Induced Hopf Galois structures

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Group theoretical

G-stable regular
Subgroups of Perm(G)

Hopf algebra
Hopf action

Skew left braces
With multiplicative group
isomorphic to G

Hopf Galois structures of
Galois field extensions with
Galois group isomorphic to G

Yang Baxter
Equations

Local or global fields
Arithmetic questions

Semidirect products: $G = J \rtimes G'$ $(N,.)=J\times G'$ $(N,\circ)=G$

Group theoretical Hopf-Galois

G group , G' a core-free subgroup

$\lambda_{G,G'}: G \to \mathrm{Perm}(G/G')$ left action on left cosets

A Hopf-Galois structure for (G,G') is N a regular subgroup of Perm(G/G') normalized by $\lambda_{G,G'}(G)$

Almost classical N $\subset \lambda_{G,G'}(G)$

Equivalently, N normal complement of $\lambda_{G,G'}(G')$ (Prop 4.1 G-P)

Almost classical

(G,G') almost classical iff G' has normal complement

Normal complements $\longleftrightarrow$ Almost classical structures

J $\longleftrightarrow$ N = $\lambda_{G,G'}$ (J)

G= $D_{16} = \langle r, s \mid r^8 = s^2 = e, srs = r^{-1} \rangle$ G'= $\langle s \rangle$

$J_1 = \langle r \rangle \simeq \mathbb{Z}_8$ $J_2 = \langle r^2, rs \rangle \simeq D_8$

Hopf-Galois "correspondence"

5.1. THEOREM [2, Theorem 7.6]. *If we define for a k-sub-Hopf algebra W of H*

$$\mathrm{Fix}(W) = \{x \in K \mid \mu(w)(x) = \varepsilon(w) \cdot x \text{ all } w \in W\},$$

then the map Fix:

$$\{W \subset H \text{ sub-Hopf algebra}\} \to \{E \mid k \subset E \subset k,\ E \text{ field}\}$$

is injective and inclusion-reversing.

Thm 5.2 Almost classical $\Longrightarrow$ $\exists$ Hopf Galois structure sucht that also surjective

$N^{opp} = \mathrm{Cent}_{Perm(G/G')}(N)$

Group theoretical formulation

Theorem 2.3. *If K/k is a Hopf Galois extension with Hopf algebra $H = \widetilde{K}[N]^G$ for a regular subgroup N of* $\mathrm{Perm}(G/G')$, *then the map*

$$\begin{aligned} \mathcal{F}_N\colon \{\textit{Subgroups } N' \subseteq N \textit{ stable under } \lambda(G)\} &\longrightarrow \{\textit{Fields } E \mid k \subseteq E \subseteq K\} \\ N' &\longmapsto K^{\widetilde{K}[N']^G} \end{aligned}$$

is injective and inclusion reversing.

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ON THE GALOIS CORRESPONDENCE THEOREM IN SEPARABLE HOPF GALOIS THEORY

