Hopf-Galois module theory

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Abstract

We apply the perspective provided by Hopf-Galois theory to some classical notions of Galois module theory. The notions of integrals and Hopf orders are introduced, as well as the notion of associated order in an arbitrary Hopf-Galois structure of a finite extension of fields. For a commutative ring R, we shall study a generalization of the tame condition for an R-algebra S which is an H-module algebra for some finite cocommutative R-Hopf algebra H. Finally, we prove different results concerning logical relations between different properties of S involving H, for instance H-free, H-tame and H-Galois.

The main reference for this lecture is [Chi00, Chapters 1 and 3]

1 First notions

1.1 Integrals

Let R be a commutative ring and H an R-Hopf algebra. Recall that if M is a left H-module, the submodule fixed by H is

$$M^H = \{m \in M \mid hm = \epsilon(h)m \text{ for all } h \in H\}.$$

Since H has the trivial left and right H-module structures, this inspires:

Definition 1.1.
- *A* ***left integral*** *in H is an element $\theta \in H$ such that $h\theta = \epsilon(h)\theta$ for every $h \in H$.*
- *A* ***right integral*** *in H is an element $\theta \in H$ such that $\theta h = \epsilon(h)\theta$ for every $h \in H$.*

With the previous notation, the set of left integrals is H^H. We will denote it by I (notation in [Chi00]) or by $\int_H^l$ (notation in [Und15]). Now, the trivial left H-module structure of H is given by the product in its underlying algebra, so I is a two-sided ideal of H:

$$h'(h\theta) = (h'h)\theta = \epsilon(h'h)\theta = \epsilon(h')\epsilon(h)\theta = \epsilon(h')h\theta \implies h\theta \in I,$$
$$h'(\theta h) = (h'\theta)h = \epsilon(h')\theta h \implies \theta h \in I.$$

Example 1.2.
1. *Let $H = R[G]$ for a finite group G. Then*
$$\theta = \sum_{\sigma \in G} \sigma$$
is a left and right integral of $R[G]$. Moreover, it generates the modules of left and right integrals (for a detailed proof, see [Und15, Proposition 3.2.4]).
2. *Let $H = R[G]^* = \sum_{\sigma \in G} Re_\sigma$, where $e_\sigma(\tau) = \delta_{\sigma\tau}$ for every $\sigma, \tau \in G$. Since $R[G]$ is always cocommutative, H is commutative, so the modules of left and right integrals coincide. Let us prove that e_1 is an integral. Indeed, for $\sigma \in G$,*
$$e_\sigma e_1 = \delta_{1\sigma} e_1 = \epsilon(e_\sigma)e_1.$$
As every $f \in R[G]^*$ is an R-linear combinations of the e_σ, $fe_1 = \epsilon(f)e_1$. Again, e_1 generates the module of integrals of $R[G]^*$ (see [Und15, Proposition 3.2.5]).

Recall that H acts on H^* by means of

$$x \cdot f = \sum_{(f)} f_{(1)} \langle x, f_{(2)} \rangle.$$

In particular, if $H = R[G]$, $R[G]$ acts on $R[G]^*$ by $\sigma \cdot e_\tau = e_{\tau\sigma^{-1}}$, whence $R[G]^* = R[G]e_1$ and e_1 is a generating integral of $R[G]^*$. On the other hand, $R[G]^*$ acts on $R[G]$ by $e_\sigma \cdot \tau = \delta_{\sigma\tau}\tau$. We deduce then that $R[G] = R[G]^* \left(\sum_{\sigma \in G} \sigma\right)$, where the last element is a generating integral of $R[G]$. More generally (see [Chi00, (3.3)]):

Theorem 1.3 (Larson-Sweedler)**.** *If H is a finite R-Hopf algebra, then $H^* \cong H \otimes_R \int_{H^*}^l$ as R-modules.*

Since $H^{**} \cong H$, this theorem also gives $H \cong H^* \otimes_R \int_H^l$. Assume that R is local. Then H and H^* are free of the same rank, so we have the following:

Corollary 1.4. *If R is local, $\int_H^l$ is R-free of rank one.*

1.2 Hopf orders

Let R be a Dedekind domain with fraction field K of characteristic zero. Let A be a finite (i.e, finitely generated and projective) K-Hopf algebra. We want to define consistently an R-module H in A that inherits its K-Hopf algebra structure so that it becomes an R-Hopf algebra. This notion is the one of Hopf order.

