Greither-Pareigis theorem and Byott translation

[Ref: Childs book, Chapter 2]

GOAL: DESCRIBE THE CLASSIFICATION OF HOPF GALOIS STRUCTURES (HGS) ON A FINITE AND SEPARABLE FIELD EXTENSION.

1. RECAP

Def: L/K finite and separable; H finite cocommutative K-Hopf-algebra.
L/K is H-Galois if L is an H-mod-alg and

$$J : L \otimes H \longrightarrow \mathrm{End}_K(L) \quad \text{is bijective.}$$
$$\ell \otimes h \longmapsto (m \mapsto \ell h(m))$$

BASE CHANGE. L/K finite & sep., $F \supseteq K$ finite.

1) $\boxed{L/K \ H\text{-Galois} \overset{F\otimes\cdot}{\rightsquigarrow} F\otimes L/F \ \ F\otimes H\text{-Galois}}$

2) Suppose L H-mod-alg over K

$\boxed{(F\otimes L)/F \ \ F\otimes H\text{-Galois with action induced by } H \curvearrowright L \Rightarrow L/K \ \ H\text{-Galois}}$

GALOIS DESCENT.

[Lemma 10.24, Notes] S/K H-Galois; $\ {}_K\mathcal{M} \underset{(\cdot)^H}{\overset{S\otimes_K \cdot}{\rightleftarrows}} {}_{\mathrm{End}_K(S)}\mathcal{M}$ ($\simeq$)

In particular for $M \in {}_{E}\mathcal{M} \rightsquigarrow M \simeq S \otimes M^H$

Remark: L/K Galois $\Rightarrow (\mathrm{End}_K(L), \circ) \overset{J}{\simeq} (L\otimes KG, \#)$ as algebras [Prop 10.12, Notes]

$\Rightarrow$ the $\mathrm{End}_K(L)$-mod are the $L\otimes KG$-mod

Def: L/K Galois, G; A L-vec. sp.
A is a G-compatible L-vec. s if:
- A is a KG-mod
- $\cdot s : L\otimes A \longrightarrow A$ (scalar mult.) is G-equivariant, that is,

$$\begin{tikzcd} L\otimes A \arrow[r, "\cdot s"] \arrow[dr, phantom, "\circlearrowright"] & A \\ L\otimes A \arrow[u, "G\curvearrowright"] \arrow[r, "\cdot s"'] & A \arrow[u, "G"'] \end{tikzcd}$$

PROP:

1) A is a $L\otimes KG$-mod IFF A is a G-compatible L-vec.sp.

2) $f: A \to B$ is a $L\otimes KG$-mod hom IFF f is a G-equivariant L-linear map

THEREFORE FOR A L-vec.sp, f L-linear:

- A DESCENDS $\iff$ A IS A $L\otimes KG$-MOD
  $\iff$ G ACTS ON A COMPATIBLY WITH THE L-vec.sp. STRUCT.

  (In this case $A \simeq L \otimes A^G$, where A^G K-vec.sp.)

- f DESCENDS $\iff$ f IS A $L\otimes KG$-mod HOM.
  $\iff$ f IS A G-EQUIVARIANT L-LINEAR MAP.

  (In this case $f = id \otimes f_0$, where $f_0 : A^G \to B^G$ K-linear)

WE CAN DEFINE G-compatible algebras
G-compatible Hopf algebras

IN THE SAME WAY: KG-mod + G-action compatible with structure maps,

2. GP THEOREM

BASICS.

Def: X finite set;
$N \leq Perm(X)$ is REGULAR if two of the following hold:

1) $|N| = |X|$
2) $N \curvearrowright X$ is transitive
3) $stab_N(x) = 1_N \quad \forall x \in X$

THE SPACE XE: E field, X finite set.

$$XE = E\text{-vector space } Map(X, E) = \{f : X \to E\}$$

Basis for XE $\quad \{u_x\}_{x\in X}$, def by

$$u_x : X \to E$$
$$y \mapsto u_x(y) = \delta_{x,y} = \begin{cases} 0 & \text{IF } x \neq y \\ 1 & \text{IF } x = y \end{cases}$$

XE E-alg. with componentwise multipl.

