

AN INTRODUCTION TO HOPF ALGEBRAS

BIBLIOGRAPHY:

[CHI00]: 2000, CHILDS, "TAMING WILD EXTENSIONS:
HOPF ALGEBRAS AND LOCAL GALOIS MODULE THEORY".

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HOPF ALGEBRAS".

0. MOTIVATION

L/K Galois extension of number fields or p -adic fields,
 $G = \text{GAL}(L/K)$. Denote by $\mathcal{O}_L, \mathcal{O}_K$ the ring of integers -
valuation rings.

If L/K is not tame, then $\mathcal{O}_L \not\cong \mathcal{O}_K[G]$, NO NIB.

$$L \text{ separable} \leadsto \mathcal{U}_{L/K} = \{ \lambda \in K[G] \mid \lambda \mathcal{O}_L \subseteq \mathcal{O}_L \}$$

If L/\mathbb{Q}_p is abelian, then $\mathcal{O}_L \cong \mathcal{U}_{L/K}$ as $\mathcal{U}_{L/K}$ -module.

But if $K \neq \mathbb{Q}_p$, or L/\mathbb{Q}_p not abelian, it may be not the
case.

Since $\mathcal{U}_{L/K}$ is the only \mathcal{O}_K -order in $K[G]$ over
 \mathcal{O}_L can be free, what can we do if it is not?

IDEA: Look in a different K -algebra!

- ① Introduce K -HOPF ALGEBRAS, and HOPF GALOIS EXTENSIONS
- ② generalise the associated order
- ③ Study \mathcal{O}_L as module over it.

BYOTT: example with freeness in a "nonclassical" setting, but not over \mathcal{K}_L/K .

1. HOPF ALGEBRAS

We fix R , a commutative ring (with unity) for all this lecture.

NOTATION:

- $\otimes = \otimes_R$
- If A, B are R -modules, then $A \otimes B$ is an R -module, and we denote by $\tau: A \otimes B \rightarrow B \otimes A$ the SWITCH MAP.
$$a \otimes b \mapsto b \otimes a$$

In the first part we worked widely with R -algebras:

- a ring A , with an R -module structure s.t. $\forall r \in R, a, a' \in A$,
$$r(aa') = (ra)a' = a(ra')$$
- a ring A , with a ring homomorphism
$$f: R \rightarrow A$$

s.t. $f(R)$ is contained in the center of A .

DEF: An R -algebra is (A, μ, ϵ)

where:

- A is an R -module
- $\mu: A \otimes A \rightarrow A$ R -linear

MULTIPLICATION

- $\epsilon: R \rightarrow A$ R -linear

UNITY

s.t.

ASSOCIATIVITY:

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\
 \text{id} \otimes \mu \downarrow & \hookrightarrow & \downarrow \mu \\
 A \otimes A & \xrightarrow{\mu} & A
 \end{array}$$

UNITY:

$$\begin{array}{ccccc}
 & \text{id} \otimes \epsilon & & \epsilon \otimes \text{id} & \\
 R \otimes A & \xrightarrow{\quad} & A \otimes A & \xleftarrow{\quad} & A \otimes R \\
 \downarrow \epsilon & & \downarrow \mu & & \downarrow \epsilon \\
 A & & A & & A
 \end{array}$$

DEF: R -COALGEBRA (C, Δ, ϵ)

where:

- C is an R -module
- $\Delta: C \rightarrow C \otimes C$ R -linear

MULTIPLICATION

- $\epsilon: C \rightarrow R$ R -linear

UNITY

s.t.

COASSOCIATIVITY:

$$\begin{array}{ccc}
 C \otimes C \otimes C & \xleftarrow{\Delta \otimes \text{id}} & C \otimes C \\
 \uparrow \text{id} \otimes \Delta & \hookrightarrow & \uparrow \Delta \\
 C \otimes C & \xleftarrow{\Delta} & C
 \end{array}$$

COUNTARY:

$$\begin{array}{ccccc}
 & \text{id} \otimes \epsilon & & \epsilon \otimes \text{id} & \\
 C \otimes R & \xleftarrow{\quad} & C \otimes C & \xleftarrow{\quad} & R \otimes C \\
 \downarrow \epsilon \otimes \text{id} & & \uparrow \Delta & & \downarrow \epsilon \\
 C & & C & & C
 \end{array}$$