Since R is included in K, A has also R-module structure. Let H be a finite R-submodule of A. The inclusion $H \hookrightarrow A$ induces a map

$$\begin{array}{cccc} \nu\colon & H \otimes_R H & \longrightarrow & A \otimes_K A \\ & \sum_i a_i \otimes_R b_i & \longrightarrow & \sum_i a_i \otimes_K b_i, \end{array}$$

which is an one-to-one R-linear homomorphism. To prove this, we may assume that R is local, otherwise we would work in the completions. Now, if $\{h_i\}_{i=1}^n$ is an R-basis of H, then $\{h_i \otimes_R h_j\}_{i,j=1}^n$ is an R-basis of $H \otimes_R H$, which by means of ν is mapped to the K-basis

$$\{h_i \otimes_K h_j \,|\, 1 \leq i, j \leq n\}$$

of $A \otimes_K A$. Then, ν is injective and $H \otimes_R H$ can be identified with a subset of $A \otimes_K A$.

Under these considerations, it makes sense to consider the restrictions

$$\begin{array}{rcl} \mu\colon A \otimes_K A \longrightarrow A & \rightsquigarrow & \mu|_{H \otimes_R H}\colon H \otimes_R H \longrightarrow A, \\ \iota\colon K \longrightarrow A & \rightsquigarrow & \iota|_R\colon R \longrightarrow A, \\ \Delta\colon A \longrightarrow A \otimes_K A & \rightsquigarrow & \Delta|_H\colon H \longrightarrow A \otimes_K A, \\ \epsilon\colon A \longrightarrow K & \rightsquigarrow & \epsilon|_H\colon H \longrightarrow K, \\ \lambda\colon A \longrightarrow A & \rightsquigarrow & \lambda|_H\colon H \longrightarrow A. \end{array}$$

Note that $\iota(R) \subseteq H$ by definition.

Definition 1.5. *We say that H is an R-**Hopf order in** A if $K \otimes_R H = A$ (so H is an R-order in A) and H has R-Hopf algebra with the operations induced from those on A, that is, $\mu(H \otimes_R H) \subseteq H$, $\Delta(H) \subseteq H \otimes_R H$, $\epsilon(H) \subseteq R$ and $\lambda(H) \subseteq H$.*

Example 1.6. If $A = K[G]$ for a finite group G, then $R[G]$ is an R-Hopf order in A. This is an immediate consequence of how the operations of the K-Hopf algebra $K[G]$ are defined. Actually, this is the minimal one.

Proposition 1.7. *[Chi00, (5.2)] Let H be an R-Hopf order in $K[G]$. Then, $R[G] \subseteq H$.*

Proof. We may assume that R is local, so H is finitely generated and free. Then H^* is a finite R-Hopf order in $(K[G])^*$. Now, $(K[G])^*$ is commutative, so it has a unique maximal R-order. On the other hand, it has K-basis $\{e_\sigma\}_{\sigma \in G}$, where $e_\sigma(\tau) = \delta_{\sigma\tau}$, which is the basis dual to the basis of $K[G]$ formed by the elements of G. Moreover, it is easy to check that this is a basis of primitive pairwise orthogonal idempotents of $(K[G])^*$. Thus, the unique maximal order is $(R[G])^*$, with basis $\{e_\sigma\}_{\sigma \in G}$. Since H is an R-order, $H^* \subseteq (R[G])^*$, whence $R[G] \subseteq H$. □

When $A = K[G]$, the condition of Hopf order reduces to check that the comultiplication restricts correctly.

Proposition 1.8. *[Tru09, Proposition 2.3.12] Let R be a Dedekind domain such that* $\mathrm{char}(K) = 0$*, where $K = \mathrm{Frac}(R)$. Let G be a finite group of order n and let H be an R-order in $K[G]$. If $\Delta_H(H) \subseteq H \otimes_R H$, then H is an R-Hopf order.*

1.3 Associated orders in Hopf algebras

Let L/K be an A-Galois extension of fields, where K is the fraction field of a Dedekind domain R and $\mathrm{char}(K) = 0$. Let S be the integral closure of R in L. When L/K is Galois and $A = K[G]$ is the classical Galois structure, we consider the module structure of S over its associated order

$$\mathfrak{A}_{K[G]} = \{\lambda \in K[G] \,|\, \lambda S \subseteq S\}.$$

In a complete analogy, for an arbitrary Hopf Galois structure, we have:

Definition 1.9. *The* ***associated order of*** S ***in*** A *is defined as*

$$\mathfrak{A}_A = \{\alpha \in A \,|\, \alpha S \subseteq S\}.$$

Its algebraic properties are pretty similar as the ones for the classical case. Clearly $\mathfrak{A}_A$ is an R-algebra and actually it is an R-order in A. However, it is not necessarily an R-Hopf algebra. For instance, if $L/\mathbb{Q}$ is an abelian extension with group G which is wildly ramified at some odd prime, then $\mathfrak{A}_{K[G]}$ is not an R-Hopf algebra (see [Chi87, Corollary 5.6].

