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19/04/2021 Explicit Galois module structure of weakly ramified extensions of local fields

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<u>Setup</u> finite Galois extension of local fields

$$G\left(\begin{array}{l} L \\ | \\ K \end{array}\right. \quad \begin{array}{l} O_L \rhd P_L \\ \\ O_K \rhd P_K \end{array} \quad \begin{array}{l} \bar{L} = O_L/P_L \\ | \\ \bar{K} = O_K/P_L \end{array}$$

Residue fields assumed to be finite.

Can just think of Galois extn of p-adic fields.

<u>Concerned with</u>

(i) P_L^n as an $O_K[G]$-module

(ii) O_L as an $\mathfrak{A}_{L/K}$-module $\quad \mathfrak{A}_{L/K} = \{x \in K[G] : x O_L \subseteq O_L\}$.

<u>Ramification groups</u>

For $i \geq -1$ $\quad G_i := \{\sigma \in G : (\sigma - 1)(O_L) \subseteq P_L^{i+1}\}$

Hence

L/K unramified $\Leftrightarrow G_0 = 1$

L/K tamely ramified $\Leftrightarrow G_1 = 1$

L/K weakly ramified $\Leftrightarrow G_2 = 1$.

<u>Hilbert's formula</u>

$$v_L(\mathcal{D}_{L/K}) = \sum_{i=0}^{\infty} (|G_i| - 1) \quad \text{(note sum is finite).}$$

$$\mathcal{D}_{L/K}^{-1} = \{x \in L : Tr_{L/K}(xy) \in O_K \ \forall y \in O_L\}$$

Theorem Suppose L/K is (at most) tamely ramified.
Then $\forall n \in \mathbb{Z}$ $\quad \mathfrak{p}_L^n$ free over $\mathcal{O}_K[G]$

Attributed to Noether*
Kawamoto (1986) constructed explicit generators.

Unramified case

$$G\left(\begin{array}{c} L \\ | \\ K \end{array}\right. \qquad G\left(\begin{array}{c} \bar{L} \\ | \\ \bar{K} \end{array}\right.$$

By Normal Basis Theorem $\exists \bar{\beta} \in \bar{L}$ s.t. $\bar{L} = \bar{K}[G] \cdot \bar{\beta}$.
Using Nakayama's Lemma, for any lift $\beta \in \mathcal{O}_L$ of $\bar{\beta}$,
$\mathcal{O}_L = \mathcal{O}_K[G] \cdot \beta$ (Can make into an iff statement).
(note cyclic $\Rightarrow$ free in this situation)

Totally & tamely ramified case

$G\left(\begin{array}{c} L \\ | \\ K \end{array}\right.$ Let $e = [L:K]$. $\exists\, \pi_L$ & π_K uniformizers s.t. $\pi_L^e = \pi_K$ & $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$

Let $\alpha \in \mathcal{O}_L$. Then $\alpha = u_0 + u_1 \pi_L + \dots + u_{e-1}\pi_L^{e-1}$ for some $u_i \in \mathcal{O}_K$.
Moreover, $\quad \mathcal{O}_L = \mathcal{O}_K[G] \cdot \alpha \iff u_i \in \mathcal{O}_K^\times \;\; \forall i$

Proof Use that π_L is a Kummer generator; determinant calculation. □

Idea of Kawamoto's proof: "glue" the two cases together.

Weakly ramified case (will give explicit examples later).

Ullom
(i) if $\exists n \in \mathbb{Z}$ s.t. $\mathcal{B}_L^n$ free over $\mathcal{O}_K[G]$ then L/K weakly ramified
(ii) if L/K totally & weakly ramified then P_L free over $\mathcal{O}_K[G]$.

Köck $\mathcal{B}_L^n$ free over $\mathcal{O}_K[G]$
$\iff$ L/K weakly ramified and $n \equiv 1 \mod |G_1|$

Proof uses cohomological triviality argument
(Erez's work on square root of inverse different uses similar ideas).