Teresa Crespo, Anna Rio, and Montserrat Vela

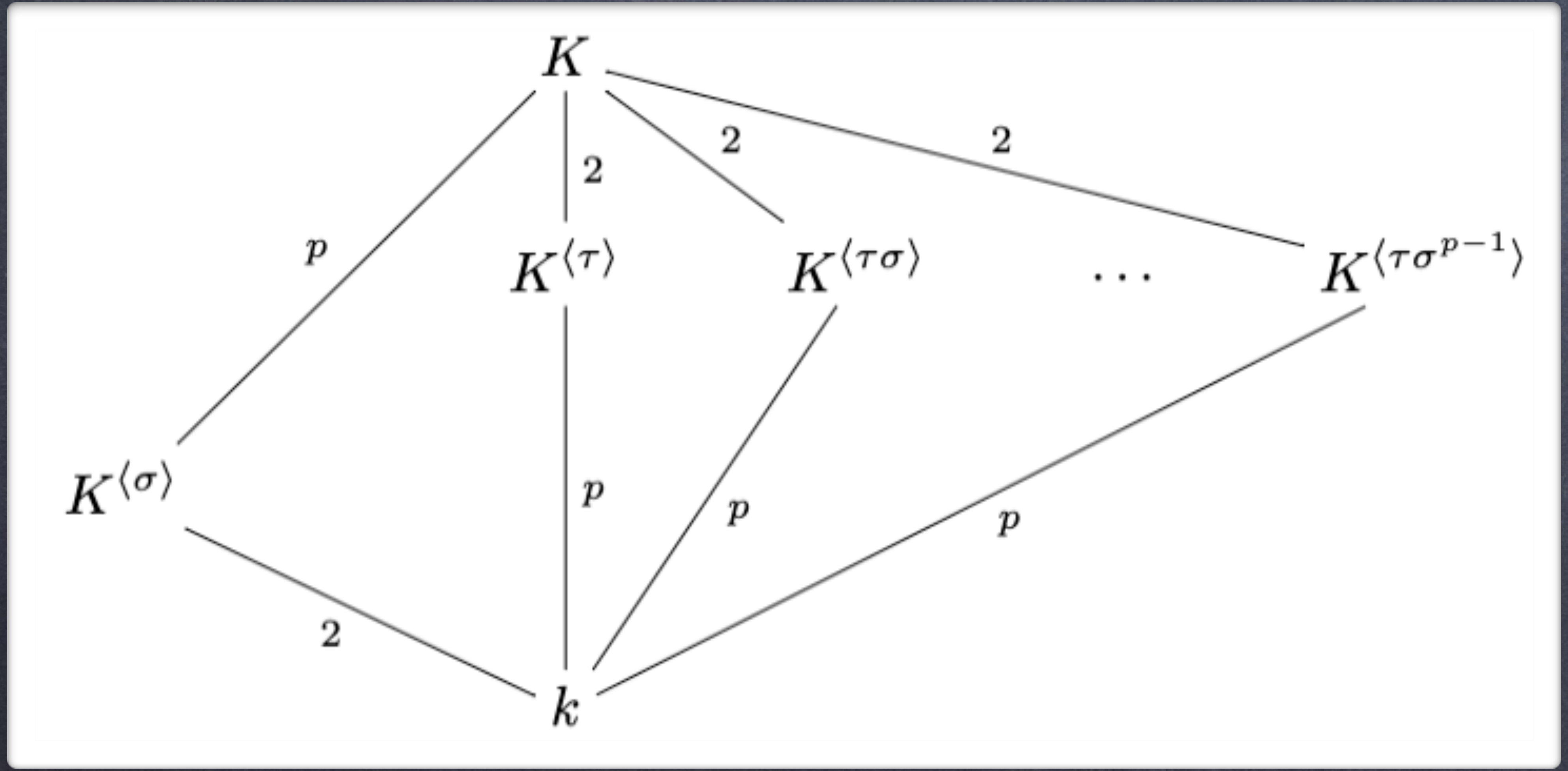
Theorem 3.4. *Let $p \geq 5$ be a prime number, d a nontrivial divisor of $p-1$. Let K/k be an extension of degree pd such that its Galois closure $\widetilde{K}$ has Galois group over k the Frobenius group $F_{p(p-1)}$. We can endow K/k with a non almost classically Galois Hopf Galois structure of type C_{pd} such that the Galois correspondence is one-to-one. We can also endow K/k with a Hopf Galois structure of type F_{pd}, which is almost classically Galois exactly when $\gcd((p-1)/d, d) = 1$ and such that the Galois correspondence is always one-to-one.*

Corollary 3.5. *There exist Hopf Galois extensions which are not almost classically Galois but may be endowed with a Hopf Galois structure such that the Galois correspondence is one-to-one.*

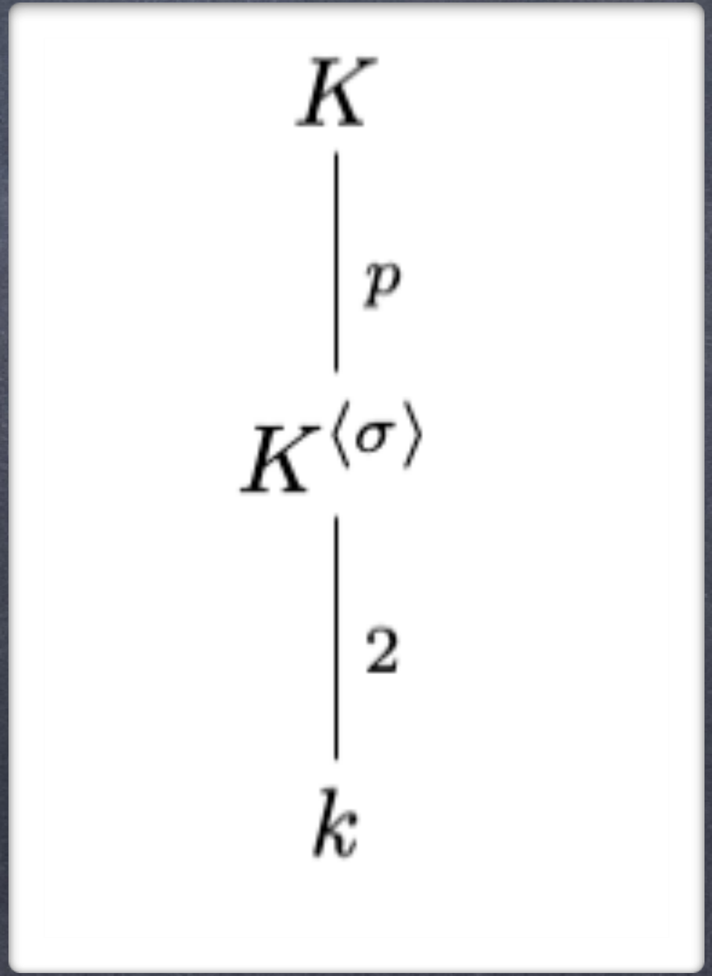
Theorem 5.4. *There exist separable Hopf Galois extensions K/k such that the Galois correspondence is not bijective for any of its Hopf Galois structures.*

