarrow \varepsilon_H \in H^*)$
- $\Delta_{H^*}(f)(h \otimes h') = f(hh')$
- $\varepsilon_{H^*}(f) = f(1_H)$
- $\lambda_{H^*}(f)(h) = f(\lambda_H(h))$

FACTS: 1) H commutative $\Leftrightarrow$ H^* cocommutative

2) H cocommutative $\Leftrightarrow$ H^* commutative

3) $H \to H^{**}$ isom. of R-Hopf algebras.

EX: $H = \mathbb{R}[G]$, G finite, $H^* = \mathbb{R}[G]^*$

basis: $\{ e_\sigma : \sigma \in G \}$ $\quad e_\sigma(\tau) = \delta_{\sigma,\tau}$.

$r \in \mathbb{R}$, $\quad \sigma, \tau, \rho \in G$,

- $(e_\sigma e_\tau)(\rho) = (e_\sigma \otimes e_\tau)(\rho \otimes \rho) = e_\sigma(\rho)\, e_\tau(\rho)$

$\Rightarrow$

$e_\sigma e_\tau = \delta_{\sigma\tau}\, e_\sigma$ ($\Rightarrow$ PAIRWISE ORTHOGONAL IDEMPOTENT)

- $u_{H^*}(r) = r \sum_{\sigma \in G} e_\sigma$

- $\Delta(e_\sigma)(\rho \otimes \tau) = e_\sigma(\rho\tau) = \delta_{\sigma,\rho\tau} \Rightarrow$

$\Delta(e_\sigma) = \sum_{\sigma = \rho\tau} e_\rho \otimes e_\tau$

- $\lambda_{H^*}(e_\sigma) = e_{\sigma^{-1}}$ $\quad (\lambda_H(\sigma) = \sigma^{-1})$

RK: $\mathbb{R}[G]^*$ always commutative

$\mathbb{R}[G]^*$ cocommutative $\Leftrightarrow$ G abelian.

3. GROUPLIKE ELEMENTS

Recall that for $H = R[G]$, $\sigma \in G$, $\Delta(\sigma) = \sigma \otimes \sigma$

DEF: Let H be an R-Hopf algebra, $h \in H$; h is GROUPLIKE if
- $h \neq 0$
- $\Delta(h) = h \otimes h$

$G(H) = \{ h \in H : h \text{ grouplike} \}$

PROP: if $R = K$ is a field, H is a K-Hopf algebra, then distinct grouplike elements are linearly independent.

PROP: Let H be an R-Hopf algebra, and suppose that R has no nontrivial idempotents $(\neq 0, 1)$

① $\forall h \in G(H), \quad \varepsilon(h) = 1$

② $G(H) \leq H^\times$

PROOF: ① $h \in H$

COUNITARY:

$$\begin{tikzcd} R \otimes H & H \otimes H \arrow[l, "\varepsilon \otimes id"'] \\ & H \arrow[u, "\Delta"'] \arrow[ul, "s^{-1}"] \end{tikzcd}$$

$h \in G(H) \qquad \Delta(h) = h \otimes h$

$$1 \otimes h = \varepsilon(h) \otimes h \implies h = \varepsilon(h) h$$

$$\implies \varepsilon(h) = \varepsilon(\varepsilon(h) h) = \varepsilon(h)\varepsilon(h)$$

$\implies \varepsilon(h)$ idempotent

- $\varepsilon(h) = 0 \implies h = 0$ ↯
- $\varepsilon(h) = 1$

(2a) $h \in G(H)$ then h is invertible $\lambda(h)$

$$(\iota \circ \varepsilon) = \mu \circ (\mathrm{id} \otimes \lambda) \circ \Delta \qquad h \in G(H) : \begin{cases} \Delta(h) = h \otimes h \\ \varepsilon(h) = 1 \end{cases}$$

$$1_H \qquad \mu(h \otimes \lambda(h)) = h\lambda(h)$$

$$1_H = \lambda(h)h$$

(2b) $\lambda(h) \in G(H)$ if $h \in G(H)$

- $\lambda(h) \neq 0$ ✓
- $\Delta(\lambda(h)) = \lambda(h) \otimes \lambda(h)$

$h \in G(H)$, $\Delta(h) = h \otimes h$

$$\begin{tikzcd}
H \arrow[rr, "\lambda"] \arrow[d, "\Delta"'] & & H \arrow[d, "\Delta"] \\
H \otimes H \arrow[r, "\tau"'] & H \otimes H \arrow[r, "\lambda \otimes \lambda"'] & H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[rr, mapsto] \arrow[d, mapsto] & & \lambda(h) \arrow[d, mapsto] \\
h \otimes h \arrow[r, mapsto] & h \otimes h \arrow[r, mapsto] & \Delta(\lambda(h)) = \lambda(h) \otimes \lambda(h)
\end{tikzcd}$$

$\lambda(h) \in G(H)$

(2c) $h, h' \in G(H), \overset{?}{\Rightarrow} hh' \in G(H)$

Δ R-algebra hom.

$$\underset{H \otimes H}{\overset{\cup}{\Delta(hh')}} = \Delta(h)\Delta(h') = (h \otimes h)(h' \otimes h')$$
$$= hh' \otimes hh' \qquad \square$$

RK: H finite Hopf algebra $H^* \ni f \neq 0$

we an: $\Delta(f) = f \otimes f \in H^* \otimes H^*$

$$\Delta(f)(h \otimes h') = (f \otimes f)(h \otimes h')$$
$$\overset{\shortparallel}{f(hh')} \qquad \overset{\shortparallel}{f(h)f(h')} \qquad \in R$$

if f R-alg. hom $\Rightarrow$ f grouplike

if f grouplike $\Rightarrow$ $f(hh') = f(h)f(h')$

$$f(1_H) = f(1_H)\, f(1_H)$$

if $R = K$ $\Rightarrow$ $f(1_H) = 1$ $\Rightarrow$ grouplike $\Leftrightarrow$ algebra hom.

