Hopf Orders as Associated Orders

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§1: The set-up:

Recall from Daniel's talk last week:

Theorem 1

Let K be a finite extension of $\mathbb{Q}_p$ and let L/K be a finite extension which is H-Galois for some cocommutative K-Hopf algebra H.
[Motivating case: $H = K[G]$ with L/K Galois and $G = \mathrm{Gal}(L/K)$.]

Let

$$\mathcal{A}_H := \{h \in H : h \cdot O_L \subseteq O_L\}$$

be the associated order of O_L in H.

If $\mathcal{A}_H$ is a Hopf order, then O_L is $\mathcal{A}_H$-tame and is therefore $\mathcal{A}_H$-free.

Moreover, if $\mathcal{A}_H$ is a local ring (not necessarily commutative), then O_L is $\mathcal{A}_H$-Galois.

In this talk, we investigate the consequences of $\mathcal{A}_H$ being a Hopf order and give construct some examples.

§2: Hopf orders in $K[C_p]$

Fix notation: K is a finite extension of $\mathbb{Q}_p$ with valuation v_K, valuation ring O_K, and $\mathfrak{p}_K$ is the maximal ideal of O_K.

Let $\pi = \pi_K$ be a uniformiser of K, so $v_K(\pi) = 1$ and $pO_K = \pi^e O_K$ where $e = e_{K/\mathbb{Q}_p}$ is the ramification index.

Now let $G = \langle \sigma \rangle = C_p$ and write $X = \sigma - 1 \in O_K[G]$.

Expanding $(X+1)^p = \sigma^p = 1$, we find

$$X^p + pX^{p-1} + \binom{p}{2} X^{p-2} + \cdots + pX = 0.$$

Also

$$\Delta(X) = \sigma \otimes \sigma - 1 \otimes 1 = X \otimes 1 + 1 \otimes X + X \otimes X.$$

So $O_K[X]$ is a Hopf order in $K[G]$.

This is not surprising since $O_K[X] = O_K[G]$!

Now let $i \in \mathbb{Z}$ and write

$$X_i = \pi^{-i} X, \qquad H_i = O_K[X_i].$$

For which i is H_i a Hopf order?

For H_i to be a finitely generated O_K-module, we need X_i to satisfy a monic equation over O_K. Now

$$X_i^p + p\pi^{-i} X_i^{p-1} + \binom{p}{2} \pi^{-2i} X_i^{p-2} + \cdots + p\pi^{-(p-1)i} X_i = 0.$$

$$\text{So } H_i \text{ is an order in } K[G] \Leftrightarrow p^{-(p-1)i} \in O_K \Leftrightarrow i \leq e/(p-1).$$

For H_i to be a Hopf order, we also need $\Delta(X_i) \in H_i \otimes_{O_K} H_i$. But

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i + \pi^i X_i \otimes X_i.$$

$$\text{So } \Delta(H_i) \subset H_i \otimes_{O_K} H_i \Leftrightarrow i \geq 0.$$

Thus we get a family of Hopf orders $H_i \subset K[G]$ for $0 \leq i \leq \lfloor e/(p-1) \rfloor$.
Then H_i is a local ring unless $i = e/(p-1)$.
In fact, these are the only Hopf orders in $K[C_p]$ (proof omitted).

We now investigate when these Hopf orders H_i occur as associated orders.

Let L/K be a Galois extension of degree p, with ramification break t:

$$t = \max\{j : (\sigma - 1) \cdot O_L \subseteq \mathfrak{p}_L^{j+1}\}$$

Then $-1 \leq t \leq ep/(p-1)$, and $p \nmid t$ unless $(p-1) \mid e$ and $t = ep/(p-1)$.

If $t > 0$ then L/K is totally (and wildly) ramified.
If $t = -1$ then L/K is unramified.