G=D_{2p} Structures

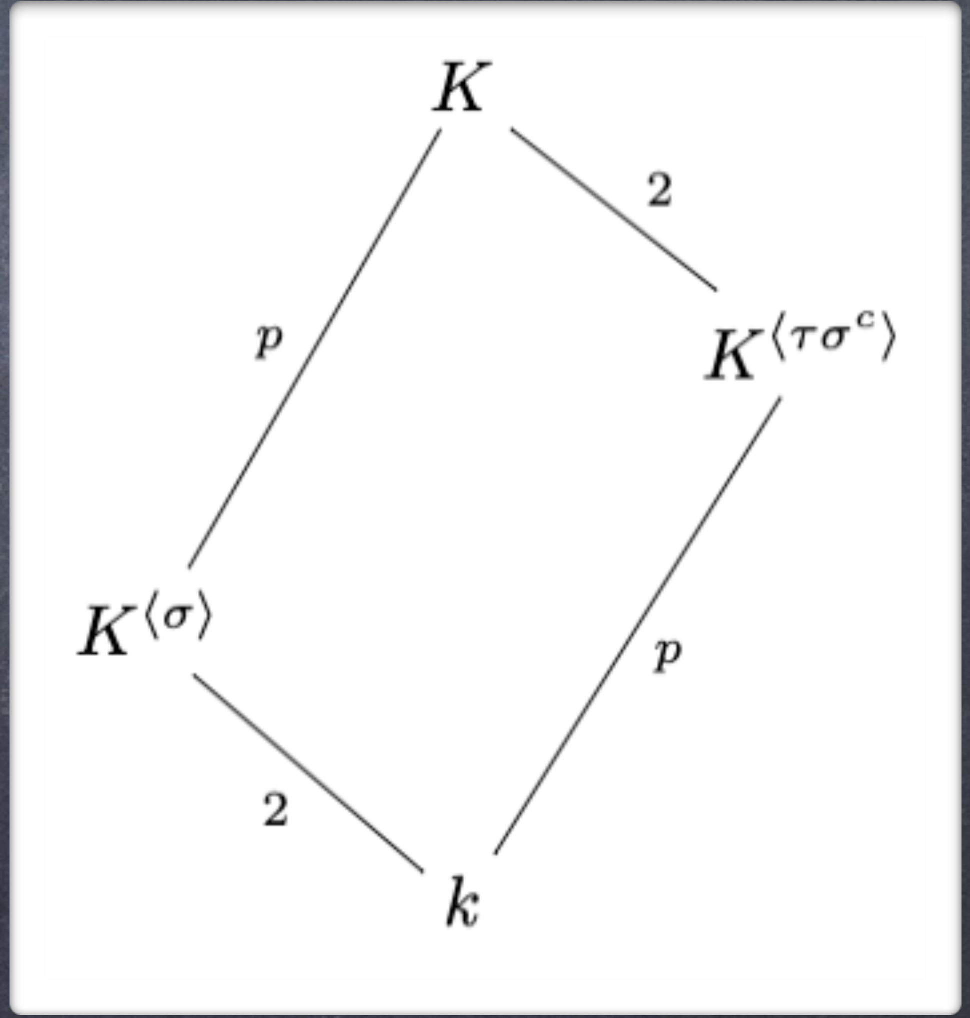
Classical



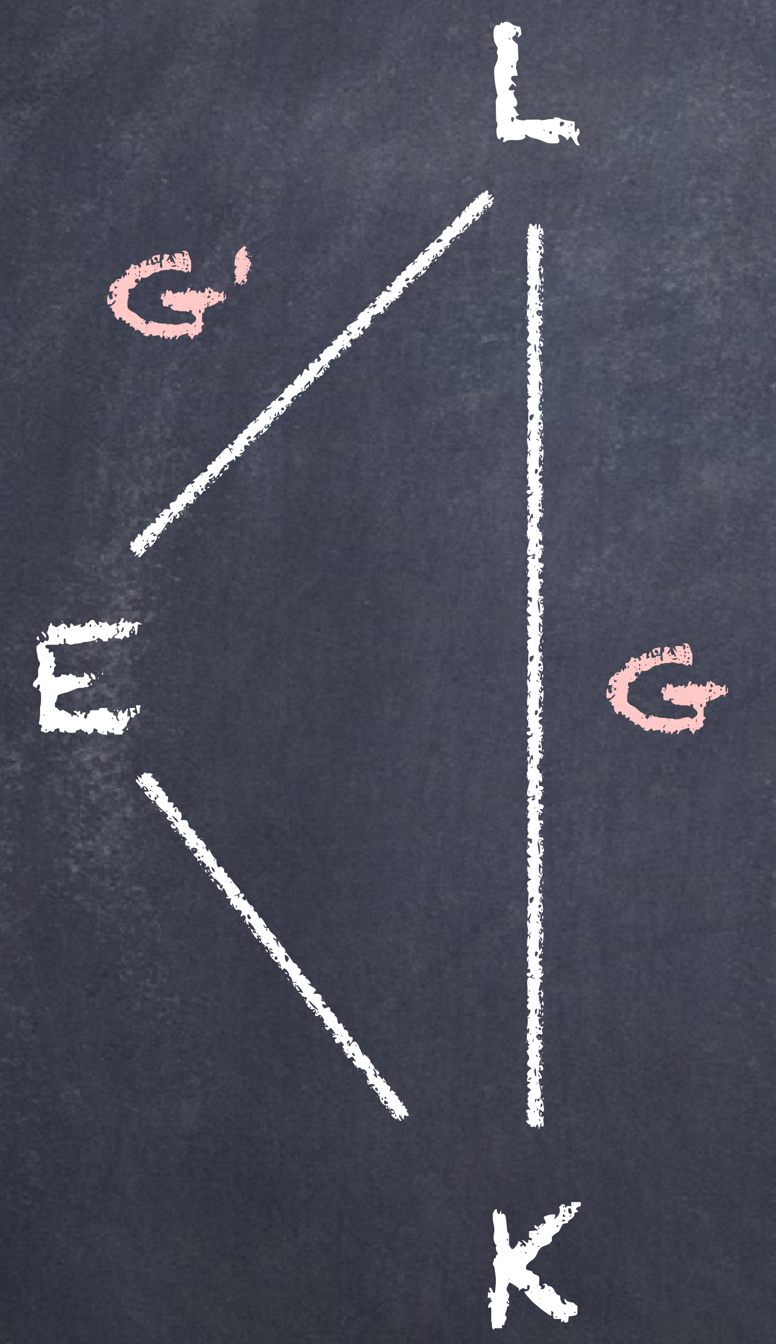
Canonical Non Classical



Cyclic (Induced)



Related? Hopf Galois structures



(G,G') Hopf Galois structures for E/K

(G,1) Hopf Galois structures for L/K

(G',1) Hopf Galois structures for L/E

Related? Hopf Galois structures

(G,G') Hopf Galois structures for E/K
