LOCAL VERSIONS OF GLOBAL RESULTS

Th (Leopoldt / Bergé / Martinet) : $K/\mathbb{Q}$ abelian or dihedral of degree $2p$ or Q_8 $\Rightarrow$ $\mathcal{O}_K$ free over $\mathfrak{O}_{K/\mathbb{Q}}$.

Can we deduce to $L/\mathbb{Q}_p$ with same Gal. group?

- I believe that their proofs work as they are, and that this is the main folklore (verified for Q_8 - extensions)
- Q: $L/\mathbb{Q}_p$ Gal., G. $\overset{?}{\exists}$ $K/\mathbb{Q}$ Gal., G s.t. $K_\wp = L$? false $p=2$, OK p odd (non-trivial, Henniart, 2001)
- More simply: Lettl '98 $\Rightarrow$ p-adic Leopoldt

  $K/\mathbb{Q}_p$ abelian

  Bergé '79 $\Rightarrow$ p-adic dihedral

  or Jaulent '81

  $C_p \rtimes C_n$

  $n = 2$

§2 OUR SCHEME

$$\begin{array}{ccc} \mathfrak{P} & K \\ & G\,\big| \\ \mathfrak{p} & F \end{array} \quad \longleftrightarrow \quad \begin{array}{cc} K_{\mathfrak{P}} \\ D\,\big| \\ F_{\mathfrak{p}} \end{array}$$

$\mathcal{O}_K$ free/$\Omega_{K/F}$

PROP. 3.38 (projective)

$\mathcal{O}_{K_\mathfrak{P}}$ free/$\Omega_{K_\mathfrak{P}/F_\mathfrak{p}}$

(**) $\ell = 0$ we'll see

(projective)

(*)

$\mathrm{Ind}_D^G \Omega_{K_\mathfrak{P}/F_\mathfrak{p}}$ ring (Prop 3.2?)

$\mathrm{Ind}_D^G \Omega_{K_\mathfrak{P}/F_\mathfrak{p}}$ free (Cor 3.36)

$\mathcal{O}_{K,\mathfrak{p}}$ free over $\Omega_{K/F,\mathfrak{p}}$

$$\mathcal{O}_{K,\mathfrak{p}} = \mathrm{Ind}_D^G \mathcal{O}_{K_\mathfrak{P}} \qquad \Omega_{K/F,\mathfrak{p}} = \Omega(F_\mathfrak{p}[G], \mathrm{Ind}_D^G \mathcal{O}_{K_\mathfrak{P}})$$

in $F_\mathfrak{p}[D]$

more generally: $\mathcal{O}_{F_\mathfrak{p}}$ ~~$F_\mathfrak{p}[D]$~~-order Λ, N Λ-lattice

Γ an $\mathcal{O}_{F_\mathfrak{p}}$-order in $F_\mathfrak{p}[G]$

s.t.

$$\Lambda \subseteq \Gamma \subseteq \mathrm{Ind}_D^G \Lambda$$

$$F_\mathfrak{p}[D] \hookrightarrow F_\mathfrak{p}[G]$$

Then $\mathrm{Ind}_D^G N$ projective over $\Gamma \Rightarrow$

$\Rightarrow N$ projective over Λ

$$\left(\begin{array}{l} \Lambda = \mathcal{O}_{F_\mathfrak{p}}[D] \\ \Gamma = \mathcal{O}_{F_\mathfrak{p}}[G] \end{array}\right)$$

§3 TAMENESS AND PROJECTIVITY

TH: K/F Gal. of # fields, gal. gr. G tame ff.

$\mathcal{O}_K$ proj. / $\mathcal{O}_F[G]$

PROOF: $\boxed{\Rightarrow}$ tame $\Rightarrow \mathcal{O}_K$ loc. free / $\mathcal{O}_F[G]$

$\Rightarrow \mathcal{O}_K$ loc. proj. / $\mathcal{O}_F[G]$

($\mathcal{O}_{K,\mathfrak{p}}$ proj. / $\mathcal{O}_{F_\mathfrak{p}}[G]$ $\forall \mathfrak{p} \subseteq \mathcal{O}_F$)

$\Rightarrow \mathcal{O}_K$ proj. / $\mathcal{O}_F[G]$

$\boxed{\Leftarrow}$ $\mathcal{O}_K$ proj. / $\mathcal{O}_F[G] \Rightarrow$

$\Rightarrow \mathcal{O}_{K,\mathfrak{p}}$ proj. / $\mathcal{O}_{F_\mathfrak{p}}[G]$ $\forall \mathfrak{p} \subseteq \mathcal{O}_F$

