Local risk minimization and numéraire

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Abstract

The *"change of numéraire"* technique has been introduced by Geman, El Karoui and Rochet for pricing and hedging contingent claims in the case of complete markets. In this article we study the "c. of n.", according to the *"locally risk minimizing approach"*, when the market is not complete. We prove that, if the stochastic process which represents the prices is continuous, the l.r.m. strategy is invariant by a change of numéraire (this result is false in the right-continuous case, as it is shown by some counterexamples).
We also give an extension of Merton's formula to the case of stochastic volatility.

Key words: Stochastic integrals, models of financial markets, change of numéraire, locally risk-minimizing strategies, minimal martingale measure.

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1 Introduction

Hedging and pricing of contingent claims are two major issues in both theoretical and applied finance (see for instance [9] for general definitions): when the market is complete, any sufficiently integrable contingent claim H is the final value of a self-financing portfolio. More precisely, we have that

$H = V_0 + \int_0^T \xi_s \cdot dX_s$, where the multidimensional stochastic process X_t represents the random evolution of financial assets, the value V_0 is the *"arbitrage price"* of contingent claim H and the predictable process ξ_t represents the *"hedging strategy"*.

The *"change of numéraire"* technique, introduced by Geman, El Karoui and Rochet in [6] (see also [1] and [9]), turned out to be very powerful both for pricing and hedging contingent claims. In [6] they are mainly concerned with the case of complete markets; in [3], Delbaen and Schachermayer consider the connections between the existence of equivalent martingale measures and the change of numéraire, while in [7] Gouriéroux, Laurent and Pham investigate the case of incomplete markets according to the *"mean-variance hedging"* criterium.

In this paper we study the *"change of numéraire"* in the case of incomplet markets according to the *"locally risk minimizing"* (shortly l.r.m.) criterium: the l.r.m. strategies were introduced in [5] for the martingale case and extensively developped in the general case in [4] and in [11]. Differently from [6] (where *numéraire* is whatever strictly positive stochastic process), but according to the definition given in [7], a *numéraire* is for us the value of a strictly positive self-financing portfolio (usually a particulare asset, or a "index" or a combination of assets).

We remark that the definition of local risk minimizing strategy used in this paper is slightly different from the usual one: this is because , according to [5], the components of a l.r.m. strategy are predictable in the risky asset but only adapted in the riskless asset. This definition cannot evidently be invariant if one chooses another asset as a numéraire: we will give the link between our definition and the original one.

The paper is organized as follows: in section 2 we introduce the model and the definitions.

Section 3 contains the main result: if the stochastic process X_t, which models the asset prices, is a continuous multidimensional semimartingale, the l.r.m. strategy (if it exists) is invariant under a change of numéraire.

This result is false if X_t is only right-continuous: in section 4 we give two counterexamples. The second one shows also that even a good property of the filtration, such as *"quasi-left continuity"*, doesn't guarantee this invariance property.

Finally, section 5 contains an application of the previous results: we illustrate a generalization of the well-known *"Merton's formula"* to the case of stochastic volatility.

2 General definitions

We consider a financial market where the price fluctuation of assets is given by a d-dimensional stochastic process

$$X_t = (X_t^1, \dots, X_t^d), \quad 0 \le t \le T, \quad d \ge 2$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a right-continuous filtration $(\mathcal{F}_t)_{0 \le t \le T}$. We assume that every component X_t^i is a strictly positive and continuous semimartingale (for general definitions on stochastic integration, we refer to [2] or [10]).