Regular subgroups of Perm(G/G') normalized by $\lambda_{G,G'}(G)$

(G,1) Hopf Galois structures for L/K
Regular subgroups of Perm(G) normalized by $\lambda_G(G)$

(G',1) Hopf Galois structures for L/E
Regular subgroups of Perm(G') normalized by $\lambda_{G'}(G')$

Normal complements

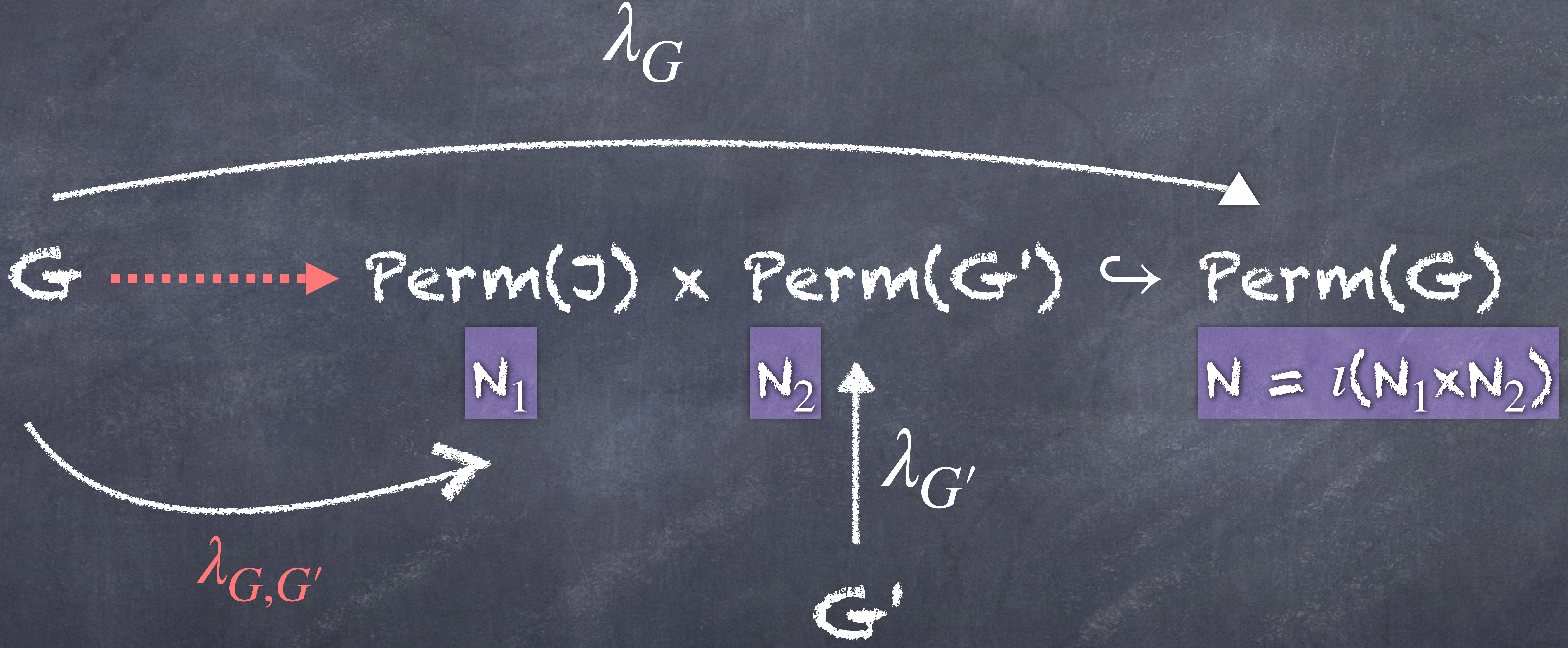
A normal complemement J for G' provides a transversal for G/G' and an embedding