If $t > 0$ then, for any $\rho \in L^\times$, we have

$$v_L\big((\sigma - 1) \cdot \rho\big) \quad \begin{cases} = v_L(\rho) + t & \text{if } p \nmid v_L(\rho), \\ > v_L(\rho) + t & \text{if } p \mid v_L(\rho). \end{cases}$$

We consider 4 cases:

(i) $t > 0$ and $t \equiv -1 \pmod{p}$, say $t = pi - 1$ with $1 \leq i \leq e/(p-1)$.

Pick any $\rho \in L$ with $v_L(\rho) = p - 1$.

Recall that $X_i = \pi_K^{-i}(\sigma - 1)$. So

$$v_L(X_i^s \cdot \rho) = p - 1 - s \text{ for } 0 \leq s \leq p-1, \text{ and } v_L(X_i^p \cdot \rho) \geq 0.$$

Thus the elements $X_i^s \cdot \rho$ form an O_K-basis for O_L.

So O_L is a free module over the Hopf order $H_i = O_K[X_i]$ on the generator ρ.

Also, O_L is a Galois H_i-extension.

(ii) For $t = -1$, L/K is unramified and O_L is free over $H_0 = O_K[G]$, and is a Galois H_0-extension.

[So all our Hopf orders H_i occur as associated orders of valuation rings O_L.]

(iii) If $t = pi - a$ for $2 \leq a \leq p-1$ then

$$H_{i-1} \subsetneq \mathcal{A}_{L/K} \subsetneq H_i$$

so $\mathcal{A}_{L/K}$ is not a Hopf order.

Then O_L is free over $\mathcal{A}_{L/K}$ if and only if $(p-a)$ divides $p-1$, unless $t+1 \geq ep/(p-1)$.

(iv) If $t = pe/(p-1)$, we have $\zeta_p \in K$ and $L = K(\alpha)$ with $\alpha = \sqrt[p]{\pi}$ for some choice of π.

Then O_L is free over H_i for $i = e/(p-1)$ on the generator

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}.$$

H_i is the maximal order, and it is not a local ring.

O_L is H_i-tame but not H_i-Galois.

Summary: O_L is H-Galois for some Hopf order $H \Leftrightarrow t \equiv -1 \pmod{p}$.

The only other case where the associated order is a Hopf order is when $t = ep/(p-1)$.

§3: A weak congruence for the ramification breaks

From now on, L/K is a totally ramified Galois extension of degree p^n.
Then $G = \mathrm{Gal}(L/K)$ has ramification filtration

$$G_j = \{\sigma \in G : (\sigma - 1) \cdot x \in \mathfrak{p}_L^{j+1} \text{ for all } x \in O_L\}.$$

The ramification breaks (in the lower numbering) are the t with $G_{t+1} \neq G_t$.
We will list these "with multiplicity": $t_1 \leq t_2 \leq \cdots \leq t_n$ where

$$t_i = \max\{j : |G_j| > p^{n-i}\}.$$

Then the inverse different

$$\mathcal{D}_{L/K}^{-1} = \{x \in L : \mathrm{Tr}_{L/K}(xO_L) \subseteq O_K\}$$

is determined by the ramification breaks: $\mathcal{D}_{L/K}^{-1} = \mathfrak{p}_L^{-w}$ where

$$w = (p^n - 1)(t_1 + 1) + \sum_{i=1}^{n-1} (t_{i+1} - t_i)(p^{n-i} - 1) \equiv -(t_n + 1) \pmod{p}.$$

Theorem 2

Suppose that $\mathcal{A}_{L/K} = H$ *is a* local *Hopf order in* $K[G]$. *Then*

$$t_i \equiv -1 \pmod{p^i} \text{ for } 1 \leq i \leq n.$$

Proof.

Use induction on n. We have just seen this for $n = 1$.

Let $N \lhd G$ with $|N| = p$ and $N \leq G_{t_n}$. Let $F = L^N$.

Since N is chosen to be compatible with the ramification filtration, L/F has ramification break t_n and F/K has ramification breaks $t_1, \ldots, t_{n-1}$.