Proposition 1.10. *[Chi00, (12.5)] If L/K is A-Galois and H is an R-order in A such that S is H-free, then $H = \mathfrak{A}_A$.*

Proof. Let t be a generator of S as H-module. Tensorizing by K, t is a generator of L as A-module. Let $\alpha \in \mathfrak{A}_A$. Since $\alpha t \in S$, there is some $h \in H$ such that $\alpha \cdot t = h \cdot t$. This equality in particular holds in $L = A \cdot t$, so uniqueness of coordinates gives $\alpha = h \in H$. □

Example 1.11. [Chi00, (12.6)] We consider the extension $L = \mathbb{Q}(\sqrt{2})$ of $K = \mathbb{Q}$. Then L/K is Galois with group $G = \{1, \sigma\}$, where $\sigma(\sqrt{2}) = -\sqrt{2}$. The ring of integers of L is $S = R[\sqrt{2}]$, where $R = \mathbb{Z}$. We know by definition that L/K is tamely ramified at a prime p if and only if p does not divide the ramification index of L/K at that prime. An equivalent definition is that the image of the trace $\mathrm{tr}\colon S \longrightarrow R$ is not divisible by p. Since $\mathrm{tr}(a + b\sqrt{2}) = 2a$, $\mathrm{tr}(S) = 2R$ and we see that L/K is wildly ramified at 2. By Noether's theorem, S is not locally free as $R[G]$-module, so it is not $R[G]$-free. However, S is $\mathfrak{A}_{K[G]}$-free. Indeed, the elements

$$e_1 = \frac{1+\sigma}{2}, \quad e_{-1} = \frac{1-\sigma}{2}$$

are primitive pairwise orthogonal idempotents of $K[G]$ and then $R[e_1, e_{-1}]$ is the maximal R-order in $K[G]$. In particular, $\mathfrak{A}_{K[G]} = R[e_1, e_{-1}]$. Let $\alpha = 1 + \sqrt{2}$. Then,

$$e_1(\alpha) = 1, \quad e_{-1}(\alpha) = \sqrt{2},$$

so $S = \mathfrak{A}_{K[G]}\alpha$ and hence S is $\mathfrak{A}_{K[G]}$-free.

2 H-tame extensions

We know that a Galois extension of p-adic fields L/K is tamely ramified if p does not divide the ramification index $e(L/K)$. In this section we generalize this notion to extensions of rings by rewriting the previous definition in a suitable way so that it is natural to replace the classical Galois structure by an arbitrary one.

Assume that L/K is an extension of local fields with valuation rings S and R. Then, L/K is tamely ramified if and only if the trace map $\mathrm{tr}\colon S \longrightarrow R$ is surjective. But the definition of the trace map is $\mathrm{tr}(a) = \theta(a)$, where $\theta = \sum_{\sigma \in G} \sigma$ is a left integral in $K[G]$. Now, recall that θ generates $\int_{K[G]}^{l}$. Then, L/K is tamely ramified if and only if $\int_{K[G]} S = R$. This motivates the following definition:

Definition 2.1. *Let R be a commutative connected ring and let H be a cocommutative R-Hopf algebra which is finite as R-module. Let S be a finite R-algebra and assume that S is an H-module algebra with $S^H = R$. We say that S is **a tame H-extension of** R or H**-tame** if:*

1. $\mathrm{rank}_R(S) = \mathrm{rank}_R(H)$.
2. *S is a faithful H-module.*
3. $\int_H^l S = R$.

The first two conditions are compatibility conditions which in the Galois case are assured. Actually, they always hold when S and R are valuation rings of an A-Galois extension L/K of local fields. The condition of R being connected (i.e, it has no non-trivial idempotents) is required so that projective modules have a well defined rank. Indeed, the rank is a function defined at $\mathrm{Spec}(R)$ which is locally constant, and if R is connected, this function is constant.

