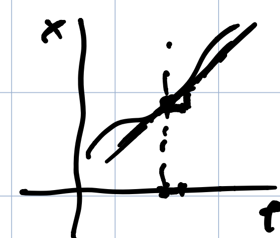
$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \qquad \det J \neq 0$$

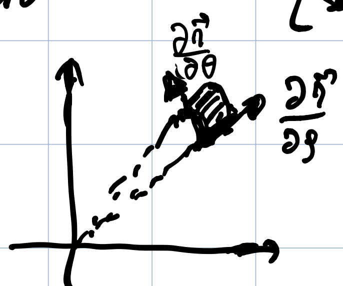
$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(u,v), y(u,v))\,|\det J|\,du\,dv$$

$$\int_a^b f(x)\,dx \qquad x = x(t) \qquad dx = x'(t)\,dt$$



$$\int_{a'}^{b'} f(x(t))\,x'(t)\,dt$$

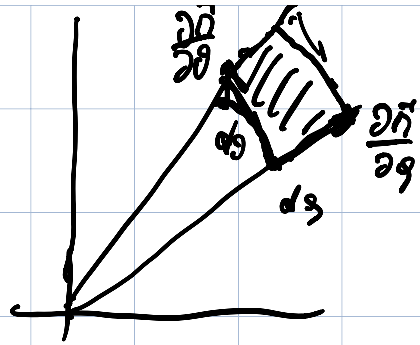
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases} \qquad J = \begin{bmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \rho\cos\theta \end{bmatrix} \qquad |\det J| = \rho$$



$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad \vec{r} = \vec{r}(u,v)$$

$u \to \vec{r}(u,v)$ Tangente: $\frac{\partial \vec{r}}{\partial u}$

$v \to \vec{r}(u,v)$ Tangente: $\frac{\partial \vec{r}}{\partial v}$



area $dA \sim$ area del parallelogramma individuato da $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}$

$$\text{Area} \sim \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| =$$ determinante della matrice con colonne aventi coordinate 2 vettori

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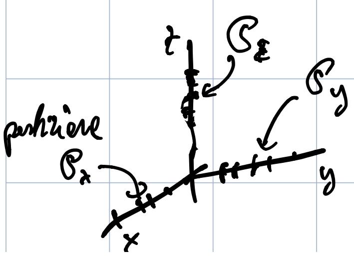
Integrali tripli

$$\iiint_D f(x,y,z)\, dV = $$

$$= \iiint_D f(x,y,z)\, dx\, dy\, dz$$

Stesse definizioni: si definisce f integrabile su un parallelepipedo $[a,b]\times[c,d]\times[e,f] =$

$$= \{(x,y,z) \in \mathbb{R}^3 \,/\, a \le x \le b,\ c \le y \le d,\ e \le z \le f \}$$



Somme di Riemann:

$$S(f,\mathcal{P}) = \sum_{i,j,k} (\text{Volume } R_{ijk}) \cdot f(P^*_{ijk}) \quad , P^*_{ijk} \in R_{ijk}$$

$$\text{Volume } R_{ijk} = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$$

Per domini più complicati (limitati) D si estende f con 0 e si dice f integrabile su D se l'estensione $\tilde{f}$ lo è in un parallelepipedo contenente D.

Formula per iterazione: le variabili si possono ordinare (ad es. x, y, z) in modo che

$$D = \{ a \le x \le b,\ \alpha(x) \le y \le \beta(x),\ \gamma(x,y) \le z \le \delta(x,y) \}$$