Corresponding to the exact sequence of groups

$$1 \to N \to G \to G/N \to 1,$$

we have an exact sequence of Hopf orders over O_K:

$$O_K \to H_1 \to H \to \overline{H} \to O_K$$

where $H_1 = H \cap K[N]$ and $\overline{H}$ is the image of H in $K[G/N]$.

Then H_1 and $\overline{H}$ are still local, and O_L, O_F are respectively tame for the Hopf orders $H_1 \otimes_{O_K} O_F = \mathcal{A}_{L/F}$ and $\overline{H} = \mathcal{A}_{F/K}$.

Applying the induction hypothesis to F/K, we get $t_i \equiv -1 \pmod{p^i}$ for $1 \leq i \leq n-1$.

Applying the induction hypothesis to L/F, we get $t_n \equiv -1 \pmod{p}$, say $t_n = pm - 1$. We need to strengthen this to a congruence mod p^n.

Now the ideal of integrals in H_1 is

$$I(H_1) = \mathfrak{a}^{-1} \sum_{\sigma \in N} \sigma$$

for some (integral) O_K-ideal $\mathfrak{a}$.

Hence the ideal of integrals in $H_1 \otimes O_F$ is

$$I(H_1 \otimes O_F) = \mathfrak{a}^{-1} O_F \sum_{\sigma \in N} \sigma$$

and $\mathrm{Tr}_{L/F}(O_L) = \mathfrak{a} O_F$.

Now $\mathcal{D}_{L/F}^{-1} = \mathfrak{P}_L^{-w}$ where $w = (p-1)(t_n+1) = (p-1)pm$, so

$$\mathfrak{p}_F^{(p-1)m} = \mathrm{Tr}_{L/F}(O_F) = \mathfrak{a} O_F.$$

But F/K is totally ramified of degree p^{n-1}, so $(p-1)m \equiv 0 \pmod{p^{n-1}}$. Then $p^{n-1} \mid m$ and

$$t_n = pm - 1 \equiv -1 \pmod{p^n}.$$

□

Corollary

If $\mathcal{A}_{L/K}$ is a local Hopf order, then $\mathcal{D}_{L/K}^{-1} = \mathfrak{a}^{-1} O_L$ for some O_K-ideal $\mathfrak{a}$.

Proof.

We have $\mathcal{D}_{L/K}^{-1} = \mathfrak{p}_L^{-w}$ where

$$w = (t_1+1)(p^n-1) + \sum_{i=1}^{n-1}(t_{i+1}-t_i)(p^{n-i}-1) \equiv -(t_n+1) \equiv 0 \pmod{p^n},$$

since $t_i \equiv -1 \pmod{p^i}$ for each i. So $\mathcal{D}_{L/K}^{-1} = \mathfrak{p}_K^{-w/p^n} O_L$. □

Remark

In particular this means that $\mathcal{D}_{L/K}^{-1}$ and O_L are isomorphic as $O_K[G]$-modules, so they both have associated order $\mathcal{A}_{L/K}$ and (by Theorem 1) both are free over $\mathcal{A}_{L/K}$.

§4: A strong congruence for the ramification breaks

Theorem 3

Let L/K be a totally ramified abelian extension of degree p^n with $\mathcal{D}_{L/K}^{-1} = \mathfrak{a}^{-1} O_L$ for some O_K-ideal $\mathfrak{a}$.
Suppose that $\mathcal{D}_{L/K}^{-1}$ (or, equivalently, O_L) is free over its associated order $\mathcal{A}$ in $K[G]$, and that $\mathcal{A}$ is a local ring. ($\mathcal{A}$ does not have to be a Hopf order.)

Then the ramification numbers t_i of L/K satisfy

$$t_i \equiv -1 \pmod{p^n} \text{ for } 1 \leq i \leq n.$$

Note the modulus is now p^n, not just p^i as before!

Proof. Let $\mathcal{A}$ have unique maximal ideal $\mathcal{M}$, and let $\mathcal{D}^{-1} := \mathcal{D}^{-1}_{L/K}$ be free over $\mathcal{A}$ with generator y.

