Lezione del 15/10/2013

Sottospazio vettoriale: è un sottogruppo rispetto alla somma, e inoltre se $\underline{v} \in W \Rightarrow \alpha \underline{v} \in W$, $\forall \alpha \in K$

Def. Un'espressione del tipo

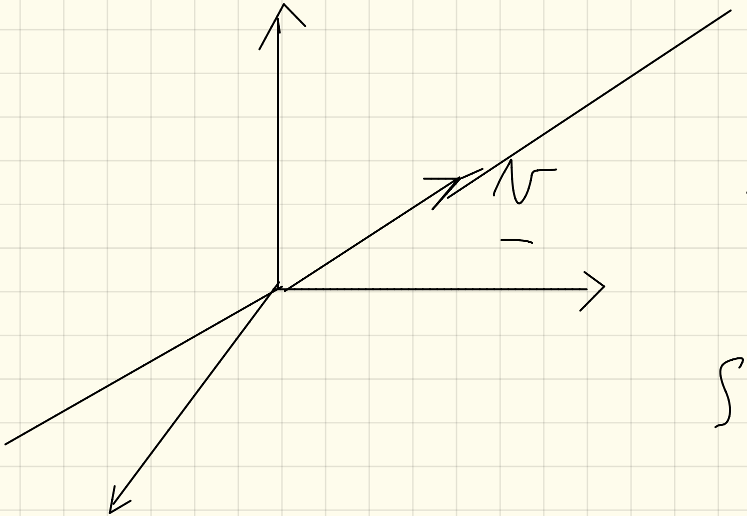
$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \dots + \alpha_n \underline{v}_n$$

dove i $\underline{v}_i \in V$ e gli $\alpha_i \in K$ si chiama una

combinazione lineare

$\alpha \underline{v}_1$ comb. lineare di 1 solo vett

$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2$ 2 vettori



$$A = \{\underline{v}\}$$

$$\mathrm{Span}(A) = \{\alpha \underline{v} \;\; \alpha \in \mathbb{K}\}$$

$$\mathrm{Span}(\{\underline{0}\}) = \{\underline{0}\}$$

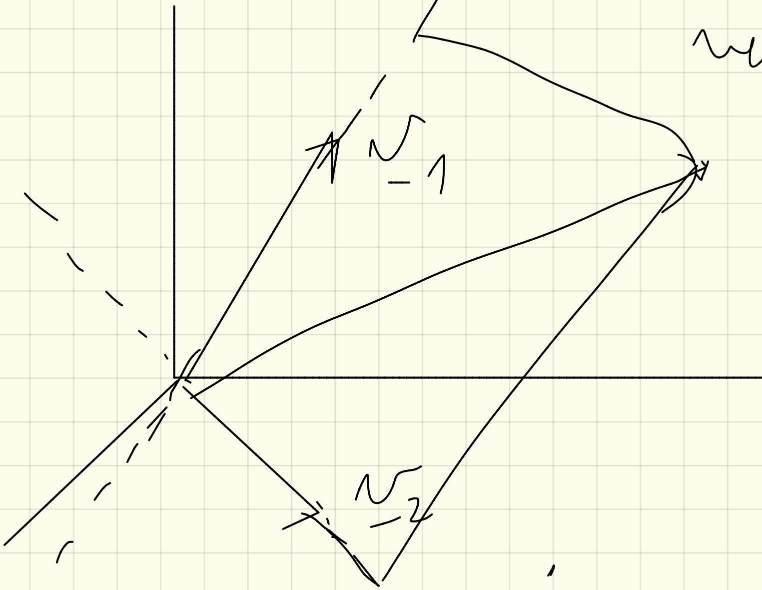
<u>Notazione</u> Dato $A \subset V$ definiamo

$$\boxed{\mathrm{Span}(A) = \{\underline{v} \in V \mid \exists\ \underline{v}_1, \dots, \underline{v}_n \in A \text{ e } \alpha_1, \dots, \alpha_n \in \mathbb{K} \text{ t.c. } \underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n\}}$$