NOTATION:

• $aa' = \mu(a \otimes a')$

• $L(\Delta_R) = \Delta_A$

$$\begin{array}{ccc} 1_R \otimes a & \xrightarrow{\quad} & i(1_R) \otimes a \\ \downarrow \triangleright & & \downarrow \triangleright \\ & & a = L(\Delta_R)a \end{array}$$

EX: R

• if A, B are R -algebras,
then $A \otimes B$ R -algebra:

$$\begin{array}{ccc} A \otimes B \otimes A \otimes B & \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} & A \otimes A \otimes B \otimes B \\ & \searrow & \swarrow \mu_A \otimes \mu_B \\ & & A \otimes B \end{array}$$

$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$

$R \xrightarrow{\cong} R \otimes R \xrightarrow{\mu_A \otimes \mu_B} A \otimes B$

$1_R \xrightarrow{\quad} \Delta_A \otimes \Delta_B$

NOTATION:

• SWEEDLER NOTATION

$c \in C, \Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$

COUNTING PROP: $C \otimes C$

$$\begin{aligned} c &= \sum_{(c)} c_{(1)} \varepsilon(c_{(2)}) \\ &= \sum_{(c)} \varepsilon(c_{(2)}) c_{(1)} \end{aligned}$$

EX: R

• if C, D are R -coalgebras,
then $C \otimes D$ //

$$\begin{array}{ccc} C \otimes D & \xrightarrow{\Delta_C \otimes \Delta_D} & C \otimes C \otimes D \otimes D \\ & \searrow & \downarrow \text{id} \otimes \tau \otimes \text{id} \\ & & (C \otimes D) \otimes (C \otimes D) \end{array}$$

$C \otimes D \xrightarrow{\mu_A \otimes \mu_B} R \otimes R \xrightarrow{\cong} R$

A, B R -algebras, $f: A \rightarrow B$
 R -linear.

DEF: f is an

- R -algebra homom. if it respects μ, ι

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \mu_A \uparrow & \hookrightarrow & \uparrow \mu_B \\
 A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
 f(a \otimes a') & = & f(a) f(a')
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \iota_A \uparrow & \hookrightarrow & \uparrow \iota_B \\
 R & & R
 \end{array}$$

R -algebra anti-homom.

if it respects ι ,

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \mu_A \uparrow & \hookrightarrow & \uparrow \mu_B \\
 A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
 f(a \otimes a') & = & f(a') f(a)
 \end{array}$$

C, D R -coalgebras, $f: C \rightarrow D$
 R -linear.

DEF: f is an

- R -coalg. homomorphism if it respects Δ, ϵ

$$\begin{array}{ccc}
 C & \xrightarrow{f} & D \\
 \Delta_C \downarrow & \hookrightarrow & \downarrow \Delta_D \\
 C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \\
 \epsilon_C \downarrow & \hookrightarrow & \downarrow \epsilon_D \\
 R & & R
 \end{array}$$

R -coalg. anti-homomorphism.

if it respects ϵ , and

$$\begin{array}{ccc}
 C & \xrightarrow{f} & D \\
 \Delta_C \downarrow & \hookrightarrow & \downarrow \Delta_D \\
 C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \\
 \epsilon_C \downarrow & \hookrightarrow & \downarrow \epsilon_D \\
 R & & R
 \end{array}$$

DEF: an R -bialgebra is $(H, \mu, \nu, \Delta, \epsilon)$ where:

- 1) (H, μ, ν) R -algebra.
- 2) (H, Δ, ϵ) R -coalgebra.
- 3) Δ, ϵ are R -algebra homomorphisms.