For any $\alpha \in \mathcal{A}$,

$$\mathcal{A} \cdot (\alpha \cdot y) = \mathcal{D}^{-1} \Leftrightarrow \alpha \notin \mathcal{M}.$$

Let $y_1, y_2, \ldots y_{p^n}$ be any O_K-basis of $\mathcal{D}^{-1}$.

Then we can write $y_j = \alpha_j \cdot y$ for $\alpha_j \in \mathcal{A}$, and $\alpha_1, \ldots, \alpha_{p^n}$ is an O_K-basis of $\mathcal{A}$.

So there is some j with $\alpha_j \notin \mathcal{M}$ and hence $\mathcal{A} \cdot y_j = \mathcal{D}^{-1}$.

We will assume for a contradiction that some $t_i \not\equiv -1 \pmod{p^n}$, and construct a basis $y_1, \ldots, y_{p^n}$ for $\mathcal{D}^{-1}$ over O_K so that none of the y_j satisfy $\mathcal{A} \cdot y_j = \mathcal{D}^{-1}$. This will show $\mathcal{D}^{-1}$ cannot be free over $\mathcal{A}$.

To show that $\mathcal{A} \cdot y_j \neq \mathcal{D}^{-1}$, it is enough to find $\beta_j \in \mathcal{A}$ so that $\beta_j \cdot y_j \in \pi\mathcal{D}^{-1}$ but $\beta_j \notin \pi\mathcal{A}$. We can do this without determining $\mathcal{A}$.

Write $\mathcal{D}^{-1} = \mathfrak{p}_L^{-w}$ with $w \equiv 0 \pmod{p^n}$. Then

$$\mathrm{Tr}_{L/K}(\mathcal{D}^{-1}) = O_K \text{ and } \mathrm{Tr}_{L/K}(\mathfrak{p}_L^{-1}\mathcal{D}^{-1}) = \mathfrak{p}_K^{-1}.$$

So

$$\mathrm{Tr}_{L/K}(\pi\mathcal{D}^{-1}) = \mathfrak{p}_K \text{ and } \mathrm{Tr}_{L/K}(\pi\mathfrak{p}_L^{-1}\mathcal{D}^{-1}) = O_K.$$

Choose $z \in \pi\mathfrak{p}_L^{-1}\mathcal{D}^{-1}$ with $\mathrm{Tr}_{L/K}(z) = 1$.

Then $v_L(z) = p^n - w - 1$.

Now choose $z_1 = z$, z_2, $\ldots z_{p^n}$ with $v_L(z_j) = p^n - w - j$. Then the z_j form an O_K-basis for $\mathcal{D}^{-1}$.

We adjust this to get a nicer basis:

$$y_1 = z_1, \qquad y_j = z_j - \mathrm{Tr}_{L/K}(z_j)z_1 \text{ for } j \geq 2.$$

Then $\mathrm{Tr}_{L/K}(y_j) = 0$ for $j \geq 2$, and we still have $v_L(y_j) = p^n - w - j$.

For each j, we need to find $\beta_j \in \mathcal{A} \backslash \pi\mathcal{A}$ with $\beta_j \cdot y_j \in \pi\mathcal{D}^{-1}$.

For $j \geq 2$, take

$$\beta_j = \beta := \pi^{-w/p^n} \sum_{\sigma \in G} \sigma.$$

Then $\beta \cdot y_j = \pi^{-w/p^n} \mathrm{Tr}_{L/K}(y_j) = 0$.

But $\beta \cdot y_1 = \pi^{-w/p^n}$. Since $\mathcal{D}^{-1} = \pi^{-w/p^n} O_L$, we have $\beta \in \mathcal{A}$ but $\beta \notin \pi\mathcal{A}$, as required.

It remains to find a β_1.

