On the algebra structure of Hopf algebras occurring in Hopf-Galois theory

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$$\mathrm{Forms}_{L/K}(A) \to \mathrm{H}^1(G, \mathrm{Aut}_L(L \otimes_K A)).$$

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This agrees with Childs' decription, where the semilinear automorphisms are extracted from an action of the entire endomorphism ring $End_K(L)$.

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This confirms our observation: In the category of finite-dimensional vector spaces, all forms are trivial.

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The are reminiscent of cosine and sine. Indeed they satisfy $c^2 + s^2 = 1$, and one can check $B = K[c, s]$.

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$M = K(w)$ with $w^4 = u \in K^\times$. (We assume $[M : K] = 4$.)

Then there is an action, making M into a B-Hopf Galois extension.
We have for example: $cw = 0, sw = w, cw^2 = -w^2, sw^2 = 0$.

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For every finite group N and every N-extension L/K, the anti-classical Hopf form $B = B_N(L/K)$ of $A = K[N]$ is isomorphic to A as a K-algebra.

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(2) ϑ is liftable, if there is a homomorphism $\Theta : G \to N$ with $\vartheta_g = C(\Theta(g))$, all $g \in G$.

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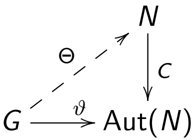

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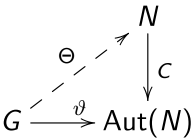

Obviously, liftable cocycles are inner. Moreover, both notions "inner" and "liftable" make also sense for cohomology classes (behave well with respect to equivalence of cocycles).

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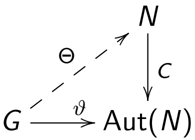

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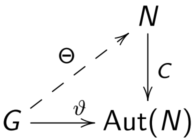

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We remark in passing: If N is centerless, then $\mathrm{Inn}(N)$ identifies with N (via C), and then "inner" and "liftable" are the same.

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The proof will use a bit of cohomology (this will be explained, cheating just a little ...), plus a straightforward generalization of Hilbert 90, which will be stated but not proved.

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So we will pretend that A is already a simple algebra (which is never literally true!) It will suffice that the form B defined by the given cocycle is isomorphic as an algebra to A.

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So Θ is equivalent to the trivial cocycle $G \to A_L^\times$; applying c gives that ϑ is equivalent to the trivial cocycle $G \to \mathrm{Aut}_L(A_L)$. Looks like cheating, but it works! — q.e.d.

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So the conclusion of the main theorem does not hold; of course ϑ is not inner.

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D_4 has two generators s and t, of order 4 and 2 respectively, and we have the rule

$$tst = s^3 (= s^{-1}).$$

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So what really matters is what happens to the non-abelian part $M_2(K)$; the corresponding factor B' of B will also be a central simple K-algebra of dimension 4, and the question is whether it is a matrix algebra or a skew field.

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Then Q_8 and D_4 are in a way similar enough for ϑ to be a group homomorphism but not similar enough for ϑ to be liftable through $C : N \to \mathrm{Inn}(N)$.

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