ι: Perm(G/G')xPerm(G')=Perm(J)xPerm(G') → Perm(G)

$$\iota(\sigma, \tau)(g) = \iota(\sigma, \tau)(xy) = \sigma(x)\tau(y)$$

$\iota(\sigma, \tau)(g) \in$ Perm(G)
ι morphism, ι injective

Due to unique decompositions
g = xy, x in J, y in G'



λ_G
G
Perm(J) x Perm(G') ↪ Perm(G)
N_1
N_2
$N = \iota(N_1 \times N_2)$
$\lambda_{G'}$
$\lambda_{G,G'}$
G'

Regular representation of semidirect product

Action of element g=xy   x in J   y in G'

For a in G $ga = xyx_a y_a = xyx_a y^{-1} y y_a = x x_a^{\,y} \, y y_a$

G ⇢ Perm(J) x Perm(G')

g = xy ⟶ ($\lambda_J(x)\Phi_y$, $\lambda_{G'}(y)$)

$\Phi_y \in$ Aut(J)

$xy \cdot x_1 G' = xyx_1 y^{-1} G' = x x_1^{y} G'$

Action on cosets

Induced Hopf Galois structures of G

Theorem

If N_1 is a Hopf Galois structure for (G,G') and N_2 is a Hopf Galois structure for $(G',1)$, then $N = \iota(N_1 \times N_2)$ is a Hopf Galois structure for $(G,1)$

Corollary

If $G = J \rtimes G'$, then $(G,1)$ has Hopf Galois structures of type $J \times G'$

(Almost classical for (G,G') and classical for $(G',1)$)

Dihedral group of order 16 has Hopf Galois structures of type $\mathbb{Z}_8 \times \mathbb{Z}_2$ and structures of type $D_8 \times \mathbb{Z}_2$

Dihedral group of order 8 has Hopf Galois structures of type $\mathbb{Z}_4 \times \mathbb{Z}_2$ and structures of type $V_4 \times \mathbb{Z}_2$

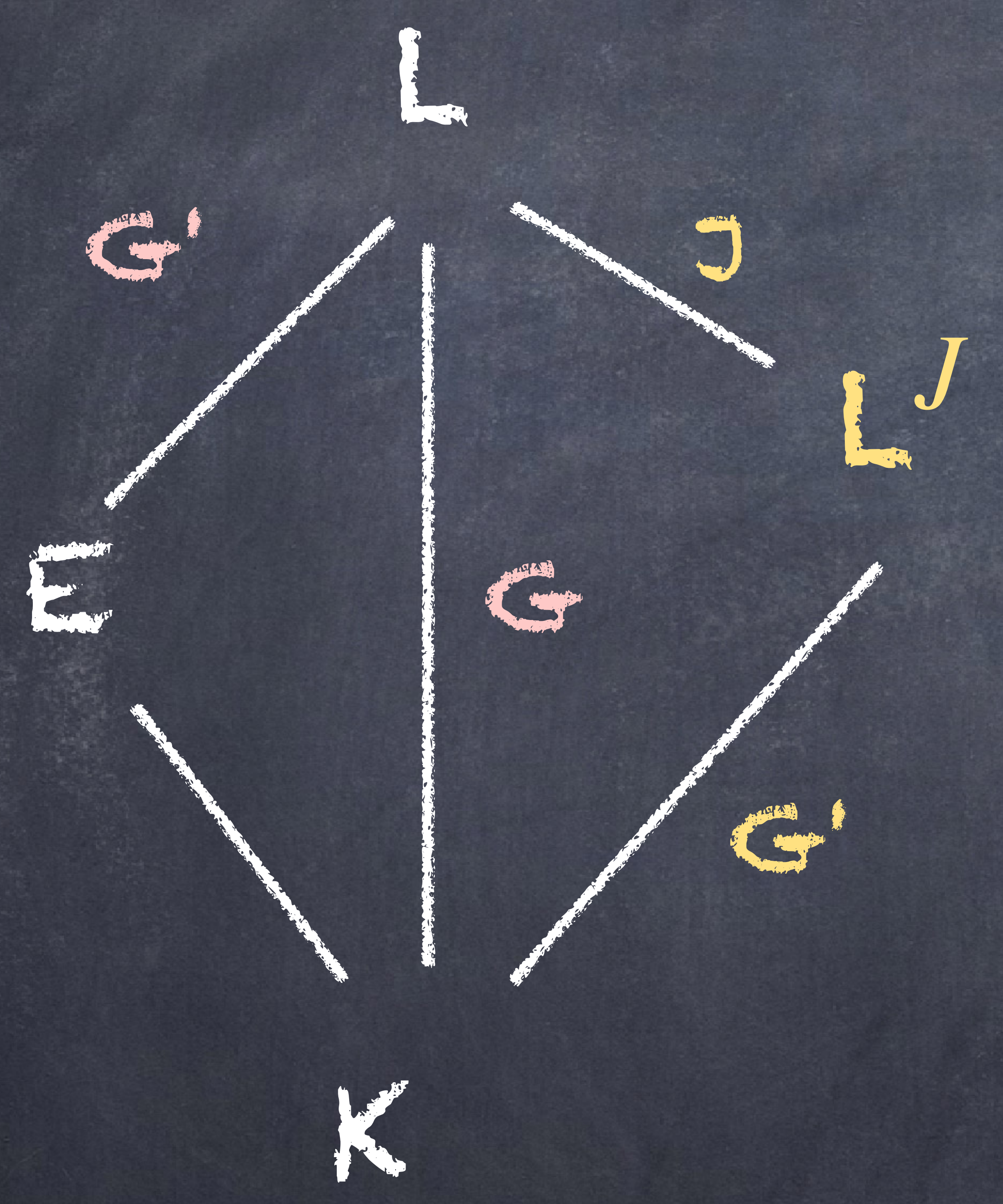
- Galois extensions with Galois group $S_3 = C_3 \rtimes C_2$ have induced Hopf Galois structures of cyclic type $C_6 = C_3 \times C_2$.
- Galois extensions with Galois group $D_{2n} = C_n \rtimes C_2$ have induced Hopf Galois structures of type $C_n \times C_2$.
- Galois extensions with Galois group $S_n = A_n \rtimes C_2$ have induced Hopf Galois structures of type $A_n \times C_2$.
- Galois extensions with Galois group $A_4 = V_4 \rtimes C_3$ have induced Hopf Galois structures of type $V_4 \times C_3$.
- Galois extensions with Galois group a Frobenius group $G = H \rtimes G'$, where H is the Frobenius kernel and G' a Frobenius complement, have induced Hopf Galois structures of type $H \times G'$. Let us note that Sonn [17] has proved that all Frobenius groups occur as Galois groups over $\mathbb{Q}$.