Recall some $t_i \not\equiv -1 \pmod{p^n}$. Let $t_i = p^n m + a$ with $1 \leq a \leq p^n - 2$, and take

$$\beta_1 = \pi^{-m}(\sigma - 1) \text{ where } \sigma \in G_{t_i} \backslash G_{t_i+1}.$$

So for any $y \in L^\times$,

$$v_L(\beta_1 \cdot y) \quad \begin{cases} = v_L(y) + a & \text{if } p \nmid v_L(y), \\ > v_L(y) + a & \text{if } p \mid v_L(y). \end{cases}$$

Thus $\beta_1 \in \mathcal{A}_{L/K}$.

Now

$$v_L(\beta_1 \cdot y_1) \geq v_L(y_1) + a = p^n - w - 1 + a \geq p^n - w,$$

so $\beta_1 \cdot y_1 \in \pi \mathcal{D}^{-1}$.

On the other hand

$$v_L(\beta_1 \cdot y_{p^n-1}) = \big(p^n - w - (p^n - 1)\big) + a \leq p^n - w - 1,$$

so $\beta_1 \cdot y_{p^n-1} \notin \pi \mathcal{D}^{-1}$, and $\beta_1 \notin \pi \mathcal{A}_{L/K}$.
Hence none of the y_j is a generator for $\mathcal{D}^{-1}$ over $\mathcal{A}$, and $\mathcal{D}^{-1}$ cannot be free.

□

Summary so far:

If K is a finite extension of $\mathbb{Q}_p$ and L/K is a totally ramified Galois extension of degree p^n such that $\mathcal{A}_{L/K}$ is a Hopf order then O_L is free over $\mathcal{A}_{L/K}$.

Moreover, if $\mathcal{A}_{L/K}$ is a local ring, then O_L is $\mathcal{A}_{L/K}$-Galois, $\mathcal{D}_{L/K}^{-1} = \mathfrak{a}^{-1} O_L$ for some O_K-ideal $\mathfrak{a}$, and the ramification numbers of L/K (counted with multiplicity) satisfy

$$t_i \equiv -1 \pmod{p^n}.$$

§5: Formal Groups and Hopf Orders

We want a way of creating Hopf orders.

In geometry, the ring of regular functions on an algebraic group is a Hopf algebra. In more sophisticated language, a commutative Hopf algebra is the representing algebra for an affine group scheme.

This suggests we should start with some kind of group operation "defined over O_K".

Definition

A (one-dimensional) formal group over O_K is a power series $F(X,Y) \in O_K[[X,Y]]$ such that

(i) $F(X,Y) \equiv X + Y$ *mod deg* 2;

(ii) $F(F(X,Y),Z) = F(X,F(Y,Z))$;

(iii) $F(X,0) = X$, $F(0,Y) = Y$;

(iv) $F(X,Y) = F(Y,X)$;

(v) *there exists a series $i(X) \in O_K[[X]]$ with $i(0) = 0$ such that*

$$F(X, i(X)) = 0.$$

(In fact, (iv) and (v) follow from (i)–(iii).)

For any algebraic extension E of K, $(\mathfrak{p}_E, +_F)$ is an abelian group, where

$$x +_F y = F(x,y).$$

This makes sense as $K(x,y)$ is complete. The identity element is 0 and the inverse of x is $i(x)$.

Examples

(i) The additive formal group

$$F(X, Y) = X + Y :$$

this gives the usual addition.

(ii) The multiplicative formal group

$$F(X, Y) = X + Y + XY = (1 + X)(1 + Y) - 1 :$$

this gives the usual operation of multiplication on $1 + \mathfrak{p}_E \subseteq O_E^\times$, *shifted to make the identity element* 0.

Definition

A homomorphism $\phi : F \to G$ *between formal groups is a power series* $\phi(X) \in O_K[[X]]$ *such that* $\phi(X) = 0$ *and* $\phi(F(X, Y)) = G(\phi(X), \phi(Y))$.

This gives a homomorphism $\phi : (\mathfrak{p}_E, +_F) \to (\mathfrak{p}_E, +_G)$ for each E.