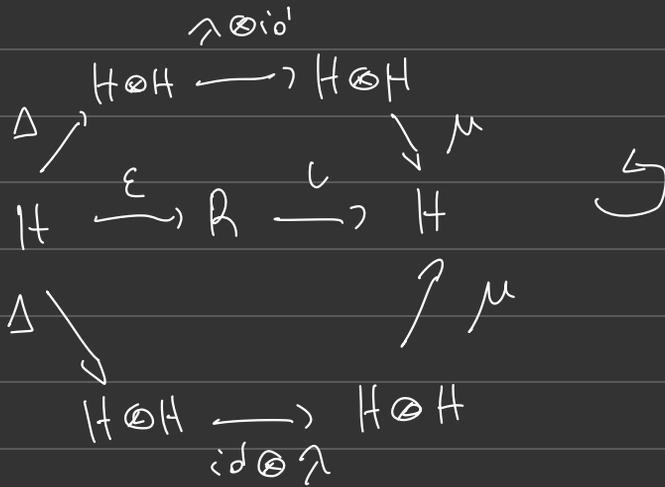
$$\begin{array}{ccc}
 H & \xrightarrow{\Delta} & H \otimes H \\
 \mu \downarrow & \hookrightarrow & \uparrow \mu_{H \otimes H} \\
 H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H
 \end{array}$$

(rk: Δ, ϵ R -alg. homom. $\Leftrightarrow \mu, \nu$ are R -coalg. homom.)

DEF: A \mathbb{K} -Hopf algebra is $(H, \mu, \nu, \Delta, \varepsilon, \lambda)$
 where

1) $(H, \mu, \nu, \Delta, \varepsilon)$ bi algebra.

2) $\lambda: H \rightarrow H$ \mathbb{K} -linear, ANTIPODE, i.T.



RL: λ is the inverse of id_H w.r.t \ast convolution product

RK: In [CH100], it is required that λ is an anti-homom. of R -algebras and R -coalgebras

In [UND15], it is shown that this fact is "free" from our definition

DEF: H, H' R -Hopf algebras, $f: H \rightarrow H'$ R -linear.

f is an R -Hopf algebra homom. iff it respects

$$\begin{array}{ccc} H & \xrightarrow{f} & H' \\ \lambda_H \downarrow & \subset & \downarrow \lambda_{H'} \\ H & \xrightarrow{f} & H' \end{array}$$

DEF: An R -Hopf algebra H is

1) COMMUTATIVE iff $\mu(a \otimes a') = \mu(a' \otimes a) \Rightarrow \mu \circ \tau = \mu$

2) COCOMMUTATIVE iff $\tau \circ \Delta = \Delta, \sum_{(h)} h_{(1)} \otimes h_{(2)} = \sum_{(h)} h_{(2)} \otimes h_{(1)}$

3) ABELIAN iff it both,

Ex: Let G be a finite group. $R[G]$ free R -module with basis $\{\sigma : \sigma \in G\}$

① $R[G]$ R -algebra: $\sigma, \tau \in G$
 $\mu(\sigma \otimes \tau) = \sigma\tau \quad (\text{in } G)$
 $\mathcal{L}(1_R) = 1_G$

② $R[G]$ R -coalgebra
 $\Delta(\sigma) = \sigma \otimes \sigma$
 $\varepsilon(\sigma) = 1_R \quad \left(\varepsilon \left(\sum_{\sigma \in G} \sigma \right) = \sum_{\sigma \in G} 1_R \quad \text{AUGMENTATION MAP} \right)$

③ $R[G]$ R -Hopf algebra
 $\lambda(\sigma) = \sigma^{-1}$

$$(\varepsilon \circ \Delta) = \mu \circ (\lambda \otimes \text{id}) \circ \Delta$$

$$\cdot \mathcal{L}(\varepsilon(\sigma)) = \mathcal{L}(1_R) = 1_{R[G]}$$

$$\cdot \Delta(\sigma) = \sigma \otimes \sigma \quad \mu(\lambda(\sigma) \otimes \sigma) = \mu(\sigma^{-1} \otimes \sigma) = 1_{R[G]}$$

PK: 1) $\mathcal{H}[G]$ is always cocommutative $\tau \circ \Delta = \Delta$

$$\Delta(\sigma) = \sigma \otimes \sigma$$

2) $\mathcal{H}[G]$ commutative $\Leftrightarrow G$ abelian

$$3) \tau^2 = \text{id} \quad (\tau(\sigma) = \sigma^{-1})$$

(\mathcal{H} commutative or cocommutative $\Rightarrow \tau^2 = \text{id}$)

2. DUAL OF HOPF ALGEBRAS

Let H be an R -module, and $H^* = \text{Hom}_R(H, R)$, also an R -module.

DEF: A FINITE R -module is a finitely generated projective R -module.

LEMMA: Let H be a finite R -module.