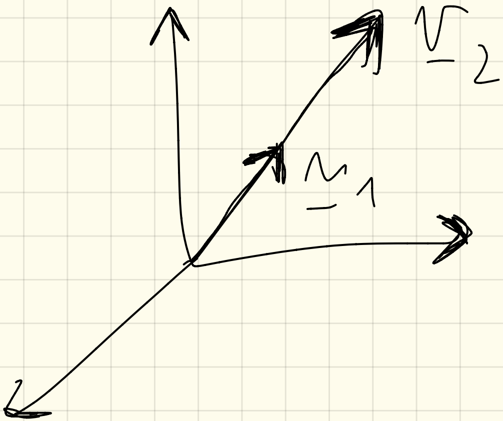
$\mathrm{Span}(\{\underline{v}_1, \underline{v}_2\})$

Se $\underline{v}_1$ e $\underline{v}_2$ non allineati

$\{\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2\}$ è il piano che contiene i due vettori



Se $\underline{v}_1$ e $\underline{v}_2$ sono allineati



$\{\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2\}$

retta che contiene $\underline{v}_1$ e $\underline{v}_2$

$A \subset V$, $\underline{0} \notin A$,

$$\mathrm{Span}(A \cup \{\underline{0}\}) = \mathrm{Span}(A)$$

$\subseteq$

$\supseteq$

Es 12 $A \subset B \subset V \Rightarrow$

$$\mathrm{Span}(A) \subset \mathrm{Span}(B)$$

Sia $\underline{v} \in \mathrm{Span}(A)$. Per def.

$\exists\ \underline{v}_1, \dots, \underline{v}_n \in A$, $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ t.c.

$$\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$$

Siccome $\underline{v}_1, \dots, \underline{v}_n \in B$ $\uparrow$ è comb. lin. di vettori di B e quindi $\underline{v} \in \mathrm{Span}(B)$

Dimostriamo che $\mathrm{Span}(A \cup \{\underline{0}\}) \subseteq \mathrm{Span}(A)$

Sia $\underline{v} \in \mathrm{Span}(A \cup \{\underline{0}\})$. Per def. $\exists$

$\underline{v}_1, \dots, \underline{v}_n \in A \cup \{\underline{0}\}$ e $\alpha_1, \dots, \alpha_n \in \mathbb{K}$

t.c. $\underline{v} = \underline{\alpha_1 \underline{v}_1 + \dots} + \alpha_n \underline{v}_n$

Se nessun $\underline{v}_i = \underline{0}$ ho finito

Supponiamo ad es. che $\underline{v}_n = \underline{0}$

$\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_{n-1} \underline{v}_{n-1} + \alpha_n \cdot \underline{0}$

$$\alpha\, \underline{0} = \underline{0} \qquad \forall \alpha \in \mathbb{K}.$$

$$\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_{n-1} \underline{v}_{n-1} + \underline{0} =$$

$$= \alpha_1 \underline{v}_1 + \dots + \alpha_{n-1} \underline{v}_{n-1}$$

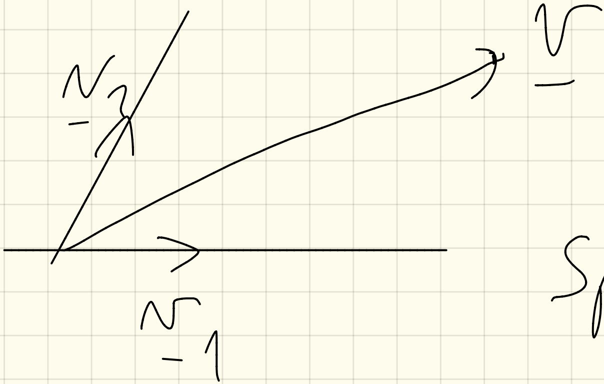
dove $\underline{v}_1, \dots, \underline{v}_{n-1} \in A$

<u>oss</u> Span(A) è un sottosp. vett. di V.