If $F(X, Y)$ is a formal group over O_K, we can make $O_K[[T]]$ into a (completed) Hopf algebra as follows.

Let $U = T \otimes 1$ and $V = 1 \otimes T$. Define continuous algebra maps Δ, ϵ, inv

$$\Delta : O_K[[T]] \to O_K[[U, V]] = O_K[[U]]\hat{\otimes} O_K[[V]], \qquad T \mapsto F(U, V);$$

$$\epsilon : O_K[[T]] \to O_K, \qquad T \mapsto 0;$$

$$\mathrm{inv} : O_K[[T]] \to O_K[[T]], \qquad T \mapsto i(T).$$

These satisfy the usual Hopf algebra axioms.

(Note that $1 + UV + U^2V^2 + \cdots$ is an element of $O_K[[U]]\hat{\otimes} O_K[[V]]$ but not of $O_K[[U]] \otimes O_K[[V]]$.)

Now suppose $\phi(X) = a_1X + a_2X^2 + \cdots$ is a homomorphism $F \to G$ such that, for some d, we have

$$a_j \in \mathfrak{p}_K \text{ for } 1 \leq j \leq d-1, \qquad a_d \notin \mathfrak{p}_K.$$

By the Weierstrass Preparation Theorem, $\phi(X) = f(X)u(X)$ where $f(X)$ is a monic polynomial of degree d with $f(X) \equiv X^d \pmod{\mathfrak{p}_K}$, and $u(X) \in O_K[[X]]^\times$.

Then the quotient O_K-algebra $H = O_K[[X]]/(\phi) \cong O_K[[X]]/(f)$ is a free O_K-module of rank d, and Δ induces a comultiplication $H \to H \otimes_{O_K} H$, making H into an O_K-Hopf algebra of rank d.

H represents $\ker(\phi)$: for each algebraic extension E of K, the O_K-algebra homomorphisms $H \to L$ correspond to $x \in \mathfrak{p}_E$ with $\phi(x) = 0$, and these form a group under $+_F$.

Now let $c \in \mathfrak{p}_K$ and write $S = O_K[[T]]/(\phi(T) - c)$. We have a well-defined function

$$S \to S \otimes_{O_K} H, \qquad T \mapsto F(T \otimes 1, 1 \otimes X).$$

This makes S into a Galois H-*object* and so into a Galois H^*-*extension*, where H^* is the dual O_K-Hopf order to H.

If $v_K(c) = 1$ then $\phi(T) - c \in O_K[[T]]$ is "morally" an Eisenstein polynomial of degree d, and $S = O_L$ for some totally ramified extension L/K of degree d.

So we have created an extension L/K for which O_L is free over some commutative, cocommutative Hopf order H^*. The underlying K-Hopf algebra $H^* \otimes K$ is a group algebra if and only if $\ker(\phi) \subseteq K$.

If we apply this to the multiplicative formal group

$$F(X, Y) = X + Y + XY$$

and the (shifted) p^n-power homomorphism

$$\phi(X) = (1 + X)^{p^n} - 1 = p^n X + \cdots + X^{p^n},$$

then $H^* \subseteq K[C_{p^n}] \Leftrightarrow \zeta_{p^n} \in K$.

If $v_K(c) = 1$ we get $S = O_K[\sqrt[p^n]{c}]$, which is the valuation ring in $L = K(\sqrt[p^n]{c})$, and is free over the Hopf order H^*.

For example, if we start with $K = \mathbb{Q}_p(\zeta_p)$ and take $c = \zeta_p - 1$, we get $L = \mathbb{Q}_p(\zeta_{p^{n+1}})$ which is certainly a Galois extension of K, but H^* gives some *non-classical* Hopf-Galois structure if $n > 1$.

§6: Lubin-Tate formal groups

Change of notation: Start with a finite extension k of $\mathbb{Q}_p$ with residue field of order $q = p^f$. Let $\pi = \pi_k$ be a uniformiser of k. Write $\overline{\mathfrak{p}}$ for the maximal ideal in the valuation ring of the algebraic closure of k.