- The elements of the form "u_x for some $x \in X$" are PRIMITIVE
- the u_x's are ORTHOGONALS and IDEMPOTENTS :

$$(u_x u_z)(y) = u_x(y)\, u_z(y) = \delta_{x,y}\, \delta_{z,y}$$

IF $x \neq z \Rightarrow u_x u_z = 0$

OTHERWISE $x = z$: $(u_x u_x)(y) = u_x^2(y) = \delta_{x,y}^2 = \delta_{x,y} = u_x(y)$

$$\rightsquigarrow \boxed{u_x u_z = \begin{cases} 0 & \text{IF } x \neq z \\ u_x & \text{IF } x = z \end{cases}} \rightsquigarrow$$

the u_x's are a basis of primitive orthogonal idempotents.

TOWARDS THE CLASSIFICATION THEOREM

THEOREM (SPECIAL CASE): E field, X finite set

1) XE/E H-Galois $\Rightarrow H = EN$ where N is (identified with) a regular subgroup of $\mathrm{Perm}(X)$.

2) N reg subgr. of $\mathrm{Perm}(X) \Rightarrow XE/E$ EN-Galois.

Pf (SKETCH):

1) • $H = EN$, $N = \{$grouplike elem. of $H\}$

$$H \simeq H^{**}$$
$$N \longleftrightarrow \mathcal{N} = \{\nu_i\}_{i=1,\dots,n}$$

ν_i = grouplike elem.s

$$\underbrace{E\times\cdots\times E}_{n^2\text{-times}} \simeq \mathrm{Map}(X\times X, E) \simeq XE \otimes_E XE \simeq XE \otimes_E H^* \simeq \underbrace{H^* \times \cdots \times H^*}_{n\text{-times}}$$

$n = |X|$

XE/E H-Galois $\Rightarrow XE/E$ H^*-object

$\Rightarrow H^* \simeq \underbrace{E\times\cdots\times E}_{n\text{-times}}$ as E-algebras

$$\nu_i : H^* \xrightarrow{\simeq} \underbrace{E\times\cdots\times E}_{n} \xrightarrow{\pi_i} E \quad \text{homomorph. of } E\text{-alg.}$$

RECALL [Remark 9.36, Notes]:

$f \in H^* = \mathrm{Hom}_K(H,K)$ IS GROUPLIKE IFF f IS AN ALGEBRA HOM.

$\rightsquigarrow \nu_i$'s are grouplike elements of H^{**}

N IS THE GROUP OF ALL GROUPLIKE EL.S OF H

• $N \leq \mathrm{Perm}(X)$

$EN \curvearrowright XE$ mod-alg.

$$\begin{cases} 0 & \text{IF } x \neq y \\ \nu(u_x) & \text{IF } x = y \end{cases} = \nu\cdot(u_x u_y) \overset{\text{mod-alg.}}{=} \mu(\overbrace{\Delta(\nu)}^{\text{grouplike}}(u_x \otimes u_y)) = \nu(u_x)\,\nu(u_y)$$

$\rightsquigarrow \forall x \in X\ \exists y \in X$ s.t. $\nu(u_x) = u_y$

$$\begin{aligned} N &\hookrightarrow \mathrm{Perm}(X) \\ \nu &\mapsto \gamma : x \mapsto y \quad \text{if } \nu(x) = y \end{aligned}$$

- ONE CAN CHECK THAT N IS ALSO REGULAR AND 2) HOLDS. $\square$

GP THEOREM.

SETUP:

$$\begin{tikzcd} E \arrow[d, dash] \arrow[d, bend left, dash, "G'"] \arrow[dd, bend right=60, dash, "G"'] \\ L \arrow[d, dash] \\ K \end{tikzcd} \qquad X = G/G' \text{ set of cosets}$$

L/K finite & sep.
E normal closure
$(\star)$

Def: the translation map is

$$\begin{aligned} \lambda : G &\longrightarrow \mathrm{Perm}(X) \\ \sigma &\longmapsto \lambda_\sigma : \bar{\tau} \mapsto \overline{\sigma\tau} \end{aligned}$$

LEMMA: λ IS INJECTIVE

THEOREM (GP):
Assume $(\star)$, then

$$\boxed{\{\text{HGS on } L/K\} \longleftrightarrow \left\{\begin{array}{l}\text{REGULAR SGR } N \leq \mathrm{Perm}(X) \\ \text{NORMALIZED BY } \lambda(G)\end{array}\right\}}$$

P (SKETCH):

- A HGS ON L/K IDENTIFIES A REGULAR SGR NORMALIZED BY $\lambda(G)$

L/K H-Galois $\overset{\text{base change}}{\rightsquigarrow}$ $E\otimes L/E$ $E\otimes H$-Galois