$\underset{\subseteq K}{\mathcal{O}} \big| \underset{\subseteq F}{\mathfrak{p}}$

"Prop. 3.38"

$\Rightarrow \mathcal{O}_{K\mathfrak{P}}$ proj. / $\mathcal{O}_{F_\mathfrak{p}}[D]$

$\Rightarrow$ " free " $\Rightarrow$ tame (at $\mathfrak{p}$) $\forall \mathfrak{p}$

§4 MORE ON (*)

$$\begin{tikzcd} \mathfrak{P} \quad K_\sigma \arrow[d, dash, "G"'] & & K_\sigma \arrow[d, dash, "D"'] \\ p \quad \mathbb{Q} \arrow[rr, shorten=1em] & & \mathbb{Q}_p \end{tikzcd}$$

D dihedral

Suppose $K_\sigma/\mathbb{Q}_p$ has almost-maximal ramification $\overset{\text{Berge '79}}{\Longrightarrow}$ $\mathcal{O}_{K_\sigma}$ free$/\mathfrak{A}_{K_\sigma/\mathbb{Q}_p}$

and $\mathfrak{A}_{K_\sigma/\mathbb{Q}_p} = \mathbb{Z}_p[D][e_{G_i}]_{i \geq 1}$

If $\mathrm{Ind}_D^G \mathfrak{A}_{K_\sigma/\mathbb{Q}_p}$ ring $\overset{\text{Prop. 3.32}}{\Longrightarrow}$ $\mathcal{O}_{K,p}$ free$/\mathfrak{A}_{K/\mathbb{Q},p}$

This is the case if $G_i \trianglelefteq G$ $\forall i$, that is, if $G \cong N \rtimes C_2$, $G_1 \subseteq N$.

<u>TH</u>: If $K_\sigma/\mathbb{Q}_p$ has almost-maximal ramification and is dihedral, and $G_1 \subseteq N$, then $\mathcal{O}_K$ is loc. free at p over $\mathfrak{A}_{K/\mathbb{Q}}$.

• $K/\mathbb{Q}$ A_4-extension

$$\begin{array}{ll} \mathcal{O} & K \\ & \big|\, A_4 \\ 2 & \mathbb{Q} \end{array}$$

$$D = \begin{cases} V_4 \trianglelefteq A_4 \rightsquigarrow \text{klein analysis} \\ C_2 \end{cases}$$

$$(D, G_0) \begin{cases} (V_4, V_4) \\ (V_4, C_2) \\ \cancel{(C_2, C_2)} \end{cases}$$

CLAIM: $\mathcal{O}_{K,2}$ not free / $\Omega_{K/\mathbb{Q},2}$

$$\Omega_{K\mathcal{O}/\mathbb{Q}_2} = \mathbb{Z}_2[V_4] + \frac{1}{2}\mathbb{Z}_2[V_4]\,\mathrm{Tr}_{G_0 \cong C_2}$$

$$\rightsquigarrow \mathrm{Ind}_{V_4}^{A_4} \Omega_{K\mathcal{O}/\mathbb{Q}_2} = \mathbb{Z}_2[A_4] + \frac{1}{2}\mathbb{Z}_2[A_4]\,\mathrm{Tr}_{G_0}$$

not a ring.

$G_0 \cong C_2 \ntrianglelefteq A_4 \implies$

$V_4 \trianglelefteq A_4$, abelian $\rightsquigarrow$ Prop. 3.45 ((2) $\Rightarrow$ (3))

$\implies \mathcal{O}_{K,2}$ not free / $\Omega_{K/\mathbb{Q},2}$

INSIGHT: $\Omega(F[H], N) = \sum \frac{1}{\pi^{n_i}} R[H]\,\mathrm{Tr}_{K_i}$ (unif of R)

$$\underbrace{\mathrm{Ind}_H^G}_{\text{a ring}} \Omega(F[H],N) = \sum \frac{1}{\pi^{n_i}} R[G]\,\mathrm{Tr}_{K_i} \quad \begin{pmatrix} K_1 \subseteq K_2 \subseteq \cdots \\ n_1 \geq n_2 \geq \cdots \end{pmatrix}$$

a ring $\iff K_i \trianglelefteq G$ $\forall i$ (N free $\Rightarrow \mathrm{Ind}_H^G N$ free)

