LECTURE 3

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1. Brief recall on the tame case

We recall the following.

Theorem 1.1. *Let K/F be a finite Galois extension of number fields. Let $G = \mathrm{Gal}(K/F)$ and let $\mathfrak{p}$ be a prime of F that is tamely ramified in K/F. Then $\mathcal{O}_{K,\mathfrak{p}}$ is free over $\mathfrak{A}_{K/F,\mathfrak{p}} = \mathcal{O}_{F_\mathfrak{p}}[G]$.*

Recall that if K/F is a tame Galois extension of number fields with Galois group G, then $\mathcal{O}_K$ defines a class in $\mathrm{Cl}(\mathcal{O}_F[G])$. In particular if $\mathrm{Cl}(\mathcal{O}_F[G]) = 0$ and $F[G]$ has locally free cancellation, then K/F automatically has normal integral basis. If $F = \mathbb{Q}$, $\mathrm{Cl}(\mathbb{Z}[G]) = 0$ is only true if G is among certain abelian groups, certain dihedral groups, A_4, S_4, A_5 (see [RU74] and [EH79]). In such cases we automatically have locally free cancellation. However we have already seen the following consequence of Fröhlich's conjecture, which was proved by Taylor [Tay81] and tells us much more without assuming the locally free class group is trivial.

Theorem 1.2. *Let $K/\mathbb{Q}$ be a finite tame Galois extension of $\mathbb{Q}$ with Galois group G. Suppose G is abelian, dihedral, of odd order, alternating or symmetric. Then $K/\mathbb{Q}$ has a normal integral basis.*

Indeed, in the hypotheses of Theorem 1.2 G has no irreducible symplectic character, which means that the class of $\mathcal{O}_K$ in $\mathrm{Cl}(\mathbb{Z}[G])$ is trivial, and locally free cancellation, which implies that $\mathcal{O}_K$ is free as a $\mathbb{Z}[G]$-module. Note that this in particular generalizes Hilbert-Speiser theorem, and that it permits us to conclude that a sufficently nice (i.e. whose Galois group does not have to do with quaternions) tame non-abelian extension of $\mathbb{Q}$ has normal integral basis.

1.1. **Clean orders.**

Definition 1.3. Let R be a Dedekind domain with field of fractions F, let G be a finite group and let Λ be an R-order in $F[G]$. We say that Λ is a clean order if it satisfies the following property: if M is a projective Λ-lattice which spans a free $F[G]$-module, then M is a free Λ-lattice.

Example 1.4. The following are clean:

- $R[G]$ when R is a DVR. This is Swan's theorem, for instance see [CR81, Theorem (32.1)].
- Whenever G is abelian and R is a discrete valuation ring with characteristic zero and finite residue field: due to Hattori [Hat65], also see [Rog70, IX Corollary 1.5].

From Swan's theorem we have the following.

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Corollary 1.5. *Let K/F be a Galois extension of number fields with Galois group G. Then K/F is tame if and only if $\mathcal{O}_K$ is a projective $\mathcal{O}_F[G]$-module.*

Now we will review what we know in the wild case, starting from the framework of p-adic fields.

2. Freeness results for Galois extensions of p-adic fields

Let p be a rational prime. We start with a consequence of Leopoldt's theorem.

Theorem 2.1. [Leo59] *Let $K/\mathbb{Q}_p$ be a finite abelian extension. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}}$.*

Lettl further generalized the result.

Theorem 2.2. [Let98] *Let K/F be an extension of p-adic fields such that $K/\mathbb{Q}_p$ is a finite abelian extension. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/F}$.*

Now we shall consider the non-abelian setting. In this framework the two first important contributions are due to Bergé and Martinet, and later to Jaulent.

