SOME A PRIORI ESTIMATES ON RIEMANNIAN MANIFOLDS WITH CONSTANT NEGATIVE CURVATURE

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Abstract. In this note we prove some a priori estimates on Riemannian manifolds with constant negative curvature,especially for the hyperbolic space $\mathbb{H}^n$ as the the model of upper branch of the hyperboloid.

1. Introduction

In this paper we will prove the following results involving Young and Hardy estimates on $\mathbb{H}^n$:

Theorem 1.1. *Let be $f_1, f_2 \in C_0^\infty(\mathbb{H}^n)$, then*

$$\|f_1 * f_2\|_{L^s(\mathbb{H}^n)} \leq \|f_1\|_{L^r(\mathbb{H}^n)} \|f_2\|_{L^p(\mathbb{H}^n)} \tag{1.1}$$

where $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{s}$

Theorem 1.2. *Let be $f \in C_0^\infty(\mathbb{H}^n)$, then*

$$\|\hat{f}\|_{L^p(\mathbb{R}\times S^{n-1}, |c_a(\lambda)|^{-2} d\omega d\lambda)} \leq C \|f e^{d_{\mathbb{H}^n}(0,\Omega)(1-\frac{2}{q})(n-1)/2}\|_{L^q(\mathbb{H}^n)} \tag{1.2}$$

where $1 \leq q \leq 2$ *and* $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 1.3. *Let $f \in C^\infty(\mathbb{H}^n)$, then*

$$\|(d_{\mathbb{H}^n}(0_{\mathbb{H}^n}, \Omega))^{-1} f\|_{L^p(\mathbb{H}^n)} \leq \|\nabla f\|_{L^p(\mathbb{H}^n)} \tag{1.3}$$

2. Young estimates on $\mathbb{H}^n$

In this section we will look for some inequalities on the hyperbolic spaces as the model of upper branch of hyperboloid in $\mathbb{R}^{n+1}$ equipped with the standard Minkowski metric. This space is considered in harmonic analysis as simmetric homogeneous space G/K where $G = SO(1,n)$ is the Lorenz group acting on $\mathbb{H}^n$ and $K = SO(n)$ is the stabilizer $0_R = (1, 0, \dots, 0)$ in $SO(1,n)$ that is

$$\mathbb{H}^n = SO(1,n)/SO(n)$$

Definition 2.1. If $f_1, f_2 \in \mathbf{C}_0^\infty(\mathbb{H}^n)$, then the convolution $f_1 * f_2$ is definied as

$$f_1 * f_2(\Omega) = \int_G f_1(gO) f_2(g^{-1}\Omega) dg.$$

Let

$$T_{f_2} f_1 \to T_{f_2}(f_1)$$

a convolution operator definied on left invariant function

$$f_1, f_2 \in \mathcal{C}_0^\infty(\mathbb{H}^n) = \mathcal{C}_0^\infty(SO(1,n)/SO(n))$$

the operator

$$T_{f_2}(f_1) = f_1 * f_2(\Omega) = \int_G f_1(gO) f_2(g^{-1}\Omega) dg,$$

where $G = SO(1,n)$ and dg is a biinvariant Haar measure definied on G and normalized by the condition

$$\int_G f(gO) dg = \int_{\mathbb{H}^n} f(\Omega) d\Omega.$$

This condition guarantees

$$\|f\|_{L^2(G)} = |f\|_{L^2(\mathbb{H}^n)}.$$

We have

$$\begin{aligned} \|T_{f_2}(f_1)\|_{L^\infty(G)} &\leq \int_G |f_1(gO)||f_2(g^{-1}\Omega)| dg \\ &\leq \|f_2\|_{L^\infty(G)} \|f_1\|_{L^1(G)} = |f_2\|_{L^\infty(\mathbb{H}^n)} \|f_1\|_{L^1(\mathbb{H}^n)} \end{aligned}$$

obviously.