1) H is a direct summand of a free R -module of finite rank.

2) H^* is finite

3) H is REFLEXIVE: $H \xrightarrow{\quad} H^{**}$ isomorphism
 $h \mapsto (f \mapsto f(h))$ of R -modules

We write $\langle , \rangle : H^* \otimes H \rightarrow R$

$$f \otimes h \mapsto \langle f, h \rangle = f(h)$$

4) H admits a PROJECTIVE COORDINATE SYSTEM (PCS):

$$\{h_i, f_i\}_{i=1}^m, \text{ with } h_i \in H, f_i \in H^*, \text{ s.t. } \forall h \in H, \\ h = \sum_{i=1}^m \langle f_i, h \rangle h_i$$

RK: H finite R -module with PCS $\{h_i, f_i\}_{i=1}^m \Rightarrow$
 H^* " " " " $\{f_i, h_i\}_{i=1}^m$

FACT: Let H be a finite \mathbb{R} -Hopf algebra. Then H^* is again a finite \mathbb{R} -Hopf algebra

Let $\alpha \in \mathbb{R}$, $h, h' \in H$, $f, f' \in H^*$.

$$\cdot (ff')(h) = (f \otimes f')(\Delta(h)) = \sum_{(h)} f(h_{(1)}) f'(h_{(2)})$$

$$\cdot \underset{H^*}{L}(\alpha) = \alpha \varepsilon_H \quad (\varepsilon_H = H \rightarrow \mathbb{R} \text{ } \mathbb{R}\text{-linear} =, \varepsilon_H \in H^*)$$

$$\cdot \underset{H^*}{\Delta}(f)(h \otimes h') = f(hh')$$

$$\cdot \varepsilon_{H^*}(f) = f(\Delta_H)$$

$$\cdot \underset{H^*}{\lambda}(f)(h) = f(\lambda_H(h))$$

FACTS: 1) H commutative $(\Leftrightarrow) H^*$ cocommutative

2) H cocommutative $(\Leftrightarrow) H^*$ commutative

3) $H \rightarrow H^*$ inv. of \mathbb{R} -Hopf algebras.

EX: $H = \mathbb{R}[G]$, G finite, $H^* = \mathbb{R}[G]^*$

basis: $\{e_\sigma : \sigma \in G\}$ $e_\sigma(\tau) = \delta_{\sigma, \tau}$.

$n \in \mathbb{R}$, $\sigma, \tau, \rho \in G$,

$$(e_\sigma e_\tau)(\rho) = (e_\sigma \otimes e_\tau)(\rho \otimes e) = e_\sigma(\rho) e_\tau(e)$$

\Rightarrow

$$e_\sigma e_\tau = \delta_{\sigma, \tau} e_\sigma \quad (\Rightarrow \text{PAIRWISE ORTHOGONAL IDEMPOTENT})$$

$$L_{H^*}(n) = n \sum_{\sigma \in G} e_\sigma$$

$$\Delta(e_\sigma)(\rho \otimes \tau) = e_\sigma(\rho \tau) = \delta_{\sigma, \rho \tau} \Rightarrow$$

$$\Delta(e_\sigma) = \sum_{\tau: \rho \tau = \sigma} e_\rho \otimes e_\tau$$

$$\lambda_{H^*}(e_\sigma) = e_{\sigma^{-1}} \quad (\lambda_H(\sigma) = \sigma^{-1})$$

EX: $\mathbb{R}[G]^\leftarrow$ always commutative

$\mathbb{R}[G]^\leftarrow$ cocommutative $\Leftrightarrow G$ abelian.

3. GROUPLIKE ELEMENTS

Recall that for $H = R[G]$, $\sigma \in G$, $\Delta(\sigma) = \sigma \otimes \sigma$

DEF: Let H be an R -Hopf algebra, $h \in H$; h is GROUPLIKE if

- $h \neq 0$
- $\Delta(h) = h \otimes h$

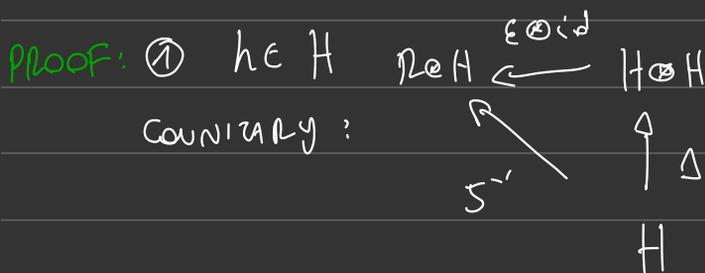
$$G(H) = \{ h \in H : h \text{ grouplike} \}$$

PROP: if $R = K$ is a field, H is a K -Hopf algebra, then distinct grouplike elements are linearly independent.