Theorem 2.3. [Ber72] *Let $K/\mathbb{Q}_p$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong D_{2p}$. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}_p}$.*

Theorem 2.4. [Mar72] *Let $K/\mathbb{Q}_p$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong Q_8$. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}_p}$.*

Theorem 2.5. [Jau81] *Let p, n and r be positive integers such that p is an odd prime, n divides $p-1$ and r is a primitive nth root modulo p. Let G be the metacyclic group with the following structure:*

$$G = \langle x, y : x^p = 1, y^n = 1, yxy^{-1} = x^r \rangle \cong C_p \rtimes C_n. \tag{2.1}$$

Let $K/\mathbb{Q}_p$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong G$. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}_p}$.

Remark 2.6. In the special case $n = 2$, the group G of (2.1) is dihedral of order $2p$.

Considering a generic base field, Johnston obtained the following.

Theorem 2.7. [Joh15] *Let K/F be a weakly ramified finite Galois extension of p-adic fields and let $G = \mathrm{Gal}(K/F)$. Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/F}$. Moreover, if K/F is both wildly and weakly ramified then $\mathfrak{A}_{K/F} = \mathcal{O}_F[G][\pi_F^{-1}\mathrm{Tr}_{G_0}]$, where π_F is a uniformizer of $\mathcal{O}_F$ and $\mathrm{Tr}_{G_0} = \sum_{\gamma \in G_0} \gamma$ is the sum of the elements of the inertia group G_0.*

One may suspect that it is always the case that, in a Galois extension of p-adic fields, the ring of integers is free over the associated order, as happens if we further assume tame ramification. We will see in a moment that this is not the case. First we define almost-maximal ramification.

For a subgroup H of G define $\mathrm{Tr}_H = \sum_{h \in H} h \in F[G]$ and $e_H = \frac{1}{|H|}\mathrm{Tr}_H \in F[G]$. Note that e_H is an idempotent. We say that K/F has almost-maximal ramification if $e_H \in \mathfrak{A}_{K/F}$ for every subgroup H of G such that $G_{t+1} \subseteq H \subseteq G_t$ for some $t \geq 1$.

Theorem 2.8. [Ber79, Proposition 7] *Let K/F be a finite dihedral extension of p-adic fields such that $F/\mathbb{Q}_p$ is unramified. Let $G = \mathrm{Gal}(K/F)$. Then $\mathcal{O}_K$ is projective over $\mathfrak{A}_{K/F}$ if and only if $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/F}$ if and only if either*

(i) *the ramification is almost-maximal, in which case $\mathfrak{A}_{K/F} = \mathcal{O}_F[G][\{e_{G_t}\}_{t \geq 1}]$, or*

(ii) *the ramification is not almost-maximal and the inertia subgroup* G_0 *is dihedral of order* $2p$, *in which case* $\mathfrak{A}_{K/F} = \mathcal{O}_F[G][2e_{G_0}]$.

More results concerning cyclic of prime order extensions and extensions with cyclic inertia group will follow later on.

3. Freeness results for Galois extensions of number fields

We start by recalling Leopoldt's theorem, which generalizes Hilbert-Speiser theorem to wildly ramified abelian extensions of $\mathbb{Q}$.

Theorem 3.1. [Leo59] *Let* $K/\mathbb{Q}$ *be a finite abelian extension. Then* $\mathcal{O}_K$ *is free over* $\mathfrak{A}_{K/\mathbb{Q}}$.

Leopoldt also specified a generator and the associated order; Lettl [Let90] gave a simplified and more explicit proof of the same result.

In the last talk we have seen a partial proof in the case K has odd conductor or is imaginary.

We also have the following result of Bergé.

Theorem 3.2. [Ber72] *Let* p *be a prime and let* $K/\mathbb{Q}$ *be a dihedral extension of degree* $2p$. *Then* $\mathcal{O}_K$ *is free over* $\mathfrak{A}_{K/\mathbb{Q}}$.

Now let $K/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong Q_8$, the quaternion group of order 8. Suppose that $K/\mathbb{Q}$ is tamely ramified. Martinet [Mar71] gave two examples of such extensions without and one with a normal integral basis. Moreover, Fröhlich [Frö72] showed that both possibilities occur infinitely often. By contrast, in the case that $K/\mathbb{Q}$ is wildly ramified, we have the following result of Martinet.