Let $f(X) \in O_k[[X]]$ satisfy

- $f(X) \equiv \pi X$ mod deg 2;
- $f(X) \equiv X^q \pmod{\mathfrak{p}_k}$.

[$f(X) = \pi X + X^q$ would do.

If $k = \mathbb{Q}_p$ and $\pi = p$, we could take
$f(X) = (1+X)^p - 1 = pX + \cdots + X^p$.]

Lubin and Tate proved there is a unique formal group $F(X, Y)$ over O_k such that $f(X)$ is an endomorphism of F.

In fact, there is an isomorphism of rings $O_k \to \mathrm{End}(F)$ given by $a \mapsto [a]$ with $[\pi](X) = f(X)$ and $[2](X) = F(X, X)$, etc.

For each $n \geq 1$, the set

$$G_n = \{x \in \overline{\mathfrak{p}} : [\pi^n](x) = 0\}$$

is an O_k-module isomorphic to $O_K/\mathfrak{p}_k^n$. In particular, it is an abelian group of order q^n.

The field

$$k_n = k(G_n)$$

is a totally ramified Galois extension of degree $q^{n-1}(q-1)$ and

$$\mathrm{Gal}(k_n/k) \cong \left(\frac{O_k}{\mathfrak{p}_k^n}\right)^\times \cong \frac{O_k^\times}{1+\mathfrak{p}_k^n}.$$

If $\omega_n \in G_n \backslash G_{n-1}$ then ω_n is a uniformiser for k_n.

If M is any finite abelian extension of k then $M \subseteq k_n k_m^{(ur)}$ for some n, m, where $k_m^{(ur)}$ is the unique unramified extension of k of degree m.

We now use the Lubin-Tate formal group F attached to f to construct some extensions of p-power degree with interesting properties.

For our p-power degree extension L/K we choose $K = k_m$ and $L = k_{m+n}$ for $n,\ m \geq 1$.

Then L/K is a totally ramified abelian extension of degree q^n with Galois group $G \cong (1 + \mathfrak{p}_k^m)/(1 + \mathfrak{p}_k^{m+n})$.

Its ramification breaks (with multiplicity) are

$$t_1 = \cdots = t_f = q^m - 1,$$

$$t_{f+1} = \cdots = t_{2f} = q^{m+1} - 1,$$

$$\vdots$$

$$t_{(n-1)f+1} = \cdots = t_{nf} = q^{m+n-1} - 1.$$

Its inverse different is $\mathcal{D}_{L/K} = \mathfrak{P}_L^{-w}$ with

$$w = (q-1)(n-m)q^{n-1} \equiv 0 \pmod{q^n}.$$

Since $L = K(\omega_{m+n})$ where ω_{m+n} is a root of $[\pi^n](X) = \omega_m$, the formal group construction gives a Hopf order H^* over O_K so that O_L is a free H^*-module of rank 1.

One can check from the ramification numbers that the associated order $\mathcal{A}_{L/K}$ of O_L in $K[G]$ is local if and only if $p - 1 < (q - 1)e_{k/\mathbb{Q}_p}$, i.e. $k \neq \mathbb{Q}_p$.

If $n \leq m$ then $H^* = \mathcal{A}_{L/K} \subseteq K[G]$, so O_L is free over its associated order in $K[G]$.

If $n > m$ then L/K is still a Galois extension but H^* is *not* contained in $K[G]$, so O_L is free over its associated order in some non-classical Hopf-Galois structure. But $t_1 = q^m - 1 \not\equiv -1 \pmod{q^n}$, so O_L is *not* free over its associated order $\mathcal{A}_{L/K}$ in $K[G]$ (provided $k \neq \mathbb{Q}_p$).

Conclusion: We have constructed a family of Galois extensions L/K where O_L is not free over its associated order in $K[G]$ but is free over its associated order in some other Hopf-Galois structure.