PROP: Let H be an R -Hopf algebra, and suppose that R has no nontrivial idempotents ($\neq 0, 1$)

$$\textcircled{1} \forall h \in G(H), \quad \varepsilon(h) = 1$$

$$\textcircled{2} G(H) \subseteq H^{\times}$$



$$h \in G(H) \quad \Delta(h) = h \otimes h$$

$$1 \otimes h = \varepsilon(h) \otimes h \Rightarrow h = \varepsilon(h)h$$

$$\Rightarrow \varepsilon(h) = \varepsilon(\varepsilon(h)h) = \varepsilon(h)\varepsilon(h)$$

$$\Rightarrow \varepsilon(h) \text{ idempotent } \begin{cases} \varepsilon(h) = 0 \Rightarrow h = 0 \\ \varepsilon(h) = 1 \end{cases}$$

$$(2c) \quad h, h' \in G(H), \quad \overset{?}{\Rightarrow} \quad hh' \in G(H)$$

Δ \mathbb{R} -algebra homom.

$$\begin{array}{c} H \otimes H \\ \downarrow \end{array}$$

$$\begin{aligned} \Delta(hh') &= \Delta(h) \Delta(h') = (h \otimes h) (h' \otimes h') \\ &= hh' \otimes hh' \quad \square \end{aligned}$$

RK: H finite Hopf algebra, $H^* \ni f \neq 0$

$$\text{we can: } \Delta(f) = f \otimes f \quad \in H^* \otimes H^*$$

$$\begin{array}{ccc} \Delta(f)(h \otimes h') & = & (f \otimes f)(h \otimes h') \\ \downarrow & & \downarrow \\ f(hh') & = & f(h)f(h') \quad \in \mathbb{R} \end{array}$$

if f \mathbb{R} -alg. hom \Rightarrow f group-like

$$\text{if } f \text{ group-like} \Rightarrow f(hh') = f(h)f(h')$$

$$f(1_H) = f(1_H) f(1_H)$$

$$\text{if } \mathbb{R} = \mathbb{K} \quad \Rightarrow \quad f(1_H) = 1 \quad \Rightarrow$$

group-like \Rightarrow algebra hom.

4. MODULES AND COMODULES

Let H be a fixed R -Hopf algebra.

DEF: a LEFT H -module is (S, α) , where

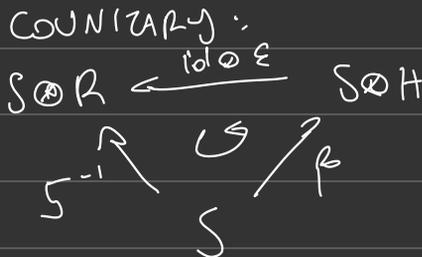
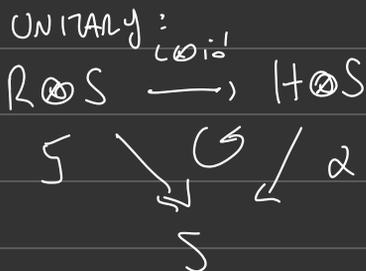
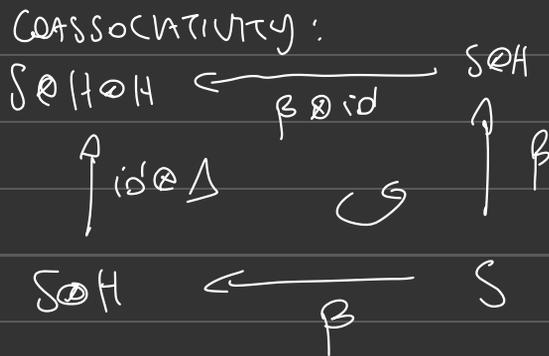
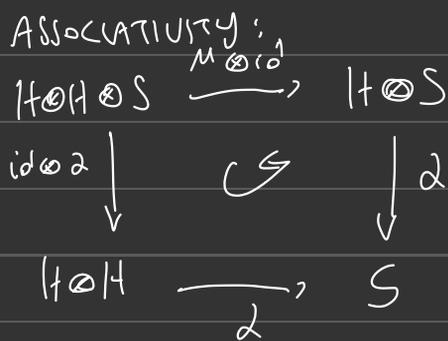
- S is an R -module
- $\alpha: H \otimes S \rightarrow S$ R -linear

\triangleright T.