Conversely, if $H \trianglelefteq G$ abelian and $\mathrm{Ind}_H^G N$ free

$\implies K_i \trianglelefteq G$ $\forall i$, in gen. $\Omega(F[G], \mathrm{Ind}_H^G N) = \sum \frac{1}{\pi^{n_i}} R[G]\,\mathrm{Tr}_{L_i}$ ($K_i \subseteq L_i$ [illegible] smallest $\trianglelefteq G$)

§6 SOMETHING ON (**)

RECALL: $K/\mathbb{Q}$ tame, G, $Cl(\mathbb{Z}[G])=0$

Then we have loc. free normal int. and $K/\mathbb{Q}$ has a NIB (local freeness $\Rightarrow$ global ")

TH: $K/\mathbb{Q}$ G-Gal., $Cl(\mathbb{Z}[G])=0$, $\mathcal{O}_K$ loc. free / $\mathfrak{A}_{K/\mathbb{Q}}$ $\Rightarrow$ $\mathcal{O}_K$ free / $\mathfrak{A}_{K/\mathbb{Q}}$.

PROOF: If $Cl(\mathfrak{A}_{K/\mathbb{Q}})=0 \Rightarrow$ OK

$\mathbb{Z}[G] \hookrightarrow \mathfrak{A}_{K/\mathbb{Q}}$ induces

$Cl(\mathbb{Z}[G]) \twoheadrightarrow Cl(\mathfrak{A}_{K/\mathbb{Q}})$ (CR(50.29)) $\square$

COR: $K/\mathbb{Q}$ Gal. with $Gal(K/\mathbb{Q}) \cong$ D_{2m} ($Cl(\mathbb{Z}[D_{2m}])=0$), A_4, S_4, A_5

Then $\mathcal{O}_K$ free / $\mathfrak{A}_{K/\mathbb{Q}}$ $\Leftrightarrow$ $\mathcal{O}_K$ loc. free / $\mathfrak{A}_{K/\mathbb{Q}}$

COR: TH 3.19

§6 RETURN TO A PURELY LOCAL SETTING

WEDDERBURN: K field with char 0

G finite group.

$$K[G] \cong \prod_i \mathrm{Mat}_{n_i}(D_i)$$

D_i ↳ skew-algebras

Indeed, here $K[G]$ separable. (CR §3B)

G abelian ⇝ $K[G] \cong \prod_{\chi \in \Phi} K(\chi)$

$\chi : G \to \overline{K}^*$

K # field or p-adic field ✪

$\underline{Q}$: Is $\underbrace{\mathcal{O}_K[G]}$ $\overset{?}{\cong}$ $\underbrace{\prod_{\chi \in \Phi} \mathcal{O}_{K(\chi)}}$ NO (G abelian)

$\subsetneq$ maximal order

usually

DEF: F # field or p-adic field, G finite, an $\mathcal{O}_F$-MAXIMAL ORDER in $F[G]$ is an $\mathcal{O}_F$-order which is maximal wrt $\subseteq$.

PROP: For every $\mathcal{O}_F$-order Λ, ∃ maximal order $\mathcal{M}$ s.t. $\Lambda \subseteq \mathcal{M}$ (CR §26)

REM: G abelian ⟹ ∃! maximal order, which is the integral closure of $\mathcal{O}_F$ in $F[G]$ ✪

PROP: $\mathcal{O}_F[G]$ maximal

iff $|G| \in \mathcal{O}_F^\times$

COR: F # field, $G \neq 1$, then $\mathcal{O}_F[G]$ never maximal order.

F p-adic field, $\mathcal{O}_F[G]$ maximal

iff $p \nmid |G|$.

TH (Reiner, "Max orders") (8.10): F p-adic field, G finite gp. $\mathfrak{M}$ maximal $\mathcal{O}_F$-order

M $\mathfrak{M}$-lattice s.t. $F \otimes_{\mathcal{O}_F} M$ free / $F[G]$.

Then M free over $\mathfrak{M}$.

COR: $\mathfrak{M}$ maximal order is clean.

COR: K/F Gal. ext. of p-adic fields.

If $\mathfrak{A}_{K/F}$ is maximal, then $\mathcal{O}_K$ free over $\mathfrak{A}_{K/F}$.

If K/F ext. of # fields, $\mathfrak{p} \subseteq \mathcal{O}_F$,

if $\mathfrak{A}_{K/F,\mathfrak{p}}$ is maximal $\mathcal{O}_{K,\mathfrak{p}}$ free over $\mathfrak{A}_{K/F,\mathfrak{p}}$.