Theorem 3.3. [Mar72] *Let* $K/\mathbb{Q}$ *be a wildly ramified Galois extension with* $\mathrm{Gal}(K/\mathbb{Q}) \cong Q_8$. *Then* $\mathcal{O}_K$ *is free over* $\mathfrak{A}_{K/\mathbb{Q}}$.

For other global freeness results we recently obtained the following results.

Theorem 3.4. *Let* n *be a positive integer and let* $p \geq 5$ *be a regular prime number such that the class number of* $\mathbb{Q}(\zeta_{p^n})^+$ *is* 1. *Let* $K/\mathbb{Q}$ *be a dihedral extension of degree* $2p^n$. *Then* $\mathcal{O}_K$ *is free over* $\mathfrak{A}_{K/\mathbb{Q}}$ *if and only if the ramification index of* p *in* $K/\mathbb{Q}$ *is a power of* p.

Corollary 3.5. *Let* $K/\mathbb{Q}$ *be a dihedral extension of degree* $2p^n$ *where* (p, n) *is* $(5, 2)$, $(5, 3)$, $(7, 2)$ *or* $(11, 2)$. *Then* $\mathcal{O}_K$ *is free over* $\mathfrak{A}_{K/\mathbb{Q}}$ *if and only if the ramification index of* p *in* $K/\mathbb{Q}$ *is a power of* p.

Similar but more complicated results hold when $p = 2$ or 3.

Theorem 3.6. *Let* $K/\mathbb{Q}$ *be a Galois extension with* $\mathrm{Gal}(K/\mathbb{Q}) \cong A_4$. *Then* $\mathcal{O}_K$ *is free over* $\mathfrak{A}_{K/\mathbb{Q}}$ *if and only if* 2 *is tamely ramified or has full decomposition group.*

Theorem 3.7. *Let* $K/\mathbb{Q}$ *be a Galois extension with* $G := \mathrm{Gal}(K/\mathbb{Q}) \cong S_4$. *Then* $\mathcal{O}_K$ *is free over* $\mathfrak{A}_{K/\mathbb{Q}}$ *if and only if one of the following conditions on* $K/\mathbb{Q}$ *holds:*

(i) 2 *is tamely ramified;*
(ii) 2 *is weakly ramified and has full decomposition group;*
(iii) 2 *has decomposition group equal to the unique subgroup of* G *of order* 12*; or*
(iv) 2 *is weakly ramified, has decomposition group of order* 8 *in* G, *and has inertia subgroup equal to the unique normal subgroup of order* 4 *in* G.

Theorem 3.8. *Let $K/\mathbb{Q}$ be a Galois extension with* $\mathrm{Gal}(K/\mathbb{Q}) \cong A_5$. *Then $\mathcal{O}_K$ is free over $\mathfrak{A}_{K/\mathbb{Q}}$ if and only if all three of the following conditions on $K/\mathbb{Q}$ hold:*

(i) 2 *is tamely ramified;*
(ii) 3 *is tamely ramified or not almost-maximally ramified; and*
(iii) 5 *is tamely ramified or not almost-maximally ramified.*

4. Local freeness results for Galois extensions of number fields

Let K/F be a finite Galois extension of number fields and let $\mathfrak{p}$ be a prime of F. We recall that $\mathcal{O}_K$ is locally free at $\mathfrak{p}$ over $\mathfrak{A}_{K/F}$ if $\mathcal{O}_{K,\mathfrak{p}} := \mathcal{O}_{F_\mathfrak{p}} \otimes_{\mathcal{O}_F} \mathcal{O}_K$ is free over $\mathfrak{A}_{K/F,\mathfrak{p}} := \mathcal{O}_{F_\mathfrak{p}} \otimes_{\mathcal{O}_F} \mathfrak{A}_{K/F}$.

