Cyclic extensions of degree p

1 Notation and statement of the result

Denote by L/K a Galois extension of p-adic fields with Galois group $G \cong \mathbb{Z}/p\mathbb{Z}$, generated by an element σ. We fix uniformisers π_K and π_L of K and L respectively and normalise the valuations by $v_K(\pi_K) = 1, v_L(\pi_L) = 1$. We will almost always assume that L/K is totally and wildly ramified. We will also use the following symbols:

1. $e = e_K = v_K(p)$, the absolute inertia index of K.
2. t for the (unique) ramification jump of the extension, that is, the unique integer such that
$$G = G_{-1} = G_0 = \cdots = G_t, \quad G_{t+1} = \cdots = \{1\};$$
it is equivalently defined by the condition
$$t = \min_{x \in \mathcal{O}_L} v_L((\sigma - 1)x) - 1,$$
where by convention $v_L(0) = +\infty$ can never be the minimum.
3. $U_{K,i} = 1 + \pi_K^i \mathcal{O}_K$, the group of i-principal units.
4. $t = pt_0 + a$, with $0 \leq a \leq p - 1$.

We will describe necessary and sufficient conditions in order for $\mathcal{O}_L$ to be free over the associated order $\mathcal{A}_{L/K}$. We will in particular prove part of the following theorem of Bertrandias, Bertrandias and Ferton [Fer74, BBF72, BF72]:

Theorem 1.1. *1. If $t \equiv 0 \pmod{p}$, then $\mathcal{A}_{L/K}$ is the maximal order of $K[G]$ and $\mathcal{O}_L$ is free over it.*

2. If $0 < t < \lfloor \frac{pe}{p-1} \rfloor - 1$, then $\mathcal{O}_L$ is free over $\mathcal{A}_{L/K}$ if and only if $a \mid p - 1$.

3. If $\lfloor \frac{pe}{p-1} \rfloor - 1 \leq t \leq \frac{pe}{p-1}$, then $\mathcal{O}_L$ is free over $\mathcal{A}_{L/K}$ if and only if the continued fraction expansion of $\frac{a}{p}$ has length at most 4 (this means that $\frac{a}{p}$ can be written as
$$\frac{a}{p} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}} = [a_0; a_1, \ldots, a_n]$$
with $a_n > 1$ and $n \leq 4$).

Remark 1.2. The papers [Fer74, BF72] are essentially announcements of results and contain few proofs. A proof of part (3) can be found in [BBF72]. The proof of parts (1) and (2) given here is obtained by following the breadcrumbs left by [BF72] and filling in the gaps with the help of [CFL20].

2 Preliminary remarks

We start with some general comments on the ramification jump t:

Proposition 2.1. *1. $t \leq \frac{ep}{p-1}$.*

2. Given K and $-1 \leq t \leq \frac{ep}{p-1}$ with $p \nmid t$, there exists L/K cyclic of degree p with ramification jump t.

3. On the other hand, if $p \mid t$, then $t = \frac{ep}{p-1}$. In this case K contains ζ_p, and there exist uniformisers π_K, π_L such that $\pi_K = \pi_L^p$.

We now sketch a proof of these facts, based on the following fundamental facts about the groups of local units (the first part is easy; for the second, see [Ser79, Chapter 5, §3, Proposition 4]):

Proposition 2.2. *The following hold:*

1. For $i \geq \frac{ep}{p-1}$, every i-principal unit is the p-th power of a $(i-e)$-principal unit. In symbols: $U_{K,i} \subseteq U_{K,i-e}^p$.

2. Define $\psi(x) = \begin{cases} x, & \text{if } x \leq t \\ t + p(x-t), & \text{if } x \geq t. \end{cases}$ Then for all $n \geq 0$ the norm map $N_{L/K}$ sends $U_{L,\psi(n)}$ into $U_{K,n}$ and $U_{L,\psi(n)+1}$ into $U_{K,n+1}$.