$\rightsquigarrow \alpha : E\otimes H \curvearrowright E\otimes L$ as mod-alg

$$\longrightarrow \quad \begin{aligned} \phi : E\otimes L &\simeq X_E = \mathrm{Map}(G/G', E) \\ e\otimes l &\mapsto (\bar{\sigma} \mapsto e\,\sigma(l)) \end{aligned}$$

ϕ is an isomorphism of E-alg.
G-modules ($= KG$-mod)

$G \curvearrowright E\otimes L$ is given by G-action on the 1st comp.
$G \curvearrowright X_E$ is given by $\sigma \cdot f = \sigma f : \bar{\tau} \mapsto \sigma f(\overline{\sigma^{-1}\tau})$

$\alpha : E\otimes H \curvearrowright X\bar{\tau}$ by special case $E\otimes H \simeq EN$
N regular sgr of $\mathrm{Perm}(X)$)

CLAIM: $G \curvearrowright E\otimes H$ translates to $G \curvearrowright EN$ given by
$$\sigma\cdot(e\nu) = \sigma(e)(\lambda_\sigma \nu \lambda_\sigma^{-1})$$

• $G \curvearrowright XE$
$$\sigma(u_{\bar\gamma})(\bar\rho) = \sigma(u_{\bar\gamma}(\overline{\sigma^{-1}\rho})) = \sigma(\delta_{\bar\gamma,\overline{\sigma^{-1}\rho}})$$
$$u_{\overline{\sigma\gamma}}(\bar\rho) = \delta_{\overline{\sigma\gamma},\bar\rho} = \delta_{\bar\gamma,\overline{\sigma^{-1}\rho}}$$
$$\rightsquigarrow \sigma(u_{\bar\gamma}) = u_{\overline{\sigma\gamma}} = u_{\lambda_\sigma(\bar\gamma)}$$

• $G \curvearrowright EN$ $\quad EN$ IS A G-COMPATIBLE E-Hopf-alg
$$\rightsquigarrow \Delta(\sigma(\nu)) = \sigma(\Delta(\nu)) = \sigma(\nu\otimes\nu) = \sigma(\nu)\otimes\sigma(\nu)$$
↳ ν grouplike

$\sigma(\nu)$ is grouplike $(\in N)$
$\Rightarrow G$ ACTS ON N

α is G-equivariant
ϕ is iso of G-mod $\quad \Longrightarrow \quad \alpha' : EN \curvearrowright X\bar\tau$ is G-equivariant

$\sigma(\nu\otimes u_{\bar\gamma}) = \sigma(\nu)\otimes\sigma(u_{\bar\gamma}) \longmapsto \sigma(\nu)\sigma(u_{\bar\gamma})$

$$\begin{tikzcd}
EN\otimes X\bar\tau \arrow[r, "\alpha"] & X\bar\tau \\
EN\otimes X\bar\tau \arrow[u, "G\curvearrowright"] \arrow[r, "\alpha'"'] \arrow[ur, phantom, "\circlearrowright"] & X\bar\tau \arrow[u, "G\curvearrowright"']
\end{tikzcd}$$

$\sigma(\nu(u_{\bar\gamma}))$ $\rightsquigarrow \sigma(\nu(u_{\bar\gamma})) = \sigma(\nu)\sigma(u_{\bar\gamma})$

$\nu\otimes u_{\bar\gamma} \longmapsto \nu u_{\bar\gamma}$

$$\sigma(\nu(u_{\bar\tau})) = \sigma(u_{\nu(\bar\gamma)}) = u_{\lambda_\sigma(\nu(\bar\gamma))}$$
$$\|$$
$$\sigma(\nu)\sigma(u_{\bar\gamma}) = \underbrace{\sigma(\nu)}_{\in N}(u_{\lambda_\sigma(\bar\gamma)}) = u_{\sigma(\nu)(\lambda_\sigma(\bar\gamma))}$$
$$\Rightarrow \lambda_\sigma(\nu(\bar\gamma)) = \sigma(\nu)\lambda_\sigma(\bar\gamma))$$
$$\Rightarrow \lambda_\sigma\nu = \sigma(\nu)\lambda_\sigma$$

$\Rightarrow \lambda_\sigma \nu \lambda_\sigma^{-1} = \sigma(\nu)$

- $N \overset{\text{REG.}}{\leq} \mathrm{Perm}(X)$ NORMALIZED BY $\lambda(G)$ CORRESPONDS TO A UNIQUE HGS

N REGULAR $\overset{\text{special case}}{\Longrightarrow}$ XE/E IS EN-Galois

$\alpha : EN \curvearrowright XE$ as mod-alg.

