PARTE I

1) $f = e^{x^3+x^2y+xy^2+y^3-x-y}$

$$\begin{cases} \dfrac{\partial f}{\partial x} = (3x^2+2xy+y^2-1)\, e^{x^3+x^2y+xy^2+y^3-x-y} = 0 \\ \dfrac{\partial f}{\partial y} = (x^2+2xy+3y^2-1)\, e^{x^3+x^2y+xy^2+y^3-x-y} = 0 \end{cases} \iff$$

$$\begin{cases} 3x^2+2xy+y^2-1=0 \\ x^2+2xy+3y^2-1=0 \end{cases}$$ sottraendo:

$$2x^2-2y^2=0 \Rightarrow x=\pm y$$

$$\begin{cases} x=y \\ 6x^2=1 \end{cases} \quad \begin{cases} x=\pm\frac{1}{\sqrt{6}} \\ y=\pm\frac{1}{\sqrt{6}} \end{cases} \quad \begin{cases} x=-y \\ 2x^2=1 \end{cases} \quad \begin{cases} x=\pm\frac{1}{\sqrt{6}} \\ y=\mp\frac{1}{\sqrt{6}} \end{cases}$$

quindi ci sono i 4 punti critici $P_1\equiv\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$, $P_2\equiv\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$

$P_3\equiv\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, $P_4\equiv\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2} = \left[6x+2y+(3x^2+2xy+y^2-1)^2\right] e^{(\cdots)} \\ \dfrac{\partial^2 f}{\partial x\partial y} = \left[2x+2y+(3x^2+2xy+y^2-1)(x^2+2xy+3y^2-1)\right] e^{(\cdots)} \\ \dfrac{\partial^2 f}{\partial y^2} = \left[2x+6y+(x^2+2xy+3y^2)^2\right] e^{(\cdots)} \end{cases}$$

Valutate nei punti critici i secondi addendi in parentesi si annullano (sono le derivate parziali) quindi:

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(P_1) = \dfrac{8}{\sqrt{6}}\, e^{g(P_1)} \\ \dfrac{\partial^2 f}{\partial x \partial y}(P_1) = \dfrac{4}{\sqrt{6}}\, e^{g(P_1)} \\ \dfrac{\partial^2 f}{\partial y^2}(P_1) = \dfrac{8}{\sqrt{6}}\, e^{g(P_1)} \end{cases}$$

$g = x^3 + x^2 y + x y^2 + y^3 - x - y$

quindi $H(P_1) =$

$$\frac{4}{\sqrt{6}} e^{g(P_1)} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} > 0$$

quindi P_1 minimo.

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(P_2) = -\dfrac{8}{\sqrt{6}}\, e^{g(P_2)} \\ \dfrac{\partial^2 f}{\partial x \partial y}(P_2) = -\dfrac{4}{\sqrt{6}}\, e^{g(P_2)} \\ \dfrac{\partial^2 f}{\partial y^2}(P_2) = \dfrac{-8}{\sqrt{6}}\, e^{g(P_2)} \end{cases}$$

$$H(P_2) = \frac{4}{\sqrt{6}} e^{g(P_2)} \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} < 0$$

quindi P_2 massimo

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(P_3) = \dfrac{4}{\sqrt{2}}\, e^{g(P_3)} \\ \dfrac{\partial^2 f}{\partial x \partial y}(P_3) = 0 \\ \dfrac{\partial^2 f}{\partial y^2}(P_3) = -\dfrac{4}{\sqrt{2}}\, e^{g(P_3)} \end{cases}$$

$$H(P_3) = \frac{4}{\sqrt{2}} e^{g(P_3)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

è indefinita, quindi P_3 sella.

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(P_4) = -\dfrac{4}{\sqrt{2}}\, e^{g(P_4)} \\ \dfrac{\partial^2 f}{\partial x \partial y}(P_4) = 0 \\ \dfrac{\partial^2 f}{\partial y^2}(P_4) = \dfrac{4}{\sqrt{2}}\, e^{g(P_3)} \end{cases}$$

$$H(P_4) = \frac{4}{\sqrt{2}} e^{g(P_3)} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

è una sella.