Sketch of proof of Proposition 2.1. For (1), assume by contradiction $t > \frac{ep}{p-1}$. Consider $\varepsilon := 1 + \pi_K^t \in U_{K,t} \subseteq (U_{K,t-e})^p$. Let $\varepsilon_1 \in U_{K,t-e}$ be a p-th root of ε. Then on the one hand $N_{L/K}(\varepsilon_1) = \varepsilon_1^p = \varepsilon = 1 + \pi_K^t$ does not belong to $U_{K,t+1}$. On the other, $N_{L/K}(\varepsilon_1) \in N_{L/K}(U_{L,p(t-e)})$. Now $p(t-e) \geq t+1$, hence $N_{L/K}(U_{L,p(t-e)}) \subseteq N_{L/K}(U_{L,t+1}) \subseteq U_{K,t+1}$ by Proposition 2.2 (2). The contradiction proves the result.

For (2), the case $t = -1$ is trivial, and for $t > 0$ one takes L to be the splitting field of $x^p - x - \alpha$, where $v_K(\alpha) = -t$.

For (3), one begins by proving (using similar tricks) that $t \geq \frac{ep}{p-1}$, hence (by (1)) that $t = \frac{ep}{p-1}$. By assumption we have $\sigma(\pi_L) - \pi_L = \vartheta \pi_L^{t+1}$, where ϑ is a unit of $\mathcal{O}_L$. Dividing through by π_L we get

$$u := \frac{\sigma(\pi_L)}{\pi_L} = 1 + \vartheta \pi_L^t.$$

Changing ϑ if necessary, since $t = \frac{ep}{p-1}$, we obtain $u = 1 + \vartheta \pi_K^{e/(p-1)}$. As the extension L/K is totally ramified we have $\mathcal{O}_L/(\pi_L) \cong \mathcal{O}_K/(\pi_K)$, so $\vartheta \equiv \vartheta_K \pmod{\pi_L}$, where now $\vartheta_K \in \mathcal{O}_K$. Thus:

$$u \equiv 1 + \vartheta_K \pi_K^i \pmod{\pi_L^{t+1}},$$

or equivalently $u = (1 + \vartheta_K \pi_K^i) u'$ with $u' \in U_{L,t+1}$. Taking the norm of this equation we obtain

$$1 = (1 + \vartheta_K \pi_K^{e/(p-1)})^p N_{L/K}(u')$$

(notice that $1 + \vartheta_K \pi_K^{e/(p-1)}$ is an element of K), with $N_{L/K}(u') \in U_{K,t+1}$ by Proposition 2.2 (2). By Proposition 2.2 (1), $N_{L/K}(u')$ is the p-th power of unit u_0 in $U_{K,t+1-e} = U_{K,e/(p-1)+1}$. We have thus obtained

$$1 = (1 + \vartheta_K \pi_K^{e/(p-1)})^p u_0^p,$$

hence $\zeta := (1 + \vartheta_K \pi_K^{e/(p-1)})u_0 \equiv 1 + \vartheta_K \pi_K^{e/(p-1)} \pmod{\pi_K^{e/(p-1)+1}}$ satisfies $\zeta^p = 1$. In particular, ζ is a p-th root of unity in K that is not 1. The last statement then follows from Kummer theory.

□

3 The case $t \equiv 0 \pmod p$

By the above, we may assume $t = \frac{ep}{p-1}$, and choose uniformisers π_K, π_L in such a way that $\pi_L^p = \pi_K$.

Theorem 3.1. *Notation as above. In the case $a = 0$, the associated order $\mathcal{A}_{L/K}$ is the maximal order of $K[G]$, and the element $\vartheta = 1 + \pi_L + \cdots + \pi_L^{p-1}$ generates $\mathcal{O}_L$ over $\mathcal{A}_{L/K}$.*

Proof. The fact that ϑ generates a normal integral basis can be proven as the case of tame, totally ramified extensions in Henri Johnston's talk (namely, it follows from a calculation with Vandermonde determinants). We choose a different route, by first describing the associated order $\mathcal{A}_{L/K}$. We have

$$K[G] \cong K[t]/(t^p - 1) \cong \prod_{i=0}^{p-1} K,$$

with the isomorphisms being given by $\sigma \mapsto t$ and $f(t) \mapsto (f(\zeta_p^i))_{i=0,\dots,p-1}$ respectively. The unique maximal order of $\prod_{i=0}^{p-1} K$ is clearly $\prod_{i=0}^{p-1} \mathcal{O}_K$; tracing the isomorphisms backwards, we see that the maximal order of $K[G]$ is given by

$$\mathfrak{M} := \left\{ \sum_{j=0}^{p-1} a_j \sigma^j \in K[G] \mid \sum_{j=0}^{p-1} a_j \zeta_p^{ij} \in \mathcal{O}_K \text{ for all } i = 0, \dots, p-1 \right\}.$$