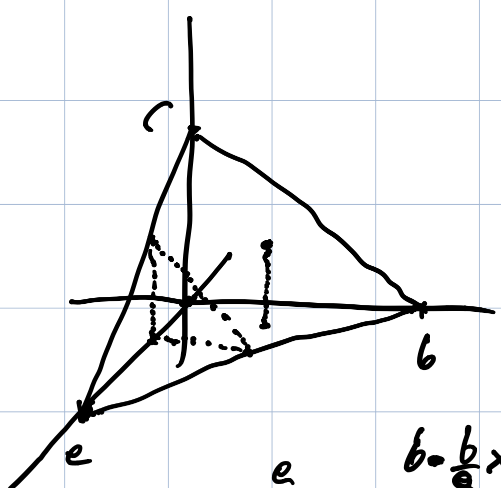
Allora: $$\iiint_D f(x,y,z)\, dV = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} \left(\int_{\gamma(x,y)}^{\delta(x,y)} f(x,y,z)\, dz \right) dy \right) dx$$

$$= \int_a^b dx \int_{\alpha(x)}^{\beta(x)} dy \int_{\gamma(x,y)}^{\delta(x,y)} f(x,y,z)\, dz$$

es: $\iiint_R x \, dV$ $\quad$ R tetraedro delimitato dai piani coordinati e da $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$a, b, c > 0$.

$$R = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x \geq 0,\ y \geq 0,\ z \geq 0,\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1 \right\}$$



$z = 0$ $\quad \frac{x}{a} + \frac{y}{b} = 1$

$y = b - \frac{b}{a} x$

$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

$$= \int_0^a dx \int_0^{b - \frac{b}{a}x} dy \int_0^{c - \frac{c}{a}x - \frac{c}{b}y} x \, dz$$

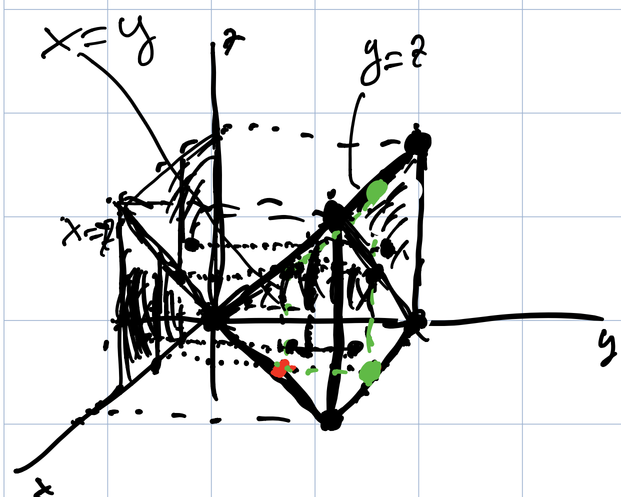
$$= \int_0^a dx \int_0^{b - \frac{b}{a}x} dy \, x\left(c - \frac{c}{a}x - \frac{c}{b}y\right) =$$

a

$$= \int_0^1 dx \left(cx\left(6-\frac{b}{a}x\right) - \frac{c}{a}x^2\left(6-\frac{b}{a}x\right) - \frac{c}{2b}\left(6-\frac{b}{a}x\right)^2 \right) = \cdots\cdots$$

$$\iiint_R \mathrm{sen}(\pi y^3)\, dV$$

R piramide con vertici $(0,0,0)$ $(0,1,0)$ $(1,1,0)$ $(1,1,1)$ $(0,1,1)$



scegliamo l'ordine x, y, z $= \int_0^1 dx \int_x^1 dy \int_0^y \mathrm{sen}(\pi y^3)\, dz$

$$= \int_0^1 dx \int_x^1 \mathrm{sen}(\pi y^3)\, y\, dy = \cdots\cdots$$

scegliamo l'ordine x, z, y :

$$= \int_0^1 dx \int_x^1 dz \int_z^1 \operatorname{sen}(\pi y^3)\, dy \qquad (1^o \text{ pezzo}$$

$$+ \int_0^1 dx \int_0^x dz \int_x^1 \operatorname{sen}(\pi y^3)\, dy \qquad (2^o \text{ pezzo}$$