Split structures which are induced

Theorem 9. *Let K/k be a finite Galois field extension, $n = [K : k]$, $G = \mathrm{Gal}(K/k)$. Let K/k be given a split Hopf Galois structure by a regular subgroup N of S_n such that $N = N_1 \times N_2$ with N_1 and N_2 G-stable subgroups of N. Let $F = K^{N_2}$ be the subfield of K fixed by N_2 and let us assume that $G' = \mathrm{Gal}(K/F)$ has a normal complement*

in G. Then K/F is Hopf Galois with group N_2 and F/k is Hopf Galois with group N_1. Moreover the Hopf Galois structure of K/k given by N is induced by the Hopf Galois structures given by N_1 and N_2.

Hopf Galois structures $G = J \rtimes G'$



(G,G') Hopf Galois structures for E/K

(G,1) Hopf Galois structures for L/K

(G',1) Hopf Galois structures for L/E

Hopf Galois structures for L^J/K

"Induced" Hopf algebra

(G,G') Hopf Galois structures for E/K
Regular subgroups of Perm(J) normalized by $\lambda_{G,G'}(G)$

$H_1=L[N_1]^G$

(G',1) Hopf Galois structures for L^J/K (or L/E)
Regular subgroups of Perm(G') normalized by $\lambda_{G'}(G')$

$H_2=L^J[N_2]^{G'}$

($G=J\rtimes G'$,1) induced Hopf Galois structures for L/K

$H=H_1\otimes H_2$

Induced Hopf Galois Structures and their Local Hopf Galois Modules

Daniel Gil-Muñoz, Anna Rio

To appear in Pub Mat

Proposition 5.5. *Let $N_1 \subseteq \mathrm{Perm}(J)$ be regular and normalized by $\lambda_c(G)$ and let $N_2 \subseteq \mathrm{Perm}(G/J)$ be regular and normalized by $\lambda^{G/J}(G/J)$. Then $N = \iota(N_1 \times N_2) \subseteq \mathrm{Perm}(G)$ gives the induced Hopf Galois structure of L/K. Therefore, the corresponding Hopf algebras are $H = L[N]^G$, $H_1 = L[N_1]^G$ and $\overline{H} = F[N_2]^{G/J}$. Then,*

$$H = H_1 \otimes_K \overline{H}.$$

Proof.

$$\begin{aligned} H &= L[N]^G = L[\iota(N_1 \times N_2)]^G = \big(L[\iota(N_1 \times 1) \times \iota(1 \times N_2)]\big)^G \\ &= \big(L[\iota(N_1 \times 1)] \otimes_K L[\iota(1 \times N_2)]\big)^G = \big(L[N_1] \otimes_K L[N_2]\big)^G \end{aligned}$$

