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19/04/2021 Explicit Galois module structure of weakly ramified extensions of local fields

Thank organisers: Ilaria, Lorenzo, Fabio and Elena

Reference: Proc AMS 143 (2015) 5059-5071.

Setup finite Galois extension of local fields

$$G \begin{pmatrix} L \\ | \\ K \end{pmatrix} \quad \begin{matrix} \mathcal{O}_L \supset \mathcal{P}_L \\ \\ \mathcal{O}_K \supset \mathcal{P}_K \end{matrix} \quad \begin{matrix} \bar{L} = \mathcal{O}_L / \mathcal{P}_L \\ | \\ \bar{K} = \mathcal{O}_K / \mathcal{P}_K \end{matrix} \quad \begin{matrix} \text{Residue fields} \\ \text{assumed to be finite.} \end{matrix}$$

Can just think of Galois extn of p-adic fields.

Concerned with

(i) \mathcal{P}_L^n as an $\mathcal{O}_K[G]$ -module

(ii) \mathcal{O}_L as an $\mathcal{U}_{L/K}$ -module $\mathcal{U}_{L/K} = \{x \in K[G] : x\mathcal{O}_L \subseteq \mathcal{O}_L\}$

Ramification groups

For $i \geq -1$ $G_i := \{g \in G : (g-1)(\mathcal{O}_L) \subseteq \mathcal{P}_L^{i+1}\}$

Hence

L/K unramified $\Leftrightarrow G_0 = 1$

L/K tamely ramified $\Leftrightarrow G_1 = 1$

L/K weakly ramified $\Leftrightarrow G_2 = 1$.

Hilbert's formula

$$v_L(\mathcal{D}_{L/K}^{-1}) = \sum_{i=0}^{\infty} (|G_i| - 1) \quad (\text{note sum is finite}).$$

$$\mathcal{D}_{L/K}^{-1} = \{x \in L : \text{Tr}_{L/K}(xy) \in \mathcal{O}_K \quad \forall y \in \mathcal{O}_L\}$$

(2)

Theorem Suppose L/K is (at most) tamely ramified.
Then $\forall n \in \mathbb{Z}$ \mathcal{O}_L^n free over $\mathcal{O}_K[G]$

Attributed to Noether*

Kawamoto (1986) constructed explicit generators.

Unramified case

$$G \begin{pmatrix} L \\ | \\ K \end{pmatrix} \quad G \begin{pmatrix} \bar{L} \\ | \\ \bar{K} \end{pmatrix}$$

By Normal Basis Theorem $\exists \bar{\beta} \in \bar{L}$ s.t. $\bar{L} = \bar{K}[G] \cdot \bar{\beta}$.

Using Nakayama's Lemma, for any lift $\beta \in \mathcal{O}_L$ of $\bar{\beta}$,

$\mathcal{O}_L = \mathcal{O}_K[G] \cdot \beta$ (can make into an iff statement).
(note cyclic \Rightarrow free in this situation)

Totally & tamely ramified case

$$G \begin{pmatrix} L \\ | \\ K \end{pmatrix} \quad \text{Let } e = [L:K]. \exists \pi_L \text{ \& } \pi_K \text{ uniformizers s.t. } \pi_L^e = \pi_K \text{ \& } \mathcal{O}_L = \mathcal{O}_K[\pi_L]$$

Let $\alpha \in \mathcal{O}_L$. Then $\alpha = u_0 + u_1 \pi_L + \dots + u_{e-1} \pi_L^{e-1}$ for some $u_i \in \mathcal{O}_K$.

Moreover, $\mathcal{O}_L = \mathcal{O}_K[G] \cdot \alpha \iff u_i \in \mathcal{O}_K^\times \forall i$

Proof Use that π_L is a Kummer generator; determinant calculation. \square

Idea of Kawamoto's proof: "glue" the two cases together.

(3)

Weakly ramified case (will give explicit examples later).