Applying any element of $\mathfrak{M}$ to a basis element π_L^i we obtain

$$\sum_{j=0}^{p-1} a_j \sigma^j \pi_L^i = \left(\sum_{j=0}^{p-1} a_j \zeta_p^{ij} \right) \pi_L^i \in \mathcal{O}_L.$$

This shows $\mathfrak{M} \subseteq \mathcal{A}_{L/K}$, hence (by maximality) that the associated order coincides with $\mathfrak{M}$. From the above calculation we also see that an element $\lambda = (c_0, \dots, c_{p-1}) \in \mathcal{O}_K^p \cong \mathfrak{M}$ acts on ϑ as

$$\lambda \cdot \vartheta = \sum_{i=0}^{p-1} c_i \pi_L^i,$$

which immediately implies $\mathcal{A}_{L/K} \cdot \vartheta = \bigoplus_{i=0}^{p-1} \mathcal{O}_K \pi_L^i = \mathcal{O}_L$. □

4 The fractional ideal $\mathcal{A}_\vartheta$

From now on we focus on the case $t \not\equiv 0 \pmod p$. In particular, since we already assumed $t \geq 0$, we will have $t \geq 1$. The method of Bertrandias and Ferton centres around the following object:

Definition 4.1. Let $\vartheta \in \mathcal{O}_L$ generate a normal basis for the field extension L/K. We define

$$\mathcal{A}_\vartheta := \{\lambda \in K[G] : \lambda\vartheta \in \mathcal{O}_L\}.$$

Remark 4.2. $\mathcal{A}_\vartheta$ is not necessarily a ring: for example, it can easily happen that $\frac{1}{p}$ is in $\mathcal{A}_\vartheta$, but $\mathcal{A}_\vartheta$ can never contain all the powers of p^{-1}.

Some basic properties of $\mathcal{A}_\vartheta$ are easy to establish:

Proposition 4.3.

1. *$\mathcal{A}_{L/K} \subseteq \mathcal{A}_\vartheta$, and $\mathcal{A}_\vartheta$ is a (left) fractional ideal of $\mathcal{A}_{L/K}$.*
2. *The map*
$$\begin{array}{ccc} \mathcal{A}_\vartheta & \to & \mathcal{O}_L \\ \lambda & \mapsto & \lambda \cdot \vartheta \end{array}$$
is an isomorphism of left $\mathcal{A}_{L/K}$-modules (in particular, it is surjective).
3. *The following are equivalent:*
   (a) *$\mathcal{O}_L$ is free over $\mathcal{A}_{L/K}$;*
   (b) *there exists $\vartheta \in \mathcal{O}_L$, generating a normal basis for L/K, such that $\mathcal{A}_\vartheta = \mathcal{A}_{L/K}$;*
   (c) *for all $\vartheta \in \mathcal{O}_L$ that generate a normal basis for L/K, the left $\mathcal{A}_{L/K}$-ideal $\mathcal{A}_\vartheta$ is principal.*

Before discussing the structure of $\mathcal{A}_\vartheta$ for some interesting choices of ϑ we identify a useful element of $K[G]$ and describe its action on $\mathcal{O}_L$:

Lemma 4.4. *Denote by f the element $\sigma - 1 \in \mathcal{O}_K[G]$. The following hold:*

1. *the powers $1, f, f^2, \ldots, f^{p-1}$ of f form an $\mathcal{O}_K$-basis of $\mathcal{O}_K[G]$;*
2. *the equality $f^p = -\sum_{j=1}^{p-1} \binom{p}{j} f^j$ holds.*

Suppose in addition $a \neq 0$. Then:

3. *$v_L(f^i \pi_L^a) = a + it$ for $i = 0, \ldots, p-1$;*
4. *the element π_L^a generates a normal basis for L/K. Explicitly, $\{f^i \pi_L^a\}_{i=0,\ldots,p-1}$ is a K-basis of L;*
5. *we have $v_L(f^p \pi_L^a) = ep + t + a$;*
6. *for every $x \in \mathcal{O}_L$ we have $v_L(fx) \geq v_L(x) + t$.*