- XE IS A G-COMP. E-ALG.
- EN IS A G-COMP. E-Hopf-ALG.
- $\alpha : EN \curvearrowright XE$ is G-equivariant

check!

$\Rightarrow$ XE, EN are $E \otimes KG$-mod; α is a $E\otimes KG$-mod homomorph.

BY GALOIS DESCENT WE GET

$$\underbrace{\alpha^G}_{\text{mod-alg. action}} : \underbrace{(EN)^G}_{K\text{-Hopf-alg}} \otimes \underbrace{(XE)^G}_{K\text{-alg.}} \longrightarrow (XE)^G$$

$\rightsquigarrow (XE)^G / E^G = K$ IS $(EN)^G$-GALOIS

CLAIM:
$$L \longrightarrow (XE)^G \quad \simeq$$
$$\ell \longmapsto \sum_{\tau \in X} \sigma(\ell) u_\sigma$$

check!

$\square$

EXAMPLES.

1 APPLICATION TO GALOIS EXTENSIONS

L/K Galois, G. $(E = L,\ X = G)$

- $\lambda : G \longrightarrow \mathrm{Perm}(G)$, $\sigma \longmapsto \lambda_\sigma : \tau \mapsto \sigma\tau$ — LEFT REGULAR REPRESENTATION $\rightsquigarrow$ $\lambda(G)$ IS REGULAR AND NORMALIZED BY ITSELF

- $\rho : G \longrightarrow \mathrm{Perm}(G)$, $\sigma \longmapsto \rho_\sigma : \tau \mapsto \tau\sigma^{-1}$ — RIGHT REGULAR REPRESENTATION $\rightsquigarrow$ $\rho(G)$ IS REGULAR AND NORMALIZED BY $\lambda(G)$

$$(\lambda_\sigma \rho_\pi \lambda_\sigma^{-1})(\tau) = \sigma\sigma^{-1}\tau\pi = \tau\pi = \rho_\pi(\tau)$$

$$\rightsquigarrow \quad \lambda_\sigma \rho_\pi \lambda_\sigma^{-1} = \rho_\pi$$

WE HAVE: $\lambda(G) = \rho(G) \iff G$ IS ABELIAN

($\Leftarrow$) G ab $\Rightarrow \lambda_\sigma(\tau) = \sigma\tau = \tau\sigma = \rho_{\sigma^{-1}}(\tau) \Rightarrow \lambda(G) = \rho(G)$

($\Rightarrow$) $\lambda_\pi = \rho_\sigma$; $\quad \pi = \lambda_\pi(1) = \rho_\sigma(1) = \sigma^{-1} \Rightarrow \lambda_{\sigma^{-1}} = \rho_\sigma$

suppose that $\exists\, \sigma, \tau \in G$ s.t. $\sigma\tau \neq \tau\sigma$

$\rightsquigarrow \rho_\sigma(\tau) = \tau\sigma^{-1} \neq \sigma^{-1}\tau = \lambda_{\sigma^{-1}}(\tau)$ ↯

$\square$

COR: IF L/K IS GALOIS NON ABELIAN, THEN THERE ARE AT LEAST TWO DIFFERENT HGS

(those corresponding to $\lambda(G)$ and $\rho(G)$)

$\lambda(G) \updownarrow$ "STANDARD NON-CLASSICAL" STRUCTURE

$\rho(G) \updownarrow$ GP CLASSICAL GALOIS STRUCTURE

PROP: $\rho(G)$ CORRESPONDS TO THE CLASSICAL GALOIS STRUCTURE

P: GP $\rightsquigarrow N = \rho(G) \longleftrightarrow H = (EN)^G = (L\rho(G))^G$

G ∾ L Galois
G ∾ ρ(G) is trivial

$= L^G \rho(G)$

$= K\rho(G)$

Moreover $H \curvearrowright L$ is given by $\underbrace{EN}_{\cong L\rho(G)} \curvearrowright XE = GL$