From the Lie group theory

$$\begin{aligned} \|T_{f_2}(f_1)\|_{L^1(\mathbb{H}^n)} &\leq \int_G \int_G |f_1(gO)||f_2(g^{-1}h\Omega)| dg dh \leq \|f_1\|_{L^1(G)} \|f_2\|_{L^1(G)} \\ &= \|f_1\|_{L^1(\mathbb{H}^n)} \|f_2\|_{L^1(\mathbb{H}^n)}. \end{aligned}$$

Using the interpolationon Riesz-Thorin theorem we have

$$\|f_1 * f_2\|_{L^q(\mathbb{H}^n)} \leq \|f_1\|_{L^1(\mathbb{H}^n)} \|f_2\|_{L^q(\mathbb{H}^n)}$$

for $1 \leq q \leq \infty$.
The estimate

$$\|T_{f_2}(f_1)\|_{L^\infty(\mathbb{H}^n)} \leq \|f_1\|_{L^q(\mathbb{H}^n)} \|f_2\|_{L^p(\mathbb{H}^n)}$$

with $\frac{1}{p} + \frac{1}{q} = 1$
follow immediately from the Holder inequality for the spaces $L^p(G)$.
Again, from the interpolation theorem we have

$$\|f_1 * f_2\|_{L^s(\mathbb{H}^n)} \leq \|f_1\|_{L^r(\mathbb{H}^n)} \|f_2\|_{L^p(\mathbb{H}^n)},$$

with $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{s}$
Now we focus our attemption to L^2 estimate involving the Fourier-Helgason transform [4] on $\mathbb{H}^n$.

Definition 2.2. Let be $f \in \mathcal{C}_0^\infty(\mathbb{H}^n) = \mathcal{C}_0^\infty(SO(1,n)/SO(n))$ we consider the Helgason-Fourier transform

$$\hat{f}(\lambda, \omega) = a^{(-ia\lambda + (n-1)/2)} \int_{X_a} [\Omega, \Lambda(\omega)]^{(ia\lambda - (n-1)/2)} f(\Omega) d\Omega,$$

where $\Lambda(\omega) = (1, \omega)$.

If we choose $a = 1$ we can define an operator T in the following way:

$$T : f \to \hat{f}$$

This is clearly an isometry between $L^2(\mathbb{H}^n)$ and $L^2(\mathbb{R} \times S^{n-1}, |c(\lambda)|^{-2} d\omega d\lambda)$, (for other informations on this argument we refer again to [4] or to [8]).
The Plancherel inequality give

$$\int_X |f(\Omega)|^2 d\Omega = \tag{2.1}$$

$$\frac{(2\pi)^{-n}}{2} \int_{-\infty}^{\infty} \int_{\mathbf{S}^{n-1}} |\hat{f}(\lambda, \omega)|^2 \frac{|\Gamma((n-1)/2 + ia\lambda)|^2}{|\Gamma(ia\lambda)|^2} d\omega \frac{d\lambda}{a^{n-1}},$$

using the inverse Fourier-Helgason trasform, we get

$$f(\Omega) a^{(n-1)/2} = \tag{2.2}$$

$$\int_{-\infty}^{\infty} \int_{\mathbf{S}^{n-1}} a^{(i\lambda a - (n-1)/2)} [\Omega, \Lambda(\omega)]^{(-i\lambda a - (n-1)/2)} \hat{f}(\lambda, \omega) |c_a(\lambda)|^{-2} d\omega d\lambda.$$

This allowes us to

$$\|\hat{f}\|_{L^2(\mathbb{R} \times S^{n-1}, |c(\lambda)|^{-2} d\omega d\lambda)} \leq \|f\|_{L^2(\mathbb{H}^n)}. \tag{2.3}$$

Now we need of the estimate of the term

$$[\Omega, \Lambda(\omega)]^{-(n-1)/2)}$$

. We introduce local coordinate for the radial model of $\mathbb{H}^n$:

$$\Omega = (coshr, \omega sinhr)$$

and, by the fact that we have $\Lambda(\omega) = (1, \omega)$, after a sobstitution we have

$$\begin{aligned} [\Omega, \Lambda(\omega)]^{-(n-1)/2} &= (coshr - (\omega, \omega_1) sinhr)^{-(n-1)/2} \\ &\leq (coshr - sinhr)^{-(n-1)/2)(n-1)/2)} \\ &\leq 2e^{r(n-1)/2)}. \end{aligned} \tag{2.4}$$