Theorem 4.1. [Let98] *Let K/F be an extension of number fields such that $K/\mathbb{Q}$ is a finite abelian extension. Then $\mathcal{O}_K$ is locally free over $\mathfrak{A}_{K/F}$.*

Theorem 4.2. [Jau81] *Let $K/\mathbb{Q}$ be a Galois extension such that* $\mathrm{Gal}(K/\mathbb{Q})$ *is metacyclic of type (2.1). Then $\mathcal{O}_K$ is locally free over $\mathfrak{A}_{K/\mathbb{Q}}$.*

Theorem 4.3. [Ber79, Théorème] *Let $K/\mathbb{Q}$ be a finite dihedral extension and let $G =$* $\mathrm{Gal}(K/\mathbb{Q})$. *Let p be an odd rational prime that is wildly ramified in $K/\mathbb{Q}$ and let N be the unique cyclic subgroup of G of index* 2. *Then $\mathcal{O}_{K,p}$ is projective over $\mathfrak{A}_{K/\mathbb{Q},p}$ if and only if $\mathcal{O}_{K,p}$ is free over $\mathfrak{A}_{K/\mathbb{Q},p}$ if and only if one of the following conditions holds:*

(i) *p is almost-maximally ramified in $K/\mathbb{Q}$ and $G_1 \subseteq N$, in which case*

$$\mathfrak{A}_{K/\mathbb{Q},p} = \mathbb{Z}_p[G][\{e_{G_t}\}_{t\geq 1}], \text{ or}$$

(ii) *p is not almost-maximally ramified, $|G_0| = 2p$ and $[G : G_0] \mid 2$, in which case*

$$\mathfrak{A}_{K/\mathbb{Q},p} = \mathbb{Z}_p[G][e_{G_0}].$$

Remark 4.4. In fact, Theorem 4.3 is [Ber79, Théorème] specialised to the case that p is odd and the base field is $\mathbb{Q}$; the more general statement is somewhat more complicated.

5. More on local freeness

We start by mentioning a slightly more general definition of associated order. Let R be a Dedekind domain with field of fractions F. Let G be a finite group. Let M be a full R-lattice in $F[G]$. The associated order of M in $F[G]$ is defined to be

$$\mathfrak{A}(F[G], M) = \{\lambda \in F[G] : \lambda M \subseteq M\}.$$

So with the previous notation we have $\mathfrak{A}_{K/F} = \mathfrak{A}(F[G], \mathcal{O}_K)$. Also in this general case we have that $\mathfrak{A}(F[G], M)$ is an R-order. In particular, it is the largest order Λ over which M has a structure of Λ-module, and with the same proof it is the only order over which M can possibly be free.

Now let K/F be a finite Galois extension of number fields and let $G = \mathrm{Gal}(K/F)$. Let $\mathfrak{p}$ be a prime of F. Then we have decompositions

$$K_\mathfrak{p} = F_\mathfrak{p} \otimes_F K \cong \prod_{\mathfrak{P}'|\mathfrak{p}} K_{\mathfrak{P}'} \quad \text{and} \quad \mathcal{O}_{K,\mathfrak{p}} = \mathcal{O}_{F_\mathfrak{p}} \otimes_{\mathcal{O}_F} \mathcal{O}_K \cong \prod_{\mathfrak{P}'|\mathfrak{p}} \mathcal{O}_{K'_\mathfrak{P}},$$

where $\{\mathfrak{P}' \mid \mathfrak{p}\}$ consists of the primes of $\mathcal{O}_K$ above $\mathfrak{p}$ (see [FT93, p. 109]). Fix a prime $\mathfrak{P}$ above $\mathfrak{p}$ and let D be its decomposition group in G. Then as G acts transitively on $\{\mathfrak{P}' \mid \mathfrak{p}\}$ we have

$$K_\mathfrak{p} \cong \prod_{s\in G/D} sK_\mathfrak{P} \quad \text{and} \quad \mathcal{O}_{K,\mathfrak{p}} \cong \prod_{s\in G/D} s\mathcal{O}_{K_\mathfrak{P}},$$

where the products run over a complete system of representatives of the left cosets G/D. Hence

$$\mathcal{O}_{K,\mathfrak{p}} \cong \mathrm{Ind}_D^G \mathcal{O}_{K_\mathfrak{P}} := \mathcal{O}_{F_\mathfrak{p}}[G] \otimes_{\mathcal{O}_{F_\mathfrak{p}}[D]} \mathcal{O}_{K_\mathfrak{P}},$$

and

$$\mathfrak{A}_{K/F,\mathfrak{p}} = \mathfrak{A}(F[G], \mathcal{O}_K)_\mathfrak{p} \cong \mathfrak{A}(F_\mathfrak{p}[G], \mathrm{Ind}_D^G \mathcal{O}_{K_\mathfrak{P}}).$$