$\ell \updownarrow \sum_{\sigma \in G} \sigma(\ell) u_\sigma \qquad L \updownarrow (XE)^G = (GL)^G$

$$\underbrace{\rho_\sigma}_{\in N = \rho(G)} \cdot \underbrace{\left(\sum_\tau \tau(\ell) u_\tau\right)}_{\in (XE)^G} = \sum_\tau \tau(\ell) \rho_\sigma(u_\tau) = \sum_\tau \tau(\ell) u_{\underbrace{\tau\sigma^{-1}}_{= \rho}} = \underbrace{\sum_\rho \rho\sigma(\ell) u_\rho}_{\in (XE)^G}$$

$\sum_\rho \rho\sigma(\ell) u_\rho \updownarrow \sigma(\ell) \in L$

THE ACTION $H \curvearrowright (XE)^G \longleftrightarrow G \curvearrowright L$

HOPF ACTION $K\rho(G) \curvearrowright (GL)^G \longleftrightarrow KG \curvearrowright L$

$\square$

② EX OF HOPF GALOIS, NOT GALOIS, EXT.

$K = \mathbb{Q}$

$L = \mathbb{Q}(\sqrt[3]{2})$ $\Big\}$ not Galois

$E = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$

$G \simeq S_3 \; = Gal(E/K)$

$G' \simeq C_2 \; = Gal(E/L)$

$\lambda : S_3 \longrightarrow \mathrm{Perm}(X) = \mathrm{Perm}(G/G') = S_3$

SGRPS OF $\mathrm{Perm}(X)$ NORMALISED BY $\lambda(G) \equiv S_3$

$\equiv$ NORMAL SGRPS OF S_3

A_3 NORMAL, REGULAR

③ EX. OF EXT. NOT HOPF GALOIS

$\forall k$ of degree 5 s.t. $\mathrm{Gal}(E/k) \simeq S_5$

$\lambda : S_5 \longrightarrow S_5$

SGRPS NORMALIZED BY $\lambda(G) = S_5$ $\equiv$ NORMAL SGRPS

$\rightsquigarrow$ A_5 NORMAL, BUT IS NOT REGULAR $(|A_5| \neq \cancel{5})$.

3. BYOTT TRANSLATION

Def: The **HOLOMORPH** of N is the normalizer of $\lambda(N)$ in $\mathrm{Perm}(N)$

$$\mathrm{Hol}(N) = \{\pi \in \mathrm{Perm}(X) : \pi \text{ normalizes } \lambda(N)\}$$

PROP: $\mathrm{Hol}(N) = \rho(N) \rtimes \mathrm{Aut}(N)$

GALOIS CASE. L/K Galois, G ; N regular sgr of $\mathrm{Perm}(G)$ normalized by $\lambda(G) \equiv G$

N regular $\Rightarrow$ $b : N \longrightarrow G$, $\nu \longmapsto \nu \cdot e_G$ BIJECTIVE and induces

$$i(b) : \mathrm{Perm}\,G \longrightarrow \mathrm{Perm}\,N, \quad \pi \longmapsto b^{-1}\pi b \qquad \text{isomorphism.}$$

Note:

$N \longmapsto \lambda(N)$

$\lambda(G) \longmapsto G_0$, $G_0 \simeq G$

$i(b)(\nu) = b^{-1}\nu b \;:\; \mu \longmapsto b^{-1}\nu\, b(\mu) = b^{-1}(\nu\,\mu \cdot e_G) = \nu\mu = \lambda_\nu(\mu)$

- G_0 IS REGULAR ($\lambda(G)$ is regular and $i(b)$ iso)
- $\lambda(G)$ normalizes N $\Rightarrow$ G_0 normalizes $\lambda(N)$

REGULAR SGR. N IN $\mathrm{Perm}\,X$ NORMALIZED BY $\lambda(G)$ $\rightsquigarrow$ REGULAR SGR. G_0 IN $\mathrm{Perm}\,N$ ISO TO G AND WHICH $\underbrace{\text{NORMALIZES } \lambda(N)}_{\subseteq \mathrm{Hol}\,N}$

$\rightsquigarrow$ REGULAR EMBEDDING
$G \hookrightarrow \mathrm{Hol}\, N$

GENERAL CASE.