Moreover, taking $|\hat{f}|$ and replacing the previous estimates we have

$$|\hat{f}(\lambda, \omega)| = \int_{X_a} 2e^{r(n-1)/2} f(\Omega) \mathrm{d}\Omega,$$

dove $\Lambda(\omega) = (1, \omega)$, that is

$$\|\hat{f}\|_{L^\infty(\mathbb{R}\times S^{n-1}, |c(\lambda)|^{-2} d\omega d\lambda)} \leq C \|f e^{d_{\mathbb{H}^n}(0,\Omega)\,(n-1)/2}\|_{L^1(\mathbb{H}^n)}. \tag{2.5}$$

Using as before the Riesz-Thorin theorem we obtain:

$$\|\hat{f}\|_{L^p(\mathbb{R}\times S^{n-1}, |c(\lambda)|^{-2} d\omega d\lambda)} \leq C \|f e^{d_{\mathbb{H}^n}(0,\Omega)\,\theta(n-1)/2}\|_{L^1(\mathbb{H}^n)}, \tag{2.6}$$

with

$$\frac{1}{q} = \frac{1-\theta}{2}, \qquad \frac{1}{q} + \frac{1}{p} = 1$$

we obtain

$$\theta = 1 - \frac{2}{q}$$

and the following theorem:

Theorem 2.1. *Let be $f \in C_0^\infty(\mathbb{H}^n)$, then*

$$\|\hat{f}\|_{L^p(\mathbb{R}\times S^{n-1}, |c_a(\lambda)|^{-2} d\omega d\lambda)} \leq C \|f e^{d_{\mathbb{H}^n}(0,\Omega)\,(1-\frac{2}{q})(n-1)/2}\|_{L^q(\mathbb{H}^n)}, \tag{2.7}$$

where $1 \leq q \leq 2$ e $\frac{1}{p} + \frac{1}{q} = 1$.

3. Hardy inequality

Let be $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, a similar estimate to

$$\||x|^{-1}u\|_{L^p(\mathbb{R}^n)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{3.1}$$

on $\mathbb{H}^n$.
First of all we indicate

$$d_{\mathbb{H}^n}(0_{\mathbb{H}^n}, \Omega), \tag{3.2}$$

where $0_{\mathbb{H}^n} = (1, 0 \ldots, 0)$, is the distance on the hyperboloid. Let be, as before, the geodesic polar coordinates for the radial model

$$\Omega = (coshr, \omega sinhr)$$

and obtain

$$d_{\mathbb{H}^n}(0_{\mathbb{H}^n}, \Omega) = r.$$

Let be now

$$\|(d_{\mathbb{H}^n}(0_{\mathbb{H}^n}, \Omega))^{-1} u\|_{L^p(\mathbb{H}^n)} \leq \|\nabla u\|_{L^2(\mathbb{H}^n)}, \tag{3.3}$$

after the change of coordinates we rewrite this inequality as

$$\int_0^\infty \int_{S^{n-1}} r^{-p} |f(coshr, \omega sinhr)|^p . (sinhr)^{n-1} dr d\omega \tag{3.4}$$

Let be now

$$\int_{S^{n-1}} |f(coshr, \omega sinhr)|^p d\omega = g(r)^p \tag{3.5}$$

and

$$g(r) = \int_0^r g'(s) ds. \tag{3.6}$$

The sobstitution of (3.5) and (3.6) in (3.4), gives

$$\int_0^\infty \frac{1}{r^p} |\int_0^r g'(s) \mathrm{d}s|^p (sinhr)^{n-1} \mathrm{d}r \mathrm{d}\omega. \tag{3.7}$$