Since the action of conjugation by $\lambda(G)$ on $\mathrm{Perm}(G)$ factors through $\mathrm{Perm}(J) \times \mathrm{Perm}(G/J)$ as conjugation by $\lambda_c(G)$ on the first component and conjugation by $\lambda^{G/J}(G/J)$ on the second one, we have

$$H = L[N_1]^G \otimes_K F[N_2]^{G/J} = H_1 \otimes \overline{H}$$

"Induced" Hopf action

(G,G') Hopf Galois structures for E/K

Regular subgroups of Perm(J) normalized by $\lambda_{G,G'}(G)$

$H_1 = L[N_1]^G$

$\rho_{H_1} : H_1 \to \mathrm{End}_K(E)$

(G',1) Hopf Galois structures for L^J/K (or L/E)

Regular subgroups of Perm(G') normalized by $\lambda_{G'}(G')$

$H_2 = L^J[N_2]^{G'}$

$\rho_{H_2} : H_2 \to \mathrm{End}_K(L^J)$

$(G = J \rtimes G', 1)$ induced Hopf Galois structures for L/K

$\rho_H = \rho_{H_1} \otimes \rho_{H_2}$

Proposition 5.8. *With the previous notations for induced Hopf Galois structures, let* $\rho_H : H \to \mathrm{End}_K(L)$, $\rho_{H_1} : H_1 \to \mathrm{End}_K(E)$ *and* $\rho_{\overline{H}} : \overline{H} \to \mathrm{End}_K(F)$ *be the representations obtained from the respective Hopf actions. We have* $L = E \otimes_K F$, $H = H_1 \otimes \overline{H}$ *and*

$$\rho_H = \rho_{H_1} \otimes \rho_{\overline{H}}$$

That is, for $w \in H_1$, $\eta \in \overline{H}$, $\alpha \in E$ *and* $z \in F$, $(w \otimes \eta) \cdot (\alpha \otimes z) = (w \cdot \alpha) \otimes (\eta \cdot z)$.

Proof. As $w \in L[N_1]^G$ and $\eta \in F[N_2]^{G/J}$, let us write

$$w = \sum_{i=1}^{r} c_i n_i^{(1)},\ c_i \in L, \qquad \eta = \sum_{j=1}^{u} d_j n_j^{(2)},\ d_j \in F,$$

where $N_1 = \{n_i^{(1)}\}_{i=1}^r$ and $N_2 = \{n_j^{(2)}\}_{j=1}^u$. Recall that $\iota(n_i^{(1)}, n_j^{(2)})(Id_G) = n_i^{(1)}(\mathrm{Id}_J) n_j^{(2)}(\mathrm{Id}_{G'})$. Then:

$$\begin{aligned}(w \otimes \eta) \cdot (\alpha \otimes z) &= \left(\sum_{i=1}^{r} \sum_{j=1}^{u} c_i d_j \iota((n_i^{(1)}, n_j^{(2)})) \right) \cdot (\alpha \otimes z) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{u} c_i d_j \iota((n_i^{(1)}, n_j^{(2)}))^{-1} (\mathrm{Id_G})(\alpha z) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{u} c_i d_j \iota(((n_i^{(1)})^{-1}, (n_j^{(2)})^{-1}))(\mathrm{Id_G})(\alpha z) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{u} c_i d_j (n_i^{(1)})^{-1}(\mathrm{Id_J})(\alpha)(n_j^{(2)})^{-1}(\mathrm{Id_{G'}})(z) \\ &= \left(\sum_{i=1}^{r} c_i (n_i^{(1)})^{-1}(\mathrm{Id_J})(\alpha) \right) \left(\sum_{j=1}^{u} d_j (n_j^{(2)})^{-1}(\mathrm{Id_{G'}})(z) \right) \\ &= \left(\sum_{i=1}^{r} c_i n_i^{(1)} \right) \cdot \alpha \left(\sum_{j=1}^{u} d_j n_j^{(2)} \right) \cdot z = (w \cdot \alpha) \otimes (\eta \cdot z).\end{aligned}$$

$$\rho_H : H \to \mathrm{End}_K(L)$$

$$\Big(\sum_{n \in N} c_n n\Big) x = \sum_{n \in N} c_n\, n^{-1}(\overline{1}_G)(x)$$

Suitable choice of basis, matrices of ρ_H are Kronecker products of matrices of ρ_{H_1} and ρ_{H_2}

Induced structures mimic classical direct products

Classical

Proposition 1.2. *Let L/K be a finite Galois extension with Galois group G and assume that G can be written as a direct product $G = J \times G'$. Let $E = L^{G'}$ and $F = L^J$. Then:*

1. *E/K and F/K are Galois extensions.*
2. *$L = EF$ and $E \cap F = K$.*
3. *E/K and F/K are linearly disjoint, namely the canonical map $E \otimes_K F \longrightarrow EF$ is a K-isomorphism.*
4. *$K[G] = K[J] \otimes_K K[G']$.*
5. *The Galois action of $K[G]$ on L is the Kronecker product of the Galois actions of $K[J]$ on E and $K[G']$ on F.*