4. MODULES AND COMODULES

Let H be a fixed R-Hopf algebra.

DEF: a LEFT H-module is (S, α), where
- S is an R-module
- $\alpha : H \otimes S \to S$ R-linear

s.t.

ASSOCIATIVITY:

$$\begin{tikzcd}
H \otimes H \otimes S \arrow[r, "m \otimes id"] \arrow[d, "id \otimes \alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & H \otimes S \arrow[d, "\alpha"] \\
H \otimes H \arrow[r, "\alpha"'] & S
\end{tikzcd}$$

UNITARY:

$$\begin{tikzcd}
R \otimes S \arrow[r, "\iota \otimes id"] \arrow[dr, "\xi"'] & H \otimes S \arrow[d, "\alpha"] \arrow[dl, phantom, "\circlearrowleft"] \\
& S
\end{tikzcd}$$

DEF: a RIGHT H-comodule is (S, β), where
- S R-module
- $\beta : S \to S \otimes H$ R-linear

s.t.

COASSOCIATIVITY:

$$\begin{tikzcd}
S \otimes H \otimes H \arrow[dr, phantom, "\circlearrowleft"] & S \otimes H \arrow[l, "\beta \otimes id"'] \\
S \otimes H \arrow[u, "id \otimes \Delta"] & S \arrow[l, "\beta"] \arrow[u, "\beta"']
\end{tikzcd}$$

COUNITARY:

$$\begin{tikzcd}
S \otimes R \arrow[dr, phantom, "\circlearrowleft"] & & S \otimes H \arrow[ll, "id \otimes \varepsilon"'] \\
& S \arrow[ul, "\xi^{-1}"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

NOTATION:

- $h \cdot s = \alpha(h \otimes s)$

$$(hh') \cdot s = h \cdot (h' \cdot s)$$

$$1_H \cdot s = s$$

DEF: S, S' left H-modules, $f : S \to S'$ R-linear. f is an H-module homom. if f respects α.

$$\begin{tikzcd}
S \arrow[r, "f"] & S' \\
H \otimes S \arrow[u, "\alpha_S"] \arrow[r, "id \otimes f"'] \arrow[ur, phantom, "\circlearrowleft"] & H \otimes S' \arrow[u, "\alpha_{S'}"']
\end{tikzcd}$$

$$f(h \cdot s) = h \cdot f(s)$$

NOTATION:

SWEEDLER

$s \in S$

$$\underset{\substack{\cap \\ S \otimes H}}{\beta(s)} = \sum_{(s)} \underset{\substack{\cup \\ S}}{s_{(0)}} \otimes \underset{\substack{\cup \\ H}}{s_{(1)}}$$

DEF: S, S' right H-comodules, $f : S \to S'$ R-linear. f is an H-comodule hom. if f respects β.

Suppose now that H is finite, with dual H^* and PCS $\{h_i, f_i\}_{i=1}^{n}$

· Let S be a right H-comodule: $\beta(s) = \sum_{(s)} s_{(0)} \otimes s_{(1)} \in S \otimes H$

FACT: S is also a left H^*-module

$$f \in H^*,\ s \in S, \quad f \cdot s = \sum_{(s)} \underbrace{s_{(0)}}_{\in S} \underbrace{\langle f, s_{(1)} \rangle}_{\in R} \in S$$

(S is a right H^*-comodule $\Rightarrow$ S is a left H-module)

· Let S be a left H-module: $h \cdot s \in S$, $\forall h \in H, s \in S$

FACT: S is a right H^*-comodule:

$$\underset{\substack{\cap \\ S \otimes H^*}}{\beta(s)} = \sum_{i=1}^{n} (h_i \cdot s) \otimes f_i$$

PROP: Let H be a finite R-Hopf algebra, and S be an R-module. Then

S is a left H-module $\iff$ S is a right H^*-comodule.

Moreover, the processes of going from H-module action to H^*-comodule action and viceversa are inverse.

PROOF: 1 S left H-module. $\quad h \cdot s \in S$

$\Downarrow$

S is a right H^*-comodule

$$\beta(s) := \sum_{i=1}^{n} h_i \cdot s \otimes f_i$$

$\Downarrow$

S is a left H-module

$$\alpha(h \otimes s) = \sum_{i=1}^{n} (h_i \cdot s) \langle h, f_i \rangle = \underbrace{\left(\sum_{i=1}^{n} h_i \langle h, f_i \rangle \right)}_{h} \cdot s$$

② S is a right H^*-comodule $\left(s \mapsto \sum_{(s)} \underset{\in S}{s_{(0)}} \otimes \underset{\in H^*}{s_{(1)}} \right.$

$\Updownarrow$

S is a left H-module

$$h \cdot s = \sum_{(s)} s_{(0)} \langle \underset{\in H}{h}, \underset{\in H^*}{s_{(1)}} \rangle$$

$\Updownarrow$

S is a right H^*-comodule.

$$\beta(s) = \sum_{i=1}^{n} h_i \cdot s \otimes f_i = \sum_{i=1}^{n} \left(\sum_{(s)} s_{(0)} \langle h_i, s_{(1)} \rangle \right) \otimes f_i$$

$$= \sum_{(s)} s_{(0)} \otimes \left(\sum_{i=1}^{n} \langle h_i, s_{(1)} \rangle f_i \right)$$

$$= \sum_{(s)} s_{(0)} \otimes s_{(1)}.$$