Proof. The first two parts are easy. For (3), as in the proof of Section 2 we have

$$\frac{\sigma(\pi_L)}{\pi_L} = 1 + \pi_L^t u$$

for some $u \in \mathcal{O}_L^\times$. Raising both sides to the j-th power, for any j prime to p, gives

$$\frac{\sigma(\pi_L^j)}{\pi_L^j} = 1 + \pi_L^t u_j$$

where $u_j \in \mathcal{O}_L^\times$ since $(j,p)=1$. Rearranging the previous equality gives

$$\sigma(\pi_L^j) - \pi_L^j = \pi_L^{j+t} u_j,$$

so we obtain $v_L(f\pi_L^j) = j+t$, provided that $(j,p)=1$. The claim then follows by induction. For part (4), notice that the L-valuations of the elements $\{f^i \pi_L^a\}_{i=0,\dots,p-1}$ are all distinct modulo p. Parts (5) and (6) are again easy.

□

In particular, we can consider $\mathcal{A}_\vartheta$ with $\vartheta = \pi^a$. This special choice of ϑ allows us to easily describe $\mathcal{A}_\vartheta$:

Proposition 4.5. *$\mathcal{A}_\vartheta$ is free over $\mathcal{O}_K$ with basis $\pi_K^{-\nu_i} f^i$, where $\nu_i := \lfloor \frac{a+it}{p} \rfloor$.*

Proof. An element $\sum_{i=0}^{p-1} c_i f^i \in K[G]$ is in $\mathcal{A}_\vartheta$ if and only if $\sum_{i=0}^{p-1} c_i f^i \pi_L^a$ is integral. Since the valuations of the terms $f^i \pi_L^a$ are all distinct by Lemma 4.4 (3), this happens if and only if $c_i f^i \pi_L^a$ is integral for all i. Since $v_L(f^i \pi_L^a) = a + it$, the condition is $v_L(c_i) + a + it \geq 0$, or equivalently $v_K(c_i) \geq -\frac{a+it}{p}$. □

Remark 4.6. An $\mathcal{O}_K$-basis of $\mathcal{O}_L$ is given by $\{\pi_K^{-\nu_i} f^i \pi_L^a\}_{i=0,\dots,p-1}$. Indeed, the L-valuations of these elements are all distinct modulo p, and are all between 0 and $p-1$.

The description of $\mathcal{A}_{L/K}$ is only slightly more complicated:

Proposition 4.7. *$\mathcal{A}_{L/K}$ is free over $\mathcal{O}_K$ with basis $\pi_K^{-n_i} f^i$, where*

$$n_i := \min_{0 \leq j \leq p-1-i} (\nu_{i+j} - \nu_j).$$

Sketch of proof. Take an arbitrary element $\lambda = \sum_{i=0}^{p-1} c_i f^i \in K[G]$. It is in $\mathcal{A}_{L/K}$ if and only if $\lambda\left(\pi_K^{-\nu_j} f^j \pi_L^a\right)$ is integral for all $i = 0, \dots, p-1$. One checks that this happens if and only if $\sum_{i=0}^{p-1-j} c_i \pi_K^{-\nu_j} f^{i+j} \pi_L^a$ is in $\mathcal{O}_L$ (the other summands are automatically integral). Since the valuations of the terms are all distinct, this happens if and only if $v_K(c_i) - \nu_j + \nu_{i+j} \geq 0$ for $j = 0, \dots, p-1-i$, which easily implies the result. □

5 The case $t \not\equiv 0 \pmod p$

In this section we prove part (2) of Theorem 1.1. Part (3) is conceptually similar, but technically much more involved, and will not be discussed here. There are also further extensions to cyclic extensions of degree p^n, see [Ber79].

Remark 5.1. The condition $t < \lfloor \frac{ep}{p-1} \rfloor - 1$ is equivalent to the extension L/K *not* being almost maximally ramified. The only nontrivial idempotent in $K[G]$ is $e_G := \frac{1}{p} \sum_{i=0}^{p-1} \sigma^i$. An amusing computation involving sums of binomials shows that e_G is in $\mathcal{A}_{L/K}$ if and only if $n_{p-1} \geq e$, and one sees that this is equivalent to $\frac{pe}{p-1} - 1 \leq t \leq \frac{pe}{p-1}$.