Theorem 1 (HJ) L/K weakly ramified. Let $n \in \mathbb{Z}$ s.t. $n \equiv 1 \mod |G_1|$.
Then can explicitly construct ε s.t. $\mathcal{B}_L^n = \mathcal{O}_K[G]\cdot\varepsilon$.

Theorem 2 (HJ) L/K weakly ramified. π_K any uniformizer of K.
Then $\mathfrak{A}_{L/K} = \mathcal{O}_K[G][\pi_K^{-1} Tr_{G_0}]$ and
$P_L = \mathcal{O}_K[G]\cdot\varepsilon \implies \mathcal{O}_L = \mathfrak{A}_{L/K}\cdot\varepsilon$

Idea of proof of thm 1:
Explicitly construct generators in following cases:
(i) unramified ✓
(ii) totally and tamely ramified ✓
(iii) totally and weakly ramified p-extn
Then use "splitting lemma".
"Glue" generators together.
Take trace.
This is a generalisation of Kawamoto's approach.

<u>Idea</u> of proof of Thm 2

Show that $\mathcal{O}_K[G][\pi_K^{-1}\mathrm{Tr}_{G_0}] \subseteq \mathfrak{A}_{L/K}$
Let ε be a free generator of P_L over $\mathcal{O}_K[G]$ (e.g. as in Thm 1)
Then

$$\begin{array}{c} \\ (i) \\ \\ \end{array}\left[\begin{array}{ccccc} \mathcal{O}_K[G][\pi_K^{-1}\mathrm{Tr}_{G_0}]\cdot\varepsilon & \subseteq & \mathfrak{A}_{L/K}\cdot\varepsilon & \subseteq & \mathcal{O}_L \\ \cup\!\!\mid & & & & \cup\!\!\mid \\ P_L = \mathcal{O}_K[G]\cdot\varepsilon & & & & P_L \end{array}\right] (ii) \qquad (*)$$

Show indices (i) & (ii) are equal, which forces equality in (*). $\square$

<u>Totally & weakly ramified p-extn</u>

<u>Thm</u> L/K totally & weakly ramified p-ext $(p = \mathrm{char}\,\bar{K} > 0)$
(i) G elementary abelian p-group (standard)
(ii) $\mathfrak{B}_L^n$ free over $\mathcal{O}_K[G] \Leftrightarrow n \equiv 1 \bmod |G|$ (known by Köck)
(iii) Suppose $n \equiv 1 \bmod |G|$.
Then $\exists\, \delta \in L$ free gen of $\mathfrak{B}_L^n$ over $\mathcal{O}_K[G] \Leftrightarrow v_L(\delta) = n$
(already shown by others: Vostokov, Vinatier (+Byott), Byott & Elder).

<u>Idea of proof</u> Elementary (no cohomology!)
- Use Hilbert's formula to compute different of L/K
- Obtain formula for $\mathrm{Tr}_{L/K}(P_L^n)$
- "Mod out" by $\mathfrak{B}_K$ $\rightsquigarrow$ work over $\bar{K}[G]$
- Use (minor variant of) result of Childs
- Lift using Nakayama's Lemma.

<u>Example</u>

$$\begin{array}{ccl} & K(\zeta_{p^2}) & \\ & | & \\ & L & \\ \text{weakly ram} & | & \text{degree } p \\ & K & \\ & | & \text{unramified} \\ & \mathbb{Q}_p & \end{array}$$

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Totally & weakly ramified extensions of arbitrary degree

$$\begin{tikzcd} L \arrow[d, dash] \\ K \end{tikzcd} \Big) \; I = G_0$$

$I = W \rtimes C$ by Schur-Zassenhaus

W: wild inertia $= G_1 =$ elementary abelian p-group. C: cyclic

$$\begin{tikzcd} & L \arrow[dl, dash, "W"'] \arrow[dd, dash, "I"] \arrow[dr, dash, "C"] & \\ E \arrow[dr, dash, "C"'] & & F \arrow[dl, dash] \\ & K & \end{tikzcd}$$

L/E, F/K totally & weakly ramified p-extns

L/F, E/K totally & tamely ramified extns

Define r by $|W| = p^r$, let $c = |C|$.