Thus in the context of number fields local freeness of $\mathcal{O}_K$ over $\mathfrak{A}_{K/F}$ at a prime $\mathfrak{p}$ is equivalent to saying that the induction from D to G of the ring of integers of any completion above $\mathfrak{p}$ is free over its associated order.

Now we will consider the relationship between $\mathfrak{A}(F[G], \mathrm{Ind}_D^G \mathcal{O}_{K_\mathfrak{P}})$ and $\mathrm{Ind}_D^G \mathfrak{A}_{K_\mathfrak{P}/F_\mathfrak{p}}$, as well as conditions under which the implication 'if $\mathcal{O}_{K_\mathfrak{P}}$ is free over $\mathfrak{A}_{K_\mathfrak{P}/F_\mathfrak{p}}$ then $\mathcal{O}_K$ is locally free over $\mathfrak{A}_{K/F}$ at $\mathfrak{p}$' holds.

Let G be a finite group and let H be a subgroup of G. Let N be an $R[H]$-lattice. We recall that $\mathrm{Ind}_H^G N$ is the induced module $R[G] \otimes_{R[H]} N \cong \bigoplus_{s \in G/H} sN$; in the latter expression we choose a system of representatives of the left cosets in G/H and the left $R[G]$-module structure is given by the relation $gs = th$ for some coset representative t and $h \in H$. Keep in mind that before we had $N = \mathcal{O}_{K_\mathfrak{P}}$ and $H = D$.

Our goal is to understand when we can deduce, assuming N is free over $\mathfrak{A}(F[H], N) \supseteq R[H]$, that $\mathrm{Ind}_H^G N$ is free over $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$, and more generally the relation between $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$ and $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$. First of all, if N is free over $\mathfrak{A}(F[H], N)$, then the rank of course must be 1, since this is true after we tensor with F. In particular, N and $\mathfrak{A}(F[H], N)$ are isomorphic as $R[H]$-modules, and one can easily verify that $\mathrm{Ind}_H^G N$ and $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ are isomorphic as $R[G]$-modules. However, $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ is not a ring in general and so it does not always make sense to conclude that $\mathrm{Ind}_H^G N$ is free over $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$. We start by giving an explicit description of $\mathfrak{A}(F[H], N)$ in terms of $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$.

Lemma 5.1. [Ber79, § 1.3] *We have*

$$\mathfrak{A}(F[G], \mathrm{Ind}_H^G N) = \bigcap_{g \in G} g \mathrm{Ind}_H^G \mathfrak{A}(F[H], N) g^{-1}. \tag{5.1}$$

In particular, $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ *is a ring if and only if it is equal to* $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$*.*

As a consequence one can prove what follows.

Proposition 5.2. *Suppose that* N *is free over* $\mathfrak{A}(F[H], N)$*. If* $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ *is a ring, then* $\mathrm{Ind}_H^G N$ *is a free* $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$*-module of rank* 1*.*

Remark 5.3. By the relation (5.1), in particular $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ is a ring if

(i) there exists a subgroup $K \leq G$ such that $G \cong H \times K$, or
(ii) H is contained in the center of G, or
(iii) $\mathfrak{A}(F[H], N) = R[H]$.

Thus in any of these cases Proposition 5.2 permits us to conclude that $\mathrm{Ind}_H^G N$ is a free $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$-module if N is a free $\mathfrak{A}(F[H], N)$-module.

If we wish to give a converse to Proposition 5.2, we only have the following general results.