We begin by proving that if $a \mid p-1$, then $\mathcal{A}_\vartheta = \mathcal{A}_{L/K}$ (where $\vartheta = \pi_L^a$), so that – by Proposition 4.3 – the ring of integers $\mathcal{O}_L$ is free over $\mathcal{A}_{L/K}$ in this case. We need a simple arithmetical lemma:

Lemma 5.2. *If $a \mid p-1$, then $\nu_i = it_0 + \lfloor \frac{i}{k} \rfloor$ for all $i = 0, \dots, p-1$, where $k = \frac{p-1}{a}$.*

Proof. By induction on i. □

Proposition 5.3. *Suppose $a \mid p-1$ (in particular $a \neq 0$). Then for $\vartheta = \pi_L^a$ we have $\mathcal{A}_{L/K} = \mathcal{A}_\vartheta$. In particular, $\mathcal{O}_L$ is free over $\mathcal{A}_{L/K}$.*

Proof. Recall that $\mathcal{A}_{L/K}$ is $\mathcal{O}_K$-free with basis $\pi_K^{-n_i} f^i$ and $\mathcal{A}_\vartheta$ is free with basis $\pi_K^{-\nu_i} f^i$. Thus, equality holds if and only if we have $\nu_i = n_i$ for all $i = 0, \dots, p-1$. By definition we have $\nu_i \geq n_i$, so it suffices to show the opposite inequality. Since n_i is defined as a minimum, we need to show

$$\nu_{i+j} - \nu_j \geq \nu_i \quad \text{for all indices } i, j \text{ with } i + j \leq p-1.$$

Using Lemma 5.2 we simply need to prove

$$(i+j)t_0 + \lfloor \frac{i+j}{k} \rfloor \geq it_0 + \lfloor \frac{i}{k} \rfloor + jt_0 + \lfloor \frac{j}{k} \rfloor,$$

which is obvious. □

It remains to show that (under the assumptions $t < \lfloor \frac{ep}{p-1} \rfloor - 1$ and $a \neq 0$), if the ring $\mathcal{O}_L$ is $\mathcal{A}_{L/K}$-free, then $a \mid p-1$. The final conclusion will follow from the next lemma, which we will not prove (even though it is not very hard, it is also not very interesting from the point of view of Galois theory):

Lemma 5.4. *Assume that $n_i = \nu_i$ for all $i = 0, \dots, p-1$. Then $a \mid p-1$.*

So we just need to prove that $n_i = \nu_i$ for all i in this case. To this end, we will apply characterisation (3c) of Proposition 4.3. Namely, we will assume that for our specific $\vartheta = \pi_L^a \in \mathcal{O}_L$ there is an isomorphism $\mathcal{A}_\vartheta \cong \mathcal{A}_{L/K}$ of left $\mathcal{A}_{L/K}$-modules, and deduce from this that $n_i = \nu_i$ needs to hold for each i.

Proof. Suppose that $\mathcal{A}_\vartheta \cong \mathcal{A}_{L/K}$. This means that there exists $\alpha \in \mathcal{A}_\vartheta$ such that

$$\begin{array}{rccc} \varphi : & \mathcal{A}_{L/K} & \to & \mathcal{A}_\vartheta \\ & \lambda & \mapsto & \lambda\alpha \end{array}$$

is an isomorphism. We represent φ as a matrix $M(\alpha)$ in the bases of $\mathcal{A}_\vartheta$ and $\mathcal{A}_{L/K}$ described above. If $\alpha = \sum_{i=0}^{p-1} x_i \pi_K^{-\nu_i} f^i$ (recall that $\alpha \in \mathcal{A}_\vartheta$, so the x_i are in $\mathcal{O}_K$), then $M(\alpha) = \sum_{i=0}^{p-1} x_i M(\pi_K^{-\nu_i} f^i)$. Notice that $M(\alpha) \in \mathrm{Mat}_{p \times p}(\mathcal{O}_K)$, hence it makes sense to reduce it modulo π_K, and φ is an isomorphism if and only if $M(\alpha)$ is invertible over $\mathcal{O}_K$, if and only if $\det M(\alpha) \in \mathcal{O}_K^\times$, if and only if $\det M(\alpha) \not\equiv 0 \pmod{\pi_K}$.