Induced structures mimic classical direct products

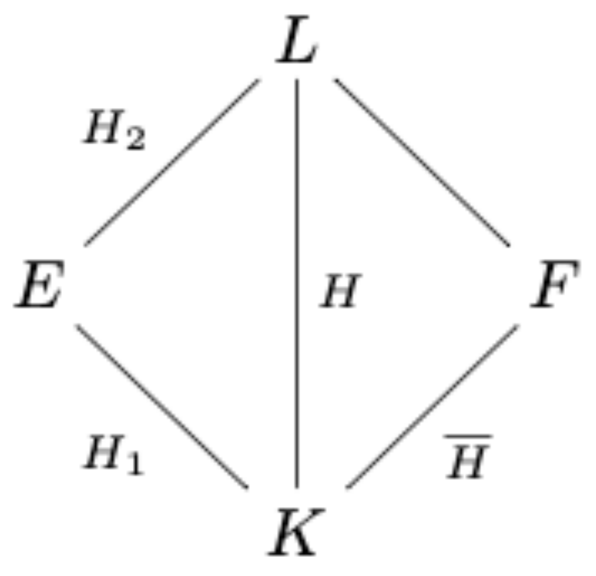
Induced

Theorem 1.3. *Let L/K be a finite Galois extension with Galois group $G = J \rtimes G'$. Let $E = L^{G'}$ and $F = L^J$. Then:*

1. *E/K and F/K are Hopf Galois extensions.*
2. *$L = EF$ and $E \cap F = K$.*
3. *E/K and F/K are linearly disjoint.*

Let E/K be H_1-Galois and let L/E be H_2-Galois. We consider the corresponding induced Hopf Galois structure of L/K. Let H be its associated Hopf algebra. Then:

4. *$H = H_1 \otimes_K \overline{H}$, where $\overline{H}$ is the Hopf algebra of the Hopf Galois structure of F/K such that $H \otimes_K E = H_2$ (see Proposition 5.3).*



5. *The Hopf action of H on L is the Kronecker product of the Hopf actions of H_1 on E and $\overline{H}$ on F.*

Arithmetic

Number fields, p-adic fields...

Freeness of ring of integers over associated order?

$$\mathfrak{A}_H = \{h \in H \ : \ h\mathcal{O}_L \subseteq \mathcal{O}_L\}$$

Arithmetic

Proposition 1.4. *Let K be the quotient field of a Dedekind domain $\mathcal{O}_K$ and let E/K, F/K be finite Galois extensions. Put $L = EF$ and suppose that E/K and F/K are arithmetically disjoint. Then:*

1. $\mathfrak{A}_{L/F} = \mathfrak{A}_{E/K} \otimes_{\mathcal{O}_K} \mathcal{O}_F$ *and* $\mathfrak{A}_{L/K} = \mathfrak{A}_{E/K} \otimes_{\mathcal{O}_K} \mathfrak{A}_{F/K}$.
2. *If there exists some* $\gamma \in \mathcal{O}_E$ *with* $\mathcal{O}_E = \mathfrak{A}_{E/K} \cdot \gamma$, *then* $\mathcal{O}_L = \mathfrak{A}_{L/F} \cdot (\gamma \otimes 1)$. *If there also exists* $\delta \in \mathcal{O}_F$ *with* $\mathcal{O}_F = \mathfrak{A}_{F/K} \cdot \delta$, *then* $\mathcal{O}_L = \mathfrak{A}_{L/K} \cdot (\gamma \otimes \delta)$.

Classic

Byott-Lettl

Theorem 1.5. *Let K be the quotient field of a principal ideal domain $\mathcal{O}_K$, L/K a finite separable Hopf Galois extension and $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L. Assume that the structure is an induced one and its Hopf algebra is $H = H_1 \otimes_K \overline{H}$. If E/K and F/K are arithmetically disjoint then the following statements hold:*

1. $\mathfrak{A}_H = \mathfrak{A}_{H_1} \otimes_{\mathcal{O}_K} \mathfrak{A}_{\overline{H}}$.
2. *If* $\mathcal{O}_E$ *is* $\mathfrak{A}_{H_1}$*-free and* $\mathcal{O}_F$ *is* $\mathfrak{A}_{\overline{H}}$*-free, then* $\mathcal{O}_L$ *is* $\mathfrak{A}_H$*-free. Moreover, an* $\mathfrak{A}_H$*-generator of* $\mathcal{O}_L$ *is the product of an* $\mathfrak{A}_{H_1}$*-generator of* $\mathcal{O}_E$ *and an* $\mathfrak{A}_{\overline{H}}$*-generator of* $\mathcal{O}_F$.

Hopf Galois

Daniel's thesis

Columns of a certain matrix give basis for the associated order

Linearly disjoint $\mathfrak{O}_L = \mathfrak{O}_E \otimes_{\mathcal{O}_K} \mathfrak{O}_F$

Induced matrix is the Kronecker product of factors matrices

Associated order is product of associated orders