THEOREM (BYOTT):
$G' \subseteq G$ finite groups, $X = G/G'$, N group of order $|X|$.
There is a bijection between:

$$\mathcal{N} = \{ \alpha : N \hookrightarrow \mathrm{Perm}\,X \text{ REGULAR EMBEDDING} \}$$
$$\mathcal{G} = \{ \beta : G \hookrightarrow \mathrm{Perm}\,N \text{ EMBEDDING} : \beta(G') = \mathrm{Stab}_{\beta(G)}(e_N) \}$$

Moreover if α, α' correspond to β, β':
1) $\alpha(N) = \alpha'(N)$ IFF $\beta(G), \beta'(G)$ ARE CONJUGATE BY AN ELEMENT IN $\mathrm{Aut}(N)$;

2) $\alpha(N)$ IS NORMALIZED BY $\lambda(G)$ IN $\mathrm{Perm}\,X$ IFF $\beta(G) \subseteq \mathrm{Hol}\,G$

$\mathcal{P}$ (SKETCH):

- $\alpha \in \mathcal{N} \Rightarrow \alpha(N)$ REGULAR $\Rightarrow \mathrm{orb}_{\alpha(N)}(\bar{e}) = \alpha(N)(\bar{e}) = X$;

$\alpha(N) \cong N$ and $\alpha : N \longrightarrow X$, $\nu \longmapsto \alpha(\nu)(\bar{e})$ BIJECTION

$i(a) : \mathrm{Perm}\,N \longrightarrow \mathrm{Perm}\,X$, $\pi \longmapsto a \pi a^{-1}$ is a iso

$i(a)^{-1} \lambda^X : G \hookrightarrow \mathrm{Perm}(X) \longrightarrow \mathrm{Perm}(N) \in \mathcal{G}$

$$f : \mathcal{N} \longrightarrow \mathcal{G}$$
$$\alpha \longmapsto i(a)^{-1} \lambda^X$$

- $\beta \in \mathcal{G}$; $b : X \longrightarrow N$, $\bar{\sigma} \longmapsto \beta(\sigma)(e_N)$

condition $\boxed{\beta(G') = \mathrm{Stab}_{\beta(G)}(e_N)} \Rightarrow$ • b well defined
• b injective

$\Rightarrow$ b bijective

$i(b) : \mathrm{Perm}(X) \longrightarrow \mathrm{Perm}(N)$, $\rho \longmapsto b \rho b^{-1}$ iso

$$i(b)^{-1}\lambda^N : N \longrightarrow \mathrm{Perm}\,N \longrightarrow \mathrm{Perm}(X) \in \mathcal{N}$$

$$g : \mathcal{G} \longrightarrow \mathcal{N}$$
$$\beta \longmapsto i(b)^{-1}\lambda^N$$

- $g \circ f = \mathrm{id}_{\mathcal{N}}$ and $f \circ g = \mathrm{id}_{\mathcal{G}}$

2) $\alpha(N)$ is normalized by $\lambda^X(G) \Rightarrow \beta(G)$ normalizes $\lambda^N(N)$

$$f(\alpha) = \beta$$
$$\| $$
$$= i(e)^{-1}\lambda^X$$

$$\lambda^X_\sigma\, \alpha(\nu)\, \lambda_{\sigma^{-1}} \in \alpha(N) \subseteq \mathrm{Perm}(X)$$

$$\Leftrightarrow i(e)^{-1}\left(\lambda^X_\sigma \alpha(\nu)\lambda^X_{\sigma^{-1}}\right) \in i(e)^{-1}\alpha(N) \subseteq \mathrm{Perm}(N)$$

and we have:

$$i(e)^{-1}(\lambda_\sigma \alpha(\nu)\lambda_{\sigma^{-1}}) = (i(e)^{-1}\lambda_\sigma)(i(e)^{-1}\alpha(\nu))(i(e)^{-1}\lambda_{\sigma^{-1}})$$
$$= \beta(\sigma)\underbrace{(i(e)^{-1}\alpha(\nu))}_{\substack{\| \\ \lambda^N_\nu}}\beta(\sigma^{-1})$$

$(i(e)^{-1}\alpha) : N \to \mathrm{Perm}\,G \to \mathrm{Perm}\,N$

$((i(e)^{-1}\alpha)(\nu))(\mu) = e^{-1}\alpha(\nu)\, e(\mu)$
$= e^{-1}\alpha(\nu)\alpha(\mu)(\bar{e})$
$= e^{-1}\alpha(\nu\mu)(\bar{e})$
$= \nu\mu = \lambda^N_\nu(\mu)$

$\Rightarrow \beta(G)$ normalizes $\lambda^N(N)$ in $\mathrm{Perm}(N)$

$\Rightarrow \beta(G) \subseteq \mathrm{Hol}(N)$

1) CHECK!

$\square$