2) $A \equiv (x,y,z)$ $\quad$ $dist(A,P) = \sqrt{(x-a)^2+y^2+z^2}$

$dist(A,Q) = \sqrt{x^2+(y-b)^2+z^2}$

$dist(A,R) = \sqrt{x^2+y^2+(z-c)^2}$

$$f(x,y,z) = (x-a)^2+y^2+z^2 + x^2+(y-b)^2+z^2 + x^2+y^2+(z-c)^2 =$$
$$= 3(x^2+y^2+z^2) - 2(ax+by+cz) + a^2+b^2+c^2$$

con vincolo $g = x+y+z-1 = 0$

$$L = f + \lambda g \qquad \begin{cases} \frac{\partial L}{\partial x} = 6x - 2a + \lambda = 0 \\ \frac{\partial L}{\partial y} = 6y - 2b + \lambda = 0 \\ \frac{\partial L}{\partial z} = 6z - 2c + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = x+y+z-1 = 0 \end{cases}$$

$\frac{1}{3}\left(a - \frac{1}{3}(a+b+c-3)\right)$

$\frac{2a - b - c + 3}{3}$

Sommando le prime 3: $\quad 6(x+y+z) - 2(a+b+c) + 3\lambda = 0$

ed usando $x+y+z=1$: $\quad 6 - 2(a+b+c) + 3\lambda = 0 \Rightarrow$

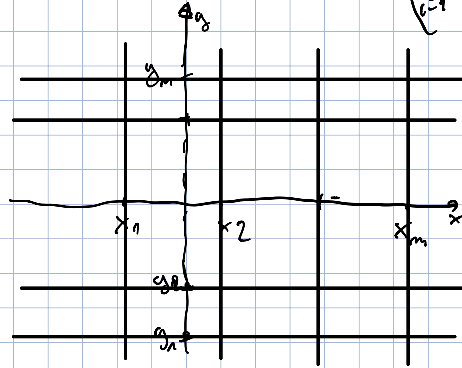
$$\lambda = \frac{2}{3}(a+b+c-3)$$

quindi:
$$x = \frac{a}{3} - \frac{1}{9}(a+b+c-3) = \frac{1}{9}(2a-b-c+3)$$
$$y = \frac{1}{9}(-a+2b-c+1)$$
$$z = \frac{1}{9}(-a-b+2c+1)$$

Poiché quando $A \to \infty$ sul vincolo $f \to +\infty$, il minimo esiste e quindi coincide col punto trovato.

3) (a) Dominio di $\vec{F}$: $\mathbb{R}^2 \setminus \left(\bigcup_{i=1}^{m} \{x = x_i\} \cup \bigcup_{j=1}^{n} \{y = y_j\} \right)$

rette verticali — rette orizzontali



(b) Ogni componente del dominio è convessa, quindi semplicemente connessa. Le derivate parziali miste sono zero, quindi uguali. Per un teorema, $\vec{F}$ è conservativo sulle componenti.

(c) Se $\vec{G} = (G_1(x,y), G_2(x,y))$ è conservativo, deve valere $\dfrac{\partial G_1}{\partial y} = \dfrac{\partial G_2}{\partial x}$. Se anche $\vec{G}' = (G_1(x,y), -G_2(x,y))$ lo è, vale anche $\dfrac{\partial G_1}{\partial y} \equiv -\dfrac{\partial G_2}{\partial x}$. Segue $\dfrac{\partial G_2}{\partial x} \equiv 0$ e quindi $\dfrac{\partial G_1}{\partial y} \equiv 0$. Ma allora $G_1 = G_1(x)$, $G_2 = G_2(y)$.

PARTE II 1)

$$M = \iiint_{D_r} \delta(x,y,z)\, dV =$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{r} e^{-k\rho} \rho^2 \operatorname{sen}\varphi \, d\rho =$$

$$= 2\pi \int_0^{\pi} \operatorname{sen}\varphi \, d\varphi \int_0^{r} e^{-k\rho} \rho^2 \, d\rho = 4\pi \left[\left(-\frac{x^2}{k} - 2\frac{x}{k^2} - \frac{2}{k^3} \right) e^{-kx} \right]_0^r$$

$$= 4\pi \left(\frac{2}{k^3} - \left(\frac{r^2}{k} + \frac{2r}{k^2} + \frac{2}{k^3} \right) e^{-kr} \right)$$

(b) $\lim_{n\to\infty} M = \frac{8\pi}{k^3} = 1 \Rightarrow k^3 = 8\pi \Rightarrow k = 2\sqrt[3]{\pi}$

2) (a) Avendo la molecola $n+3$ atomi, l'ordine di Γ vale $3(n+3)$.