Proposition 5.4. *Suppose that* N *is a free* $\mathfrak{A}(F[H], N)$*-module. Then the two left* $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$*-modules* $\mathrm{Ind}_H^G N$ *and* $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ *are isomorphic, where the latter* *structure is given by the inclusion*

$$\mathfrak{A}(F[G], \mathrm{Ind}_H^G N) \subseteq \mathrm{Ind}_H^G \mathfrak{A}(F[H], N),$$

which follows for example from Lemma 5.1.

Proof. Since N is a free $\mathfrak{A}(F[H], N)$-module, then in particular N is isomorphic to $\mathfrak{A}(F[H], N)$ as $R[H]$-modules. Then from the isomorphism we can easily construct a map

$$f : \mathrm{Ind}_H^G N \to \mathrm{Ind}_H^G \mathfrak{A}(F[H], N),$$

of $R[G]$-modules, which continues to be an isomorphism. Note that both sides also have a structure of $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$-modules. If we prove that the bijection f is also a map of $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$-modules, then we are done. The following type of argument has been widely used in the theory of lattices. Let $x \in \mathrm{Ind}_H^G N$ and $a \in \mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$. Note that there exists $r \in R$ such that $ra = a' \in R[G]$; then

$$rf(ax) = f(rax) = f(a'x) = a'f(x) = raf(x)$$

and we conclude by R-torsion-freeness that $f(ax) = af(x)$. □

Remark 5.5. Proposition 5.4, or more direct considerations, tells us that in fact, just assuming that N is a free $\mathfrak{A}(F[H], N)$-module, we have the compatibility

$$\mathfrak{A}(F[G], \mathrm{Ind}_H^G \mathfrak{A}(F[H], N)) = \mathfrak{A}(F[G], \mathrm{Ind}_H^G N).$$

Corollary 5.6. *Suppose that N is a free $\mathfrak{A}(F[H], N)$-module. Then $\mathrm{Ind}_H^G N$ is a free $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$-module if and only if $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ is free as a (left) $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$-module.*

Remark 5.7. It is not necessarily true that, if N is a free $\mathfrak{A}(F[H], N)$-module and $\mathrm{Ind}_H^G N$ a free $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$-module, then $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ is a ring, or equivalently, that $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N) = \mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$.

We end the subsection with a proposition concerning projectivity.

Proposition 5.8. [Ber79, Proposition 2] *If $\mathrm{Ind}_H^G N$ is a projective $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$-module, then N is a projective $\mathfrak{A}(F[H], N)$-module.*

5.1. **Induction when H is normal in G.** Bergé noted that we can restate some conditions if $H \triangleleft G$. In this case we can in fact define the order

$$\mathfrak{A}^* = \bigcap_{g \in G} g\mathfrak{A}(F[H], N)g^{-1} \subseteq F[H].$$

Then, using (5.1), one can verify that $\mathrm{Ind}_H^G \mathfrak{A}^* = \mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$ and the following lemma.

Lemma 5.9. *Suppose $H \triangleleft G$. Then $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ is a ring if and only if $\mathfrak{A}(F[H], N) = \mathfrak{A}^*$.*

The following result tells us something more specific than Proposition 5.8.

Proposition 5.10. [Ber79, Proposition 3] *Suppose $H \triangleleft G$. Then $\mathrm{Ind}_H^G N$ is projective over $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$ if and only if N is projective over $\mathfrak{A}^*$.*

5.2. **Induction when H is abelian and normal in G.** We continue to assume the hypotheses and notation of this section. We recall that, if H is abelian and R is a discrete valuation ring with characteristic zero and finite residue field, every R-order Λ in $F[H]$ is clean. This implies that in this setting N is projective over $\mathfrak{A}^* \subseteq F[H]$ if and only if it is free. This immediately brings to the following conclusion.