Next we claim that the matrices $M(\pi_K^{-\nu_i} f^i)$ are all lower-triangular when reduced modulo π_K, and in fact *strictly* lower-triangular unless $i = 0$. Assuming this fact, the matrix $M(\alpha)$ is congruent modulo π_K to a lower-triangular matrix whose k-th diagonal coefficient is $x_0 \pi_K^{\nu_k - n_k}$. In particular, $M(\alpha)$ is invertible if and only if $v_K(x_0) = 0$ and $\nu_k = n_k$ for all k, as desired.

It remains to show the claim about the matrices $M(\pi_K^{-\nu_i} f^i)$ being strictly lower-triangular for $i > 0$ (for $i = 0$, the matrix $M(\pi_K^{-\nu_i} f^i)$ is easily seen to be diagonal). Consider an entry of $M(\pi_K^{-\nu_i} f^i)$ strictly above the diagonal, say in position (c, d) with $d > c$. The coefficient of $M(\pi_K^{-\nu_i} f^i)$ in position (c, d) is the coefficient of $\pi_K^{-\nu_c} f^c$ in

$$\pi_K^{-n_d} f^d \cdot \pi_K^{-\nu_i} f^i.$$

Since $d > c$, if $i + d \le p - 1$ the coefficient in question is simply 0. Otherwise, using Lemma 4.4 we may rewrite the above as

$$\pi_K^{-n_d-\nu_i} f^{i+d-p} f^p = -\pi_K^{-n_d-\nu_i} \sum_{j=1}^{p-1} \binom{p}{j} f^{i+d+j-p},$$

which we claim is in $\pi_K \mathcal{A}_\vartheta$, and therefore has coefficient along $\pi_K^{-\nu_c} f^c$ divisible by π_K. To finish the proof, we only need to show that $\pi_K^{-n_d-\nu_i-1} f^{i+d-p} f^p$ is in $\mathcal{A}_\vartheta$ (notice the -1 in the exponent of π_K). Replacing $-n_d$ by $-\nu_d$ (which is larger in absolute value), this reduces to proving

$$v_L(f^{i+d-p} f^p \pi_L^a) \ge p(\nu_d + \nu_i + 1),$$

which follows in a straightforward manner from Lemma 4.4 if one uses the assumption that t is not too large.

Here are the details. We may assume $a \neq p - 1$ (since we have already handled this case), so $a < p - 1$. By part (5) of Lemma 4.4 we have $v_L(f^p \pi_L^a) = ep + t + a$. By part (6), every subsequent application of f increases the valuation by at least t, so $v_L(f^{i+d-p} f^p \pi_L^a) \ge ep + t + a + t(i + d - p)$. On the other hand,

$$p(\nu_d + \nu_i + 1) \le a + dt + a + it + p.$$

So we need to check that

$$ep + t + a + t(i + d - p) \ge a + dt + a + it + p.$$

Simplifying like terms, this is equivalent to

$$ep + t - pt \ge a + p \Longleftrightarrow ep \ge p + t(p - 1) + a.$$

By assumption $t \le \lfloor \frac{ep}{p-1} \rfloor - 2 \le \frac{ep}{p-1} - 2$, so $t(p - 1) \le ep - 2(p - 1)$. Thus it suffices to check

$$ep \ge p + ep - 2(p - 1) + a = ep + 1 + (a - (p - 1)),$$

which holds since $a \le p - 2$. □

In [CFL20] we consider, given a Galois extension L/K, the quantity

$$m(L/K) := \min_{\alpha \in \mathcal{O}_L} [\mathcal{O}_L : \mathcal{O}_K[G] \cdot \alpha].$$

By methods not too different from the above, we find an explicit formula for $m(L/K)$ when L/K is cyclic of degree p:

Theorem 5.5. *Let L/K be a ramified Galois extension of p-adic fields of degree p, with ramification jump t. Let $a \in \{0, \dots, p - 1\}$ be the residue class of t modulo p and set $\nu_i = \left\lfloor \frac{a+it}{p} \right\rfloor$. Then if $a \neq 0$ we have $v_p(m(L/K)) = f_K \left(\sum_{i=0}^{p-1} \nu_i + \min_{0 \le i \le p-1} (ie_K - (p - 1)\nu_i) \right)$, while for $a = 0$ we have $v_p(m(L/K)) = \frac{1}{2}[L : \mathbb{Q}_p]$.*

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