On the other hand, the third condition means that $\int_H^l S$ is as large as possible, because of the following result:

Proposition 2.2. *[Chi00, (13.2)] Let H be a finite cocommutative R-Hopf algebra and let S be an H-module algebra. Then $\int_H^l S \subseteq S^H$.*

Proof. Let $\xi = \sum_i \theta_i s_i \in \int_H^l S$, where $\theta_i \in \int_H^l$ and $s_i \in S$ for all i. Given $h \in H$,

$$h\xi = h\left(\sum_i \theta_i s_i\right) = \sum_i (h\theta_i)s_i = \sum_i \epsilon(h)\theta_i s_i = \epsilon(h)\xi.$$

Thus, $\xi \in S^H$. □

3 Linking notions

3.1 Maximal order implies freeness

Proposition 3.1. *[Tru09, Proposition 2.5.5] Let K be a p-adic field with valuation ring R and let A be a commutative separable K-algebra. Let $\mathfrak{M}$ be the maximal R-order in A and let S be a finite R-module which is also a $\mathfrak{M}$-module. Suppose that $S \otimes_R K$ is H-free of rank one. Then, S is $\mathfrak{M}$-free of rank one.*

In particular if L/K is A-Galois and $\mathfrak{A}_A$ is the maximal order in A, then S is $\mathfrak{A}_A$-free of rank one.

3.2 H-tame implies H-free

Theorem 3.2. *[Chi00, (13.5)] Let R be a local ring. Let H be a finite cocommutative R-Hopf algebra and S a finite R-algebra which is an H-module algebra. If S is H-tame, then S is H-projective.*

Proof. Let $I = R\theta$ be the module of left integrals of H. Since S is H-tame, then $\theta S = R$, so there exists $z \in S$ such that $\theta z = 1$. Since S is R-projective and H is R-free, the tensorized H-module $H \otimes_R S$ (that is, H acts on $H \otimes_R S$ with the product in the first factor) is projective. In order to prove that S is H-projective, we prove it is a direct summand of $H \otimes_R S$, which by H-projectiveness is a direct summand of an H-free module.

Let $\mu\colon H \otimes S \longrightarrow S$ be the R-linear map defined by $\mu(h \otimes s) = hs$, which clearly is an H-module homomorphism. If we prove that this map splits, then $H \otimes S = \mathrm{Ker}(\mu) \oplus S$ as desired. Let us define $\nu\colon S \longrightarrow H \otimes_R S$ by $\nu(s) = \sum_{(\theta)} \theta_{(1)} \otimes z(\lambda(\theta_{(2)})s)$. We need to prove that ν is an H-module homomorphism and $\mu \circ \nu = \mathrm{Id}_S$.

First, we prove that ν is an H-module homomorphism. Let $h \in H$, $s \in S$. Then,

$$\begin{aligned} h \cdot (\nu(s)) &= h\left(\sum_{(\theta)} \theta_{(1)} \otimes z(\lambda(\theta_{(2)})s)\right) \\ &= \sum_{(\theta)} (h\theta_{(1)}) \otimes z(\lambda(\theta_{(2)})s) \\ &= (1 \otimes z)\left(\sum_{(\theta)} (h\theta_{(1)}) \otimes \lambda(\theta_{(2)})\right)(1 \otimes s) \end{aligned}$$

Now, we note that $\sum_{(\theta)}(h\theta_{(1)}) \otimes \lambda(\theta_{(2)}) = (h \otimes 1)((1 \otimes \lambda)\Delta(\theta))$. By [Chi00, (3.7)], this coincides with $((1 \otimes \lambda)\Delta(\theta))(1 \otimes h) = \sum_{(\theta)} \theta_{(1)} \otimes \lambda(\theta_{(2)})h$. Then,

$$\begin{aligned} h(\nu(s)) &= (1 \otimes z)\left(\sum_{(\theta)} \theta_{(1)} \otimes \lambda(\theta_{(2)})h\right)(1 \otimes s) \\ &= \sum_{(\theta)} \theta_{(1)} \otimes z(\lambda(\theta_{(2)})(hs)) \\ &= \nu(hs), \end{aligned}$$

which proves that ν is an homomorphism of H-modules.