DEF: a RIGHT H -comodule is (S, β) , where

- S R -module.
- $\beta: S \rightarrow S \otimes H$ R -linear

\triangleright T.



NOTATION:

- $h \cdot s = \alpha(h \otimes s)$

$$(hh') \cdot s = h \cdot (h' \cdot s)$$

$$1_H \cdot s = s$$

DEF: S, S' left H -modules,

$f: S \rightarrow S'$ R -linear.

f is an H -module

homom. iff f respects

α .

$$S \xrightarrow{f} S'$$

$$\alpha_S \uparrow \quad \hookrightarrow \quad \uparrow \alpha_{S'}$$

$$H \otimes S \xrightarrow{\text{id} \otimes f} H \otimes S'$$

$$f(h \cdot s) = h \cdot f(s)$$

NOTATION:

SWEEDLER

$$s \in S \quad \begin{matrix} S \\ \cup \\ H \end{matrix}$$

$$\beta(s) = \sum_{(s)} S_{(0)} \otimes S_{(1)}$$

$$S \otimes H$$

DEF: S, S' right H -comodules,

$f: S \rightarrow S'$ R -linear.

f is an H -comodule hom.

iff respects β .

Suppose now that H is finite, with dual H^* and
 $P \subset S \quad \{h_i, f_i\}_{i=1}^m$

• Let S be a right H -comodule: $\beta(s) = \sum_{(s)} s_{(0)} \otimes s_{(1)} \in S \otimes H$

FACT: S is also a left H^* -module

$$f \in H^*, s \in S, \quad f \cdot s = \sum_{(s)} \overset{S}{s_{(0)}} \langle f, \overset{R}{s_{(1)}} \rangle \in S$$

(S is a right H -comodule $\Rightarrow S$ is a left H -module)

• Let S be a left H -module: $h \cdot s \in S, \forall h \in H, s \in S$

FACT: S is a right H^* -comodule:

$$\beta(s) = \sum_{i=1}^m (h_i \cdot s) \otimes f_i$$

\uparrow
 $S \otimes H^*$

PROP: Let H be a finite R -Hopf algebra, and S be an R -module. Then

S is a left H -module $\Leftrightarrow S$ is a right $H^{\#}$ -comodule.

Moreover, the processes of going from H -module action to $H^{\#}$ -comodule action and vice versa are inverse.

PROOF: ① S left H -module, $h \cdot s \in S$

\Downarrow

S is a right $H^{\#}$ -comodule

$$\beta(s) = \sum_{i=1}^m h_i \cdot s \otimes f_i$$

\Downarrow

S is a left H -module

$$\alpha(h \otimes s) = \sum_{i=1}^m (h_i \cdot s) \langle h, f_i \rangle = \left(\sum_{i=1}^m h_i \langle h, f_i \rangle \right) \cdot s$$

$h \cdot s$

② S is a right H^* -module $\left(S \rightarrow \sum_{(s)} S_{(s)} \otimes S_{(s)}^*$

}
 \downarrow

S is a left H -module

$$h \cdot s = \sum_{(s)} S_{(s)} \langle h, S_{(s)}^* \rangle$$

}
 \downarrow

S is a right H^* -module.

$$\beta(s) = \sum_{i=1}^n h_i \cdot s \otimes f_i = \sum_{i=1}^n \left(\sum_{(s)} S_{(s)} \langle h_i, S_{(s)}^* \rangle \right) \otimes f_i$$

$$= \sum_{(s)} S_{(s)} \otimes \left(\sum_{i=1}^n \langle h_i, S_{(s)}^* \rangle f_i \right)$$

$$= \sum_{(s)} S_{(s)} \otimes S_{(s)}^*$$