Cambiamento di variabili:
$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \quad \boxed{\vec{r} = \vec{r}(u,v,w)}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \qquad \det J \neq 0$$

colonne di J: $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}$

$$\iiint_D f(x,y,z)\, dx\, dy\, dz =$$

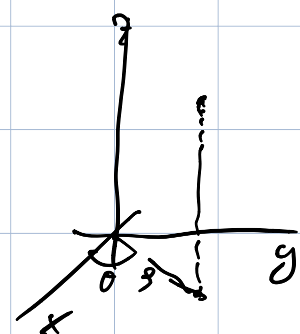
$|\det J|$ = volume del parallelepipedo con lati $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}$

$$= \iiint_{D'} f\big(x(u,v,w),\, y(u,v,w),\, z(u,v,w)\big)\, |\det J| \, du\, dv\, dw$$

<u>es.</u>

coordinate cilindriche:

(ρ, θ, z) $\quad$ ρ: distanza dall'asse z

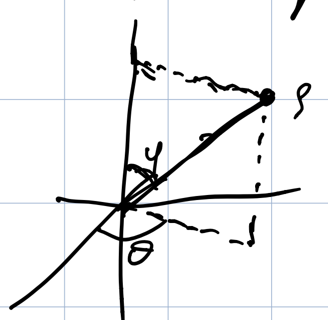


$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = z \end{cases}$$

$$J = \begin{bmatrix} \cos\theta & -\rho \operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|\det J| = \rho$$

coordinate sferiche (ρ, θ, φ)



ρ distanza dall'origine

θ: come prima $\qquad 0 \le \theta \le 2\pi$

φ: angolo che $\vec{r}$ forma con asse z

$\qquad 0 \le \varphi \le \pi$

$$\begin{cases} x = \rho \,\text{sen}\,\varphi \cos\vartheta \\ y = \rho \,\text{sen}\,\varphi \,\text{sen}\,\vartheta \\ z = \rho \cos\varphi \end{cases} \qquad J = \begin{bmatrix} \text{sen}\,\varphi \cos\vartheta & -\rho \,\text{sen}\,\varphi \,\text{sen}\,\vartheta & \rho \cos\varphi \cos\vartheta \\ \text{sen}\,\varphi \,\text{sen}\,\vartheta & \rho \,\text{sen}\,\varphi \cos\vartheta & \rho \cos\varphi \,\text{sen}\,\vartheta \\ \cos\varphi & 0 & -\rho \,\text{sen}\,\varphi \end{bmatrix}$$

$$\det J = \cos\varphi \left(-\rho^2 \text{sen}^2\vartheta \,\text{sen}\,\varphi \cos\varphi - \rho^2 \cos^2\vartheta \,\text{sen}\,\varphi \cos\varphi \right)$$

$$-\rho \,\text{sen}\,\varphi \left(\rho \cos^2\vartheta \,\text{sen}^2\varphi + \rho \,\text{sen}^2\vartheta \,\text{sen}^2\varphi \right) =$$

$$= -\rho^2 \cos^2\varphi \,\text{sen}\,\varphi - \rho^2 \,\text{sen}^3\varphi = -\rho^2 \,\text{sen}\,\varphi$$

$$|\det J| = \rho^2 \,\text{sen}\,\varphi$$

Volume sfera S_r di raggio r $\quad S_r = \{x^2+y^2+z^2 \leq 1\}$

$$V = \iiint_{S_r} 1 \, dx\,dy\,dz = \int_0^{2\pi} d\vartheta \int_0^{\pi} d\varphi \int_0^{r} \rho^2 \,\text{sen}\,\varphi \, d\rho =$$

$$= \int_0^{2\pi} d\vartheta \int_0^{\pi} \operatorname{sen}\varphi \, \frac{r^3}{3} \, d\varphi = \frac{r^3}{3} \int_0^{2\pi} d\vartheta \left[-\cos\varphi\right]_0^{\pi} =$$

$$= \frac{r^3}{3} \cdot 2 \cdot 2\pi = \frac{4}{3}\pi r^3$$