$\mathcal{O}_{F_\mathfrak{p}}$-order...

$F \quad \mathfrak{a}$

$\mathfrak{p} \mid p$

COR: local freeness at all $\mathfrak{p}$: $p \nmid |G|$

$\mathcal{O}_{F_\mathfrak{p}}[G] \overset{\text{maximal}}{\subseteq} \mathfrak{A}_{K/F,\mathfrak{p}}$

APPLICATIONS OF MAXIMAL ORDERS

1) p odd, $K/\mathbb{Q}_p$ abelian, K/F tot. ram.
$\Longrightarrow \mathfrak{A}_{K/F}$ maximal, so $\mathcal{O}_K$ free $\mathfrak{A}_{K/F}$
(Lettl '98)

2) link with almost-maximal ramification.
K/F, cycl. gr. G almost-maximal ram. if $e_H \in \mathfrak{A}_{K/F}$ $\forall\, G_{t+1} \subseteq H \subseteq G_t$

REM: $\frac{1}{|H|} \mathrm{Tr}_H \in \mathfrak{A}_{K/F} \iff$

$$\begin{tikzcd} K \arrow[d, dash, "H"] \arrow[dd, dash, bend right, "G"'] \\ M \arrow[d, dash] \\ F \end{tikzcd}$$

$\iff \frac{1}{|H|} \mathrm{Tr}_H(\mathcal{O}_K) \subseteq \mathcal{O}_M$

$\iff \mathrm{Tr}_{K/M}(\mathcal{O}_K) \subseteq |H| \mathcal{O}_M$.

$\iff \mathcal{O}_K \subseteq |H|\, \mathcal{O}_M\, \mathcal{D}_{K/M}^{-1}$ (Serre, "loc. fields", III Prop. 7)

$\iff \mathcal{D}_{K/M} \subseteq |H|\, \mathcal{O}_K$

$\iff v_{\mathcal{O}}(\mathcal{D}_{K/M}) \geq e(K/\mathbb{Q}_p)\, v_p(|H|)$

$\iff \sum_{i=0}^{\infty} (|G_i(K/M)| - 1) \geq e(K/\mathbb{Q}_p)\, v_p(H)$ (Serre, IV Prop. 4).

Berge, 1978. $1 \le t_1 < t_2 < \dots$ 10

t_i ramification jump if $G_{t_i} \supsetneq G_{t_i+1}$

PROP: K/F cycl. ext. of p-adic fields, $\overset{G_1 \text{ cyclic}}{\vee}$,

$F/\mathbb{Q}_p$ unram., $r := [G_0 : G_1]$.

Then $\frac{r}{p-1} \le t_1 \le \frac{rp}{p-1}$

$t_i = \frac{rp_i}{p-1} - \frac{rp}{p-1} + t_1$

PROP: almost-maximal $\iff \exists\, G_i \neq 1$ $e_{G_i} \in \mathfrak{A}_{K/F}$

$\iff t_1 \ge \frac{rp}{p-1} - 1$.

$(r, p) = 1$

PROP: G cyclic group, $|G| = rp^n$,

$F/\mathbb{Q}_p$ unramified, H_i subgroup s.t.

$|H_i| = p^i$, $i = 0, \dots, n$. Then

the maximal order is: $\mathcal{O}_F[G][e_{H_i}]_{i=0,\dots,n}$

COR: K/F a tot. ram. cyclic extension

s.t. $F/\mathbb{Q}_p$ unramified. If K/F

almost-maximally ramified, then $\mathfrak{A}_{K/F}$

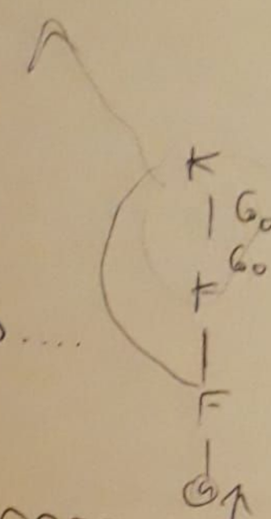
is maximal.

FULL THEOREM: Let L/F a tot ram.

cyclic ext. s.t. $F/\mathbb{Q}_p$ unram.

then O_K free / $\mathfrak{A}_{K/F}$ iff

$$k_1 > \frac{rp}{p-1} - \frac{p^n}{p^{n-1}-1}$$



If not tot. ram., G_0 cyclic

there are some sufficient conditions....