Proposition 5.11. [Ber79, Corollaire to Proposition 3] *Suppose that R is a discrete valuation ring with characteristic zero and finite residue field, $H \lhd G$ and H is abelian. Then the following are equivalent:*

(i) $\mathrm{Ind}_H^G N$ *is projective over* $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$*;*
(ii) $\mathrm{Ind}_H^G N$ *is free over* $\mathfrak{A}(F[G], \mathrm{Ind}_H^G N)$*;*
(iii) $\mathrm{Ind}_H^G \mathfrak{A}(F[H], N)$ *is a ring, and* $\mathrm{Ind}_H^G N$ *is free over it;*
(iv) N *is free over* $\mathfrak{A}(F[H], N)$ *and* $\mathfrak{A}^* = \mathfrak{A}(F[H], N)$.

Proof. (i)⇒(iv). From Proposition 5.10, N is projective over $\mathfrak{A}^*$. But the latter is a clean order and so we have freeness. We conclude since $\mathfrak{A}^*$ now has to be the associated order.

(iv)⇒(iii). It follows from Lemma 5.9 and Proposition 5.2.

(iii)⇒(ii). It is clear for instance from the last sentence of Lemma 5.1.

(ii)⇒(i). Clear. □

Let us write an application of what we did above. With the notation and results we introduced, we are now able to have a quite good understanding of local freeness in weakly ramified extensions.

Proposition 5.12. *Let K/F be a finite Galois extension of number fields with Galois group G and let $\mathfrak{P}|\mathfrak{p}$ be two primes of K/F such that $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ is weakly ramified. Assume that the inertia group $G_0 = G_0(\mathfrak{P}|\mathfrak{p})$ is normal in G. Then $\mathcal{O}_{K,\mathfrak{p}}$ is free over $\mathfrak{A}_{K/F,\mathfrak{p}}$.*

Proof. Let $D = D(\mathfrak{P}|\mathfrak{p})$ be the decomposition group. By Theorem 2.7, we know that

$$\mathfrak{A}_{K_{\mathfrak{P}}/F_{\mathfrak{p}}} = \mathcal{O}_{F_{\mathfrak{p}}}[D]\left[\frac{1}{\pi_{F_{\mathfrak{p}}}}\mathrm{Tr}_{G_0}\right] = \mathcal{O}_{F_{\mathfrak{p}}}[D] + \frac{1}{\pi_{F_{\mathfrak{p}}}}\mathcal{O}_{F_{\mathfrak{p}}}[D]\mathrm{Tr}_{G_0}.$$

We can hence show that

$$\mathrm{Ind}_D^G \mathfrak{A}_{K_{\mathfrak{P}}/F_{\mathfrak{p}}} = \mathcal{O}_{F_{\mathfrak{p}}}[G] + \frac{1}{\pi_{F_{\mathfrak{p}}}}\mathcal{O}_{F_{\mathfrak{p}}}[G]\mathrm{Tr}_{G_0}.$$

If $G_0 \lhd G$ it is easy to see that $\mathrm{Ind}_D^G \mathfrak{A}_{K_{\mathfrak{P}}/F_{\mathfrak{p}}}$ is a ring. We conclude with Theorem 2.7 and Proposition 5.2. □

Proposition 5.13. *Let K/F be a finite Galois extension of number fields with Galois group G and let $\mathfrak{p}$ be a prime of F such that the completed extension $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ is wildly and weakly ramified for every prime $\mathfrak{P}$ above $\mathfrak{p}$. Then $\mathrm{Ind}_D^G \mathfrak{A}_{K_{\mathfrak{P}}/F_{\mathfrak{p}}}$ is a ring if and only if $G_0 \lhd G$.*

Suppose furthermore that $D \lhd G$ and D is abelian. Then $\mathcal{O}_{K,\mathfrak{p}}$ is free over $\mathfrak{A}_{K/F,\mathfrak{p}}$ if and only if $G_0 \lhd G$.

Proof. For the first statement we omit the details, but it not difficult to show it given the already proved description $\mathrm{Ind}_D^G \mathfrak{A}_{K_{\mathfrak{P}}/F_{\mathfrak{p}}} = \mathcal{O}_{F_{\mathfrak{p}}}[G] + \frac{1}{\pi_{F_{\mathfrak{p}}}}\mathcal{O}_{F_{\mathfrak{p}}}[G]\mathrm{Tr}_{G_0}$. For the second one, simply apply Proposition 5.11. □

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