$\underline{v}_1, \underline{v}_2 \in \text{Span}(A) \Rightarrow \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 \in \text{Span}(A)$ $\forall \alpha_1, \alpha_2 \in \mathbb{K}$

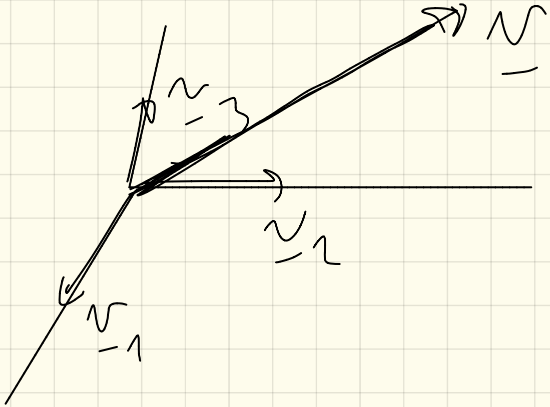
da dimostrare

tutte le possibili combinazioni lineari finite di vettori di A, in particolare quelle di 2 vettori



$$\underline{v} = x\,\underline{v}_1 + y\,\underline{v}_2$$

$$\mathrm{Span}(\{\underline{v}_1, \underline{v}_2\}) = V_2$$



$$\underline{v} = x\,\underline{v}_1 + y\,\underline{v}_2 + z\,\underline{v}_3$$

$$\mathrm{Span}(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}) = V_3$$

I idea $\{\underline{v}_1, \dots, \underline{v}_n\}$ che possa essere preso come riferimento per introdurre delle coordinate deve soddisfare

$$\mathrm{Span}(\{\underline{v}_1, \dots, \underline{v}_n\}) = V$$

cioè ogni $\underline{v} = x_1 \underline{v}_1 + \dots + x_n \underline{v}_n$

$$x_i \in \mathbb{K}$$

Se $A \subset V$ è t.c. $\mathrm{Span}\, A = V$

allora A si chiama insieme di generatori

Per l'es. visto prima:

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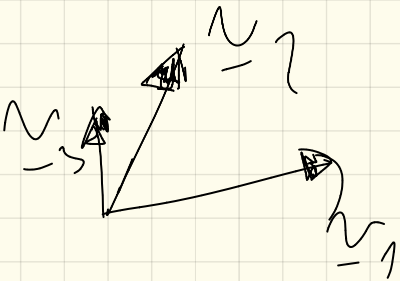
oss Se $\mathrm{Span}(A) = V \Rightarrow$
$\mathrm{Span}(B) = V$ se $A \subset B$



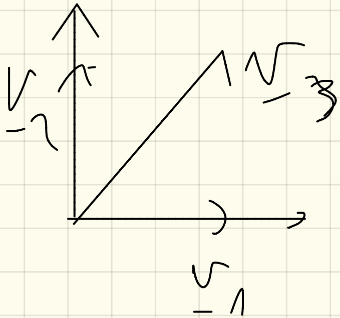
$\subset V_2$

Ogni $\underline{v}$ si scrive in infiniti modi come:

$$\underline{v} = x_1 \underline{v}_1 + x_2 \underline{v}_2 + x_3 \underline{v}_3$$

esemp. $\underline{v} = \underline{0}$

$$\underline{0} = 0\,\underline{v}_1 + 0 \cdot \underline{v}_2 + 0\,\underline{v}_3$$