§7 RETURNING TO LOCAL FREENESS: HYBRID ORDERS

$K/\mathbb{Q}$ Gal., Gal. gr. G, p prime.

$\mathfrak{A}_{K/\mathbb{Q},p}$ might not be maximal if $p \mid |G|$

EXAMPLE: $G \cong A_4, S_4$. O_K locally free at $p \neq 2,3$ over $\mathfrak{A}_{K/\mathbb{Q}}$.

At 3:
$$A_4 \cong V_4 \rtimes C_3$$
$$S_4 \cong V_4 \rtimes S_3$$

$G \cong V_4 \rtimes H$ where H is C_3 or S_3.

$V_4 \trianglelefteq G \rightsquigarrow e_{V_4}$ central idempotent

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$$\mathbb{Z}_3[G] \cong e_{V_4}\mathbb{Z}_3[G] \times (1-e_{V_4})\mathbb{Z}_3[G]$$

$$\cong \mathbb{Z}_3[H] \times \underbrace{(1-e_{V_4})\mathbb{Z}_3[G]}$$

<u>FACT</u>: $\uparrow$ is maximal ($\mathbb{Z}_3[G]$ is V_4-hybrid)

$$\mathfrak{A}_{K/\mathbb{Q},3} \cong \underset{\substack{\cup\!\shortmid \\ \mathbb{Z}_3[H]}}{\overset{\substack{\mathbb{Q}_3[H] \\ \cup\!\shortmid}}{\text{something}}} \times (1-e_{V_4})\mathbb{Z}_3[G]$$

$$\text{something} = \mathfrak{A}_{M/\mathbb{Q},3}$$

$$\begin{tikzcd}[row sep=small]
K \arrow[dd, dash, "G"'] \arrow[dr, dash, "V_4"] & \\
& M \arrow[dl, dash, "H"] \\
\mathbb{Q} &
\end{tikzcd}$$

$$\mathcal{O}_{K,3} \cong e_{V_4}\mathcal{O}_{K,3} \oplus (1-e_{V_4})\mathcal{O}_{K,3}$$

$$e_{V_4}\mathcal{O}_{K,3} \cong \mathcal{O}_{M,3}$$

free   free

$\mathcal{O}_M$ is free / $\mathfrak{A}_{M/\mathbb{Q}}$ (H < C_3, S_3)

$\implies$ $\mathcal{O}_{K,3}$ free $\mathfrak{A}_{K/\mathbb{Q},3}$

<u>COR</u>. $K/\mathbb{Q}$ A_4 or S_4 - ext. $\mathcal{O}_K$ free over $\mathfrak{A}_{K/\mathbb{Q}}$ $\overset{cl(\mathbb{Z}[A_4/S_4])=0}{\iff}$ loc. free $\iff$ loc. free at 2.

$$\begin{array}{c} K \\ | \\ F \end{array}$$ $K/\mathbb{Q}$ absolute abelian, K/F p-extension, F field, $\mathfrak{A}_{K/F,p}$ maximal $\implies$ $\mathfrak{A}_{K/F}$ maximal

Theorem 3.6. *Let $K/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong A_4$. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}}$ if and only if 2 is tamely ramified or has full decomposition group.*

2 tame $\Rightarrow$ OK

Theorem 3.7. *Let $K/\mathbb{Q}$ be a Galois extension with $G := \mathrm{Gal}(K/\mathbb{Q}) \cong S_4$. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}}$ if and only if one of the following conditions on $K/\mathbb{Q}$ holds:*

$\exists!\ L$ $|A_4$ $\mathbb{Q}_2$ weakly ram.

(i) *2 is tamely ramified;*
(ii) *2 is weakly ramified and has full decomposition group;*
(iii) *2 has decomposition group equal to the unique subgroup of G of order 12; or*
(iv) *2 is weakly ramified, has decomposition group of order 8 in G, and has inertia subgroup equal to the unique normal subgroup of order 4 in G.*

$K_8/\mathbb{Q}_2$ weakly ram.

Theorem 3.8. *Let $K/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong A_5$. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}}$ if and only if all three of the following conditions on $K/\mathbb{Q}$ hold:*

(i) *2 is tamely ramified;*
(ii) *3 is tamely ramified or not almost-maximally ramified; and*
(iii) *5 is tamely ramified or not almost-maximally ramified.*