Ullom

- (i) if $\exists n \in \mathbb{Z}$ s.t. B_L^n free over $O_K[G]$ then L/K weakly ramified
- (ii) if L/K totally & weakly ramified then P_L free over $O_K[G]$.

Köck B_L^n free over $O_K[G]$

$\Leftrightarrow L/K$ weakly ramified and $n \equiv 1 \pmod{|G|}$

Proof uses cohomological triviality argument

(Erez's work on square root of inverse different uses similar ideas).

Theorem 1 (HJ) L/K weakly ramified. Let $n \in \mathbb{Z}$ s.t. $n \equiv 1 \pmod{|G|}$.

Then can explicitly construct E s.t. $P_L^n = O_K[G] \cdot E$.

Theorem 2 (HJ) L/K weakly ramified. π_K any uniformizer of K .

Then $\text{ut}_{L/K} = O_K[G][\pi_K^{-1} \text{Tr}_{G_0}]$ and $P_L = O_K[G] \cdot E$

$P_L = O_K[G] \cdot E \Rightarrow O_L = \text{ut}_{L/K} \cdot E$

Idea of proof of Thm 1:

Explicitly construct generators in following cases:

(i) unramified ✓

(ii) totally and tamely ramified ✓

(iii) totally and weakly ramified p -extr

then use "splitting lemma".

"Glu" generators together.

Take trace.

This is a generalisation of Kawamoto's approach.

(4)

Idea of proof of Thm 2

Show that $O_K[G][\pi_K^{-1} \text{Tr}_{G_0}] \subseteq U_{L/K}$

Let ϵ be a free generator of P_L over $O_K[G]$ (eg. as in Thm 1)

Then

$$(i) \left[\begin{array}{c} O_K[G][\pi_K^{-1} \text{Tr}_{G_0}] \cdot \epsilon \subseteq U_{L/K} \cdot \epsilon \subseteq O_L \\ \cup \\ P_L = O_K[G] \cdot \epsilon \end{array} \right] (ii) \quad (*)$$

Show indices (i) & (ii) are equal, which forces equality in (*). \square

Totally & weakly ramified p-extn

Thm L/K totally & weakly ramified p-ext ($p = \text{char } \bar{K} > 0$)

(i) G elementary abelian p-group (standard)

(ii) B_L^n free over $O_K[G] \Leftrightarrow n \equiv 1 \pmod{|G|}$ (known by Kock)

(iii) Suppose $n \equiv 1 \pmod{|G|}$.

Then $\exists \delta \in L$ free gen of B_L^n over $O_K[G] \Leftrightarrow v_L(\delta) = n$

(already shown by others: Vostokov, Vinatier (+Byott), Byott & Elder)

Idea of proof Elementary (no cohomology!)

- Use Hilbert's formula to compute different of L/K !

- Obtain formula for $\text{Tr}_{L/K}(P_L^n)$

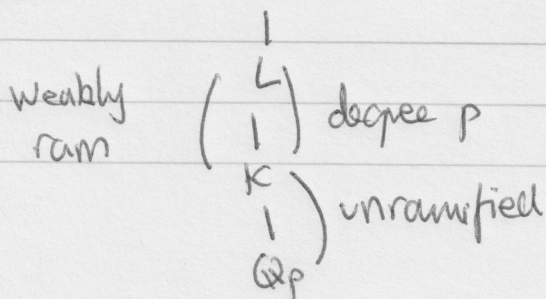
- "Mod out" by $B_x \rightarrow$ work over $\bar{K}[G]$

- Use (minor variant of) result of Childs

- Lift using Nakayama's Lemma.

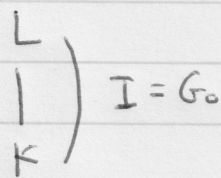
Example

$K(\mathbb{Z}_p^2)$



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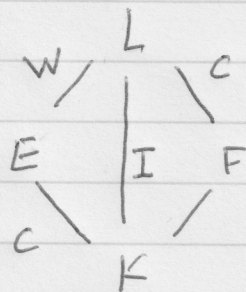
Totally & weakly ramified extensions of arbitrary degree



$I = W \times C$ by Schur-Zassenhaus

$\uparrow \quad \uparrow$ cyclic

wild inertia = G_1 = elementary abelian p -group.