$\underline{v}_1 = (1, 0)$

$\underline{v}_2 = (0, 1)$

$\underline{v}_3 = (1, 1)$

$$\underline{v}_3 = \underline{v}_1 + \underline{v}_2 \longrightarrow \underline{0} = \underline{v}_1 + \underline{v}_2 - \underline{v}_3$$

coeffic.: $1, 1, -1$

Def. I vettori $\underline{v}_1, \dots, \underline{v}_n \in V$ si dicono $\boxed{\text{LINEARMENTE DIPENDENTI}}$ se $\exists\ \alpha_1, \dots, \alpha_n \in \mathbb{K}$ non tutti nulli tali che $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$

Def. I vettori $\underline{v}_1, \dots, \underline{v}_n \in V$ si dicono $\boxed{\text{LINEARMENTE INDIPENDENTI}}$ se l'unica soluzione di $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$ è $\alpha_1 = \dots = \alpha_n = 0$

OSS. Siano $\underline{v}_1, \dots, \underline{v}_n$ linear. indipendenti.

Allora se

$$\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n$$

$$\Rightarrow \quad \alpha_1 = \beta_1, \dots, \alpha_n = \beta_n$$

$$(\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n) - (\beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n) = \underline{0}$$

$$(\alpha_1 - \beta_1) \underline{v}_1 + \dots + (\alpha_n - \beta_n) \underline{v}_n = \underline{0}$$

$$\Rightarrow \quad \alpha_1 = \beta_1, \dots, \alpha_n = \beta_n$$

DEF Una $\boxed{\underline{\text{BASE}}}$ dello spazio V

è un insieme <u>ordinato</u> di vettori $\{\underline{v}_1, \dots, \underline{v}_n\} \in V$ tale che

1) $\text{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = V$ ($\leftrightarrow$ generatori)

2) I $\underline{v}_1, \dots, \underline{v}_n$ sono linearmente indipendenti

Esempi. $V = \mathbb{R}$ sp. vett. su $\mathbb{R}$

es) $\{\underline{v}\}$ è linear. indip. $\Leftrightarrow$ $\underline{v} \neq \underline{0}$

$\Rightarrow$ se $\underline{v} = \underline{0}$ allora $1 \cdot \underline{v} = 1 \cdot \underline{0} = \underline{0}$

$\Leftarrow$ $\alpha \underline{v} \neq \underline{0}$ se $\alpha \neq 0$

$\alpha \underline{v} = \underline{0}$, $\underline{v} \neq \underline{0}$ $\Rightarrow$ $\alpha = 0$

se $\alpha \neq 0$ $\exists\, \alpha^{-1}$

$\alpha^{-1}(\alpha \underline{v}) = \alpha^{-1} \cdot \underline{0} \rightarrow (\alpha^{-1}\alpha)\underline{v} = \underline{0}$

$\boxed{\text{si ha contraddizione}} \rightarrow 1 \cdot \underline{v} = \underline{0}$

Se $\underline{0} \in A \Rightarrow A$ non è lin. indip.

CASO GENERALE $A \subset V$ (qui si ammette A infinito) si dice linearmente dipendente

se $\exists\, \underline{v}_1, \dots, \underline{v}_m \in A$ e $\alpha_1, \dots, \alpha_m \in \mathbb{K}$,

numeri non tutti nulli t.c.

$$\alpha_1 \underline{v}_1 + \dots + \alpha_m \underline{v}_m = \underline{0}$$

A indipendente se l'unica soluzione di

$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$ è la soluzione nulla $\boxed{\alpha_i = 0 \ \forall i}$

$\forall$ scelta di vettori $\underline{v}_1, \dots, \underline{v}_n \in V$

$1 \cdot \underline{0} = \underline{0}$ è combin. lineare non banale che dà $\underline{0}$

$V = \mathbb{R}$ <u>base</u> $\{\underline{v}_1\}$ $\underline{v} \neq 0$

<u>es</u> $\underline{v}_1 = 1$

$V = \mathbb{R}^2$ <u>base CANONICA</u> $\underline{e}_1 = (1, 0)$

$\underline{e}_2 = (0, 1)$

verifichiamo

1. <u>Sono generatori</u> . $\text{Span}\{\underline{e}_1, \underline{e}_2\} = \mathbb{R}^2$