(b)

	E	$2C_3$	$3\sigma_v$
θ	0	$\frac{2\pi}{3}$	0
$2\cos\theta\pm1$	3	0	1
u_n	$n+3$	n	$n+1$
$\chi(R)$	$3(n+3)$	0	$n+1$

(c) $$a_1 = \frac{1}{6}\left(3(n+3) + 0 + 3(n+1)\right) = \frac{1}{6}(6n+12) = n+2$$

$$a_2 = \frac{1}{6}\left(3(n+3) + 0 - 3(n+1)\right) = \frac{1}{6}\,6 = 1$$

$$a_3 = \frac{1}{6}\left(6(n+3) + 0 + 0\right) = n+3$$

(d) [facoltativo]. Facendo il cambiamento $\xi_1 = \sqrt{m_1}\,x_1$, $\xi_2 = \sqrt{m_2}\,x_2$ si ottiene $T = \frac{1}{2}(\dot\xi_1^2 + \dot\xi_2^2)$,

$$V = 2\left(\frac{\xi_1^2}{m_1} + \frac{\xi_1\xi_2}{\sqrt{m_1m_2}} + \frac{\xi_2^2}{m_2}\right) =$$

$$2\,[\xi_1 \;\; \xi_2]\underbrace{\begin{bmatrix}\frac{1}{m_1} & \frac{1}{2\sqrt{m_1m_2}}\\ \frac{1}{2\sqrt{m_1m_2}} & \frac{1}{m_2}\end{bmatrix}}_{B}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$$

gli autovalori di B sono dati dalle soluzioni di

$$\lambda^2 - \frac{m_1+m_2}{m_1 m_2}\lambda + \left(\frac{1}{m_1 m_2} - \frac{1}{4 m_1 m_2}\right) =$$

$$= \lambda^2 - \frac{m_1+m_2}{m_1 m_2}\lambda + \frac{3}{4 m_1 m_2} = 0$$

$$= \lambda^2 - \frac{\sigma}{p}\lambda + \frac{3}{4p} = 0$$

avendo posto

$\sigma = m_1 + m_2$

$p = m_1 m_2$

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{\sigma}{p} \pm \sqrt{\frac{\sigma^2}{p^2} - \frac{3}{p}}\right) =$$

$$= \frac{1}{2}\left(\frac{\sigma}{p} \pm \frac{1}{p}\sqrt{\sigma^2 - 3p}\right) = \frac{1}{2p}\left(\sigma \pm \sqrt{\sigma^2 - 3p}\right)$$

Un autovettore per λ_1 è soluzione di

$$\left(\frac{1}{m_1} - \lambda_1\right) x_1 + \frac{1}{2\sqrt{m_1 m_2}} x_2 = 0$$

cioè è un multiplo di $v_1 = \left(\frac{1}{2\sqrt{p}}, \lambda_1 - \frac{1}{m_1}\right) = \dots$

mentre per λ_2 si ha:

$$\frac{1}{2\sqrt{m_1 m_2}} x_1 + \left(\frac{1}{m_2} - \lambda_2\right) x_2 = 0$$

cioè i multipli di $v_2 = \left(\lambda_2 - \frac{1}{m_2}, \frac{1}{2\sqrt{p}}\right) = \dots$

Le coordinate normali si ottengono facendo la trasformazione data dalla matrice con righe gli autovettori normalizzati, cioè:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} v_1/\|v_1\| \\ \hline v_2/\|v_2\| \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} v_1/\|v_1\| \\ \hline v_2/\|v_2\| \end{bmatrix} \begin{bmatrix} \sqrt{m_1}\, x_1 \\ \sqrt{m_2}\, x_2 \end{bmatrix}$$