Finally, we show that $\mu \circ \nu$ is the identity. Let $s \in S$. Then,

$$\begin{aligned} \mu \circ \nu(s) &= \sum_{(\theta)} \theta_{(1)}(z(\lambda(\theta_{(2)})s)) \\ &= \sum_{\theta} (\theta_{(1)}z)(\theta_{(2)}(\lambda(\theta_{(3)})s)) \\ &= \sum_{\theta} (\theta_{(1)}z)(\epsilon(\theta_{(2)})\, s) \\ &= \sum_{(\theta)} ((\theta_{(1)}\epsilon(\theta_{(2)}))z)\, s \\ &= (\theta z)\, s = s, \end{aligned}$$

as we wanted. □

Theorem 3.3 (Schneider [Sch77]). *Let R be a local domain with fraction field $K = \mathrm{Frac}(R)$ of characteristic zero. Let H be a finite cocommutative R-Hopf algebra and let P and Q be finite left H-modules. If $K \otimes_R P \cong K \otimes_R Q$ as $K \otimes_R H$-modules, then $P \cong Q$ as H-modules.*

Proposition 3.4. *[Chi00, (13.6)] Let L/K be an A-Galois extension of p-adic fields and call S and R the corresponding valuation rings. Let H be an R-Hopf order in A such that $H \subseteq \mathfrak{A}_A$ and S is H-projective. Then, S is H-free (in particular, $H = \mathfrak{A}_A$).*

Proof. Let us check that we can apply Schneider theorem. Since K is p-adic, $\mathrm{char}(K) = 0$. On the other hand, H is an R-Hopf order in A, and hence a finite cocommutative R-Hopf algebra. Since L/K is A-Galois, $L \cong A$ as A-modules. But $L = K \otimes_R S$ and $A = K \otimes_R H$, so Schneider theorem gives that $S \cong H$ as H-modules, that is, S is H-free. □

Joining Theorem 3.2 and Proposition 3.4, we obtain:

Corollary 3.5. *Let L/K be an A-Galois extension of p-adic fields and call S and R the corresponding valuation rings. Let H be an R-Hopf order in A such that $H \subseteq \mathfrak{A}_A$ and S is H-tame. Then, S is H-free (in particular, $H = \mathfrak{A}_A$).*

3.3 Hopf order implies freeness

Theorem 3.6. *[Chi00, (13.3)] Let L/K be an A-Galois extension of local fields, let R be the valuation ring of K and let S be the integral closure of R in L. If $\mathfrak{A}_A$ is an R-Hopf order in A, then S is $\mathfrak{A}_A$-tame.*

Proof. Since R is local, $I = \int_{\mathfrak{A}_A}^l$ is R-free of rank one. Let θ be an R-generator of I. Since L/K is A-Galois, $L^A = K$. Then we can prove easily that $S^{\mathfrak{A}_A} = R$. By Proposition 2.2, $\theta S \subseteq S^{\mathfrak{A}_A}$, whence θS is an ideal of R. If π is an uniformizer of R, this means that $\theta S = \pi^i R$ for some $i \geq 0$, so $\frac{\theta}{\pi^i} S = R$. In particular, $\frac{\theta}{\pi^i} \in \mathfrak{A}_A$.

Let us check that $\frac{\theta}{\pi^i}$ is actually a left integral of $\mathfrak{A}_H$. Indeed, given $\alpha \in \mathfrak{A}_A$, since θ is a left integral, $\frac{\theta}{\pi^i}\alpha = \frac{\epsilon_{\mathfrak{A}_A}(\theta)}{\pi^i}\alpha$. Now,

$$\epsilon_{\mathfrak{A}_A}(\theta) = \epsilon_{\mathfrak{A}_A}\left(\frac{\theta}{\pi^i}\pi^i\right) = \epsilon_{\mathfrak{A}_A}\left(\frac{\theta}{\pi^i}\right)\epsilon_{\mathfrak{A}_A}(\pi^i) = \epsilon_{\mathfrak{A}_A}\left(\frac{\theta}{\pi^i}\right)\pi^i,$$

and joining this with the last expression gives $\frac{\theta}{\pi^i}\alpha = \epsilon_{\mathfrak{A}_A}(\frac{\theta}{\pi^i})\alpha$, as desired.