By Bézout $\exists\, a, b \in \mathbb{Z}$ s.t. $ap^r + bc = 1$.

Choose uniformizers π_E & π_K s.t. $\pi_E^c = \pi_K$, π_F any uniformizer of F.

Let $\alpha = 1 + \pi_E + \pi_E^2 + \dots + \pi_E^{c-1}$.

<u>Proposition</u> (special case for simplicity)

$\pi_F^a \cdot \pi_E^b \cdot \alpha$ is a free gen of P_L over $O_K[I]$.

<u>Proof</u>

(i) $v_L(\pi_F^b \pi_E^a) = 1$ so $P_L = O_E[W] \cdot (\pi_F^b \pi_E^a)$.

(ii) $\pi_E^a O_E = O_K[C] \cdot (\pi_E^a \alpha)$.

Do explicit calculation using semidirect product and that $\pi_F \in F = L^C$, $\pi_E, \alpha \in E = L^W$

Starts like $\quad W = \{\tau_i\} \quad C = \{\sigma_j\}$

$$\left.\begin{aligned} P_L &= O_E[W] \cdot (\pi_F^b \pi_E^a) \quad \text{by (i)} \\ &= \bigoplus_i \tau_i (\pi_F^b \pi_E^a) O_E \\ &= \bigoplus_i \tau_i (\pi_F^b) \pi_E^a O_E \\ &= \quad \dots \quad (\text{use (ii)}) \\ &= \bigoplus_i \bigoplus_j \tau_i \sigma_j (\pi_F^b \pi_E^a \cdot \alpha) \end{aligned}\right\} \text{optional}$$

$$= O_K[I] \cdot (\pi_F^b \pi_E^a \alpha)$$

<u>Rmk</u> If L/K abelian, totally & wildly ramified, not of p-power degree then L/K cannot be weakly ramified.

E.g.

$$\begin{tikzcd}[row sep=small] \mathbb{Q}_p(\zeta_{p^2}) \arrow[d, dash, "p-1 \text{ tame}"] \\ K \arrow[d, dash, "p \quad \text{weakly ramified}"] \\ \mathbb{Q}_p \end{tikzcd}$$

not weakly ramified

<u>Note</u> typo in Rmk 6.2 of paper!

However $\mathbb{Q}_3(\zeta_3, \sqrt[3]{2})/\mathbb{Q}_3$ has Galois group $\cong S_3$ and is totally and weakly ramified.

Doubly split extensions

L/K finite Galois extn of local fields
$G = \mathrm{Gal}(L/K)$ $\quad I = G_0 \quad W = G_1$
We say L/K is:

(i) split wrt inertia if $G = I \rtimes U$ for some (cyclic U)
(so L/L^U is unramified)

(ii) split wrt wild inertia if $G = W \rtimes T$ for some T
(so L/L^T tamely ramified)

(iii) doubly split if $\exists C \leq I$ s.t (i) & (ii) hold with choices of U & T s.t. $I = W \rtimes C$ and $T = C \rtimes U$ and so

$$G = W \rtimes T = W \rtimes (C \rtimes U) = (W \rtimes C) \rtimes U = I \rtimes U$$

<u>Rmk</u> Automatic in the case of totally ramified extns by Schur-Zassenhaus

<u>Idea</u> Glue generators together for doubly split extensions

<u>Lemma</u> L/K finite Galois extn of local fields
Let K'/K be the unique unramified extn of degree $[L:K]$.
Let $L' = LK'$. Then

(i) L'/K Galois
(ii) $\mathrm{Gal}(L'/K')$ is the inertia subgroup
(iii) L'/K doubly split
(iv) if L/K weak ram then L'/K weak ram.

<u>Proof</u> Group theory.

<u>Idea</u> of construction of $\mathcal{E}$ for general L/K weakly ramified.
Construct gen $\mathcal{E}'$ for L'/K (by glueing).
Then $\mathcal{E} := \mathrm{Tr}_{L'/L}(\mathcal{E})$ is gen for L/K. □