$\forall \, \underline{v} = (a, b)$ devo poter risolvere :

$\exists \, x, y \in \mathbb{R}$ t.c. $\underline{v} = x \underline{e}_1 + y \underline{e}_2$

$$\boxed{(a, b) = a(1, 0) + b(0, 1)}$$

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2. <u>Sono linearmente indipendenti</u>

devo dimostrare che l'equazione nelle incognite α_1, α_2

$$\alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 = \underline{0}$$

ha solo la soluzione $\alpha_1 = \alpha_2 = 0$

$$\alpha_1 (1, 0) + \alpha_2 (0, 1) = (0, 0)$$

$$\begin{cases} \alpha_1 \cdot 1 + \alpha_2 \cdot 0 = 0 \\ \alpha_1 \cdot 0 + \alpha_2 \cdot 1 = 0 \end{cases} \rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

in $\mathbb{R}^3$: base canonica

$\underline{e}_1 = (1,0,0)$
$\underline{e}_2 = (0,1,0)$
$\underline{e}_3 = (0,0,1)$

$\underline{v} = (a, b, c)$ $\qquad \exists\, x, y, z \in \mathbb{R}$?

$\underline{v} = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3$

$(a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$x = a,\ y = b,\ z = c$ $\qquad \mathrm{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \mathbb{R}^3$

$\mathbb{K}^n = V$

<u>base canonica</u>

$$\underline{e}_1 = (1, 0, \dots, 0)$$
$$\underline{e}_2 = (0, 1, 0, \dots, 0)$$
$$\vdots$$
$$\underline{e}_i = (0, \dots, 1, \dots, 0) \Longleftarrow$$ 1 al posto i-simo
$$\vdots$$
$$\underline{e}_n = (0, \dots, 0, 1)$$

verificare che $\quad \mathrm{Span}(\{\underline{v}_1, \dots, \underline{v}_n\}) = \mathbb{K}^n$

e sono linearm. indipendenti

$$V = \mathbb{R}[X] = \{ a_0 + a_1 X + \cdots + a_n X^n , \; a_i \in \mathbb{R} \}$$

non $\exists$ base finita

se $\underline{v}_1, \dots, \underline{v}_n$ fossero generatori

$$\text{grado}(\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n) \leq \max_i \{\text{grado } \underline{v}_i\}$$

$$\underline{\text{Base}} = \{ \underset{\underset{\underline{v}_0}{\|}}{1}, \underset{\underset{\underline{v}_1}{\|}}{x}, \underset{\underset{\underline{v}_2}{\|}}{x^2}, \underset{\underset{\underline{v}_3}{\|}}{x^3}, x^4, x^5, \dots, \underset{\underset{\underline{v}_n}{\|}}{x^n}, \dots \}$$

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$

è comb. lineare

$$p(x) = a_0 \underline{v}_0 + a_1 \underline{v}_1 + \dots + a_n \underline{v}_n$$

$$a_0 x^{n_0} + a_1 x^{n_1} + \dots + a_k x^{n_k} = \underline{0}$$

$$\Leftarrow) \quad a_0 = \dots = a_k = 0$$

n_j : esponenti diversi tra loro

$$V = \{ f : \{1, 2, 3\} \longrightarrow \mathbb{R} \}$$

es. trovare una base

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es. $\{ f : \{1, \dots, n\} \longrightarrow \mathbb{R} \}$

trovare base

abbiamo visto: $\forall A, B \subset V$ $\quad Span(A) \subset Span\{A \cup B\}$

<u>es</u> se $Span(A) = Span(A \cup B)$

allora $\forall \underline{v} \in B \implies \underline{v} \in Span(A)$

[in altri termini: $B \subset Span(A)$]