Then, we have proved that $\frac{\theta}{\pi^i} \in I$, while θ is a generator of I as R-module. Then, $i = 0$ and $\theta S = R$, so S is $\mathfrak{A}_A$-tame. □

Corollary 3.7. *If $\mathfrak{A}_A$ is an R-Hopf order, then S is $\mathfrak{A}_A$-free.*

3.4 H-Galois implies H-tame

To prove this implication, we will need Morita theory. Let R be a commutative ring with unity and H a finite cocommutative R-Hopf algebra. Let S be an H-Galois extension and let $E = \mathrm{End}_R(S)$. Then, there is an equivalence of categories (see [Chi00, (2.13)]):

$$\begin{array}{ccc} {}_R\mathcal{M} & \longrightarrow & {}_E\mathcal{M} \\ N & \longmapsto & S \otimes_R N \end{array},$$

$$\begin{array}{ccc} {}_E\mathcal{M} & \longrightarrow & {}_R\mathcal{M} \\ M & \longmapsto & M^H \end{array}.$$

Proposition 3.8. *[Chi00, (14.3)] For any left E-module M, we have*

$$M^H \cong \int_H^l M.$$

In particular, $M \cong S \otimes \int_H^l M$ as left E-modules.

Proof. Since S is H-Galois, by the proof of [Chi00, (2.13)], we have the isomorphism

$$M^H \cong \mathrm{Hom}_E(S, M)$$

for any E-module M. Now, we claim that

$$\mathrm{Hom}_E(S, M) \cong \mathrm{Hom}_E(S, E) \otimes_E M.$$

Indeed, if we fix M, the equality holds for $S = E$, hence for a free E-module, hence for a projective E-module. Then it holds for S.

Joining the two isomorphisms above, we have

$$M^H \cong \mathrm{Hom}_E(S, E) \otimes_E M.$$

In particular, for $M = E$ we have

$$E^H \cong \mathrm{Hom}_E(S, E).$$

Then,

$$M^H \cong E^H \otimes_E M.$$

It is enough to prove that $E^H \cong \int_H^l E$, because in that case

$$M^H \cong \int_H^l E \otimes_E M \cong \int_H^l M.$$

Let us prove $E^H \cong \int_H^l E$. Since S is H-Galois, $E \cong H \otimes S$ as left H-modules. Then, the last isomorphism is equivalent to $(H \otimes_R S)^H \cong \int_H^l (H \otimes_R S)$. Now, if $P = \sum_{i=1}^n Rv_i$ is a free R-module, then

$$(H \otimes P)^H = \left(\sum_{i=1}^n Hv_i \right)^H = \sum_{i=1}^n \int_H^l v_i = \sum_{i=1}^n \int_H^l (Hv_i) = \int_H^l (H \otimes P),$$

so the isomorphism holds for any free R-module. Since S is R-projective, it is the direct summand of a free R-module, and then it also holds for S.

Finally, using the previous equivalence,

$$M \cong S \otimes M^H \cong S \otimes \int_H^l M.$$

□

Proposition 3.9. *[Chi00, (14.1)] Let S be an H-Galois extension of R. Then, S is H-tame.*

Proof. Since S is H-Galois, $\operatorname{rank}_R(S) = \operatorname{rank}_R(H)$ and S is H-faithful. On the other hand, applying Proposition 3.8 to the E-module $M = S$, we have $\int_H^l S = R$. Then S is H-tame. □

3.5 H-free implies H-tame

Corollary 3.10. *[Chi00, (14.5)] Let S be an H-module algebra with $S^H = R$ and $S \cong H^*$ as H-modules. Then, S is H-tame.*

Proof. Since H^* is faithful as H-module and $S \cong H^*$ as H-modules, S is faithful as H-module. On the other hand, $\operatorname{rank}_R(S) = \operatorname{rank}_R(H^*) = \operatorname{rank}_R(H)$. In order to check that S is H-tame, it remains to prove that $\int_H^l S = R$. The image of $\int_H^l S$ under the isomorphism $S \cong H^*$ is $\int_H^l H^*$. Now, H^* is the trivial H-Galois extension, so by Proposition 3.9, H^* is H-tame. Hence, $\int_H^l H^* = R$, proving that $\int_H^l S = R$. □

In particular, when R is a local ring, we have that $H \cong H^*$ as H-modules. Then:

Corollary 3.11. *Assume that R is a local ring. If S is H-free, then S is H-tame.*

3.6 Equivalence between notions

If the Hopf algebra is local, then the three notions studied in this lecture are equivalent. Namely:

Theorem 3.12. *[Chi00, (14.7)] Let R be a local ring, let H be a local cocommutative R-Hopf algebra with module of integrals $R\theta$ and let S be a finite R-algebra which is also a faithful H-module algebra. The following are equivalent:*

1. *S is H-tame.*
2. *S is H-free.*
3. *S is H-Galois.*

If so, any element $t \in S$ satisfying $\theta t = 1$ is a free generator of S as H-module.

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