Using Holder inequality in (3.7), we have

$$\int_0^\infty \frac{1}{r^p} r^{p/p'} (sinhr)^{n-1} \int_0^r |g'(s)|^p (sinhr)^{n-1} \mathrm{d}s \mathrm{d}r, \tag{3.8}$$

then

$$\int_0^\infty \frac{1}{r^p} r^{p/p'} \int_0^r |g'(s)|^p (sinhr)^{n-1} \mathrm{d}s \mathrm{d}r \quad \leq \quad \int_0^\infty (\int_r^\infty \chi_{[suppf]}^{p/p'-p} \mathrm{d}t) |g'(r)|^p (sinhr)^{n-1} \mathrm{d}r, \tag{3.9}$$

where the function $\chi_{[suppf]}$ in the second term of the previous inequality follows from the fact that in the (3.4) we have chosen $f \in C_0^\infty(\mathbb{H}^n)$, such that we have $suppf \subset D_{m,l}(R) \subset \mathbb{H}^n$ where $D_{m,l}(R)$ are open neighborhoods definied [2] for any m, l. We obtain

$$
\begin{aligned}
\|(d_{\mathbb{H}^n}(0_{\mathbb{H}^n},\Omega))^{-1}f\|^p_{L^2(\mathbb{H}^n)} &\leq C\int_0^\infty\int_{S^{n-1}} r^p|f(coshr,\omega sinhr)|^p(sinhr)^{n-1}\mathrm{d}r\mathrm{d}\omega\\
&\leq \int_0^\infty \frac{1}{r^{-p}}|\int_0^r g'(s)\mathrm{d}s|^p(sinhr)^{n-1}\mathrm{d}r\mathrm{d}\omega\\
&\leq \int_0^\infty |g'(r)|^p(sinhr)^{n-1}\mathrm{d}r\\
&\leq \int_0^\infty \frac{1}{r^p}r^{p/p'}(sinhr)^{n-1}\int_0^r |g'(s)|^p(sinhr)^{n-1}\mathrm{d}s\mathrm{d}r\\
&\leq \|\nabla f\|_{L^p(\mathbb{H}^n)},
\end{aligned}
\tag{3.10}
$$

where in the previous estimate we used the inequalities (3.5), (3.6) and (3.9).

Consider again the partition of unity of the hyperboloid, $\{\phi_{m,l}, D_{m,l}(R)\}$, we can rewrite the function $f\in C_0^\infty(\mathbb{H}^n)$ in the followin way:

$$
f_{m,l}=\sum_{m,l}\phi_{m,l}f, \tag{3.11}
$$

from this fact comes out

$$
\begin{aligned}
\|(d_{\mathbb{H}^n}(0_{\mathbb{H}^n},\Omega))^{-1}f\|_{L^p(\mathbb{H}^n)} &\leq \sum_{m,l}\|\phi_{m,l}f\|_{L^p(\mathbb{H}^n)}\\
&\leq C'\sum_{m,l}\|\phi_{m,l}\nabla f\|_{L^p(\mathbb{H}^n)}+\\
&+ C''\sum_{m,l}\|f\nabla\phi_{m,l}\|_{L^p(\mathbb{H}^n)}.
\end{aligned}
\tag{3.12}
$$

From the fact that the sum in the right term of the previous estimete is finite and that we will use in (3.14) the following inequality due to Yau (see [3] or [9])

$$
\int_{\mathbb{H}^n}|f|^p\mathrm{d}\Omega\leq(n-1)\int_{\mathbb{H}^n}|\nabla f|^p\mathrm{d}\Omega. \tag{3.13}
$$

By the fact $\mathbb{H}^n$ is a complete simply connected manifold with sectional curvature $K<0$, we have, after the sobstitution, the following theorem:

Theorem 3.1. *Let be $f \in C_0^\infty(\mathbb{H}^n)$, then we have*

$$\|(d_{\mathbb{H}^n}(0_{\mathbb{H}^n}, \Omega))^{-1} f\|_{L^p(\mathbb{H}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{H}^n)}, \tag{3.14}$$

where ∇ is the Riemmanian connection on $(\mathbb{H}^n, h)$.

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