$L/E, F/K$ totally & weakly ramified p -extns
 $L/F, E/K$ totally & tamely ramified extns

Define r by $|W| = p^r$, let $c = |C|$.
 By Bézout $\exists a, b \in \mathbb{Z}$ s.t. $ap^r + bc = 1$.

Choose uniformizers π_E & π_K s.t. $\pi_E^c = \pi_K$, π_F any uniformizer of F .
 Let $\alpha = 1 + \pi_E + \pi_E^2 + \dots + \pi_E^{c-1}$.

Proposition (special case for simplicity)

$\pi_F^a \cdot \pi_E^b \cdot \alpha$ is a free gen of P_L over $O_K[I]$.

Proof

(i) $v_L(\pi_F^b \pi_E^a) = 1$ so $P_L = O_E[W] \cdot (\pi_F^b \pi_E^a)$.

(ii) $\pi_E^a O_E = O_K[C] \cdot (\pi_E^a \alpha)$.

Do explicit calculation using semidirect product and that $\pi_F \in F = L^C$, $\pi_E, \alpha \in E = L^W$

Starts like $W = \{\tau_i\}$ $C = \{\sigma_j\}$

$P_L = O_E[W] \cdot (\pi_F^b \pi_E^a)$ by (i)

$= \bigoplus_i \tau_i (\pi_F^b \pi_E^a) O_E$

$= \bigoplus_i \tau_i (\pi_F^b) \pi_E^a O_E$

$= \dots$ (use (ii))

$= \bigoplus_i \bigoplus_j \tau_i \sigma_j (\pi_F^b \pi_E^a \alpha)$

$= O_K[I] \cdot (\pi_F^b \pi_E^a \alpha)$

} optional

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Rmk If L/K abelian, totally & wildly ramified, not of p -power degree then L/K cannot be weakly ramified.

E.g. $\left(\begin{array}{c} \mathbb{Q}_p(S_{p^2}) \\ | \\ K \\ | \\ \mathbb{Q}_p \end{array} \right)_{p-1}$ tame
 not weakly ramified
 $\left(\begin{array}{c} | \\ | \\ \mathbb{Q}_p \end{array} \right)_p$ weakly ramified

Note typo in Rmk 6.2 of paper!

However $\mathbb{Q}_3(\zeta_3, \sqrt[3]{2})/\mathbb{Q}_3$ has Galois group $\cong S_3$ and is totally and weakly ramified.

Doubly split extensions

L/K finite Galois extn of local fields

$G = \text{Gal}(L/K)$ $I = G_0$ $W = G_1$

We say L/K is:

- (i) split wrt inertia if $G = I \rtimes U$ for some (cyclic U)
(so L/L^U is unramified)
- (ii) split wrt wild inertia if $G = W \rtimes T$ for some T
(so L/L^T tamely ramified)
- (iii) doubly split if $\exists C \leq I$ s.t. (i) & (ii) hold with choices of $U \leq T$ s.t. $I = W \rtimes C$ and $T = C \rtimes U$
and so

$$G = W \rtimes T = W \rtimes (C \rtimes U) = (W \rtimes C) \rtimes U = I \rtimes U$$

Rmk Automatic in the case of totally ramified extns by Schur-Zassenhaus

Idea Glue generators together for doubly split extensions

(7)

Lemma L/K finite Galois extn of local fields

Let K'/K be the unique unramified extn of degree $[L:K]$.

Let $L' = LK'$. Then

- (i) L'/K Galois
- (ii) $\text{Gal}(L'/K')$ is the inertia subgroup
- (iii) L'/K doubly split
- (iv) if L/K weak ram then L'/K weak ram.

Proof Group theory.

Idea of construction of E for general L/K weakly ramified.

Construct gen E' for L'/K (by gluing).

Then $E := \text{Tr}_{L'/L}(E')$ is gen for L/K . \square