Consider a d-dimensional predictable stochastic process H_t such that the vector stochastic integral $\int_0^T H_s \cdot dX_s$ is defined: in order to simplify the notations, we will substitute the expression $Y_t - Y_0 = \int_0^t H_s dX_s$ with the compact one $dY_t = H_t dX_t$. For instance, given $F \in \mathcal{C}^2(\mathbb{R})$, Ito's formula becomes:

$$dF(t, X_t) = \frac{\partial F}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial F}{x_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j}(t, X_t)d\langle X^i, X^j\rangle_t$$

Note that for two continuous semimartingales X_t^i, X_t^j the quadratic covariation $\langle X^i, X^j\rangle$ is always defined and it is invariant under a change of equivalent probability measure (see e.g.[10]): we have in fact that $\langle X^i, X^j\rangle_t = \lim_{sup_k |t_{k+1} - t_k| \to 0} \sum_k (X_{t_{k+1}}^i - X_{t_k}^i)(X_{t_{k+1}}^j - X_{t_k}^j)$, where the limit is in the sense of uniform convergence in probability.
We recall that two local martingales M, N are *orthogonal* if $M_t N_t$ is a local martingale: if they are continuous, this property is equivalent to $d\langle M, N\rangle_t = 0$. In particular, if X_t is a continuous semimartingale and M_t is a local martingale (not necessarily continuous), M_t is orthogonal to the martingale part of X_t if and only if $d\langle X, M\rangle_t = 0$.

Definition 2.1. *We call* strategy *a pair* (ξ_t, V_t) *where* ξ_t *is a d-dimensional predictable stochastic process integrable with respect to* X_t *and the portfolio value* V_t *is a càdlàg optional process such that* $V_{t-} = \xi_t \cdot X_t$. *The difference* $C_t = V_t - \int_o^t \xi_s dX_s$ *is called the* cost *accumulated up to time* t.

We point out that (according to the definition given in [5]) we consider strategies which are not in general self-financing: it is evident that the portfolio is self-financing if and only if $C_t = C_0 = V_0$. In that case, $V_t = V_0 + \int_o^t \xi_s dX_s = \xi_t \cdot X_t$.

We remark also that our definition differs from the one given in [4], where they consider a d-dimensional process S_t with a bond S_t^0 and work directly with the discounted process $X_t^i = \dfrac{S_t^i}{S_t^0}$. Following this approach, a strategy is a pair (ξ_t, η_t) where ξ_t is a d-dimensional predictable process, η_t is an optional process of dimension one and the value of the resulting portfolio V_t is given by $\dfrac{V_t}{S_t^0} = \xi_t \cdot X_t + \eta_t$. It follows that, with respect to our definition, $\eta_t = \xi_t^0 + \Delta V_t$: the components $(\xi_t^0, \ldots, \xi_t^d)$ represent a sort of "intrinsic" strategy (independent from the chosen "numéraire"). Even if X_t is a continuous semimartingale, the value of the portfolio V_t is only right-continuous (see example 5.3 in [4]): the istantaneous adjustment ΔV_t it is carried on by the bond S_t^0 according to [4], while it is carried on by anyone of the underlying assets according to our definition.

Definition 2.2. *A* numéraire *is a strictly positive stochastic process B_t, which is the value of a self-financing portfolio.*

More precisely, $B_t = \theta_t \cdot X_t = B_0 + \int_0^t \theta_s dX_s$, where θ_t is integrable with respect to X_t.

Remark 2.1. *It is usually assumed the existence of a numéraire B_t and of an equivalent probability $\mathbb{P}^*$ such that $\dfrac{X_t}{B_t}$ is a (local) martingale under $\mathbb{P}^*$: this is related to the so called* "no-arbitrage property" *under the numéraire B_t. For further informations, see, for instance, [3].*

Definition 2.3. *Given a numéraire B_t such that $\dfrac{X_t}{B_t}$ is a semimartingale of class $\mathcal{S}^2$, a strategy (ξ_t, V_t) is said to be* admissible *with respect to a numéraire B_t if:*

1. *the portfolio V_t is a square integrable stochastic process whose left limit is equal to $\dfrac{V_{t-}}{B_t} = \xi_t \cdot \dfrac{X_t}{B_t}$*
2. *the stochastic integral $\displaystyle\int_0^t \xi_s \, d\left(\frac{X_s}{B_s}\right)$ is a semimartingale belonging to the class $\mathcal{S}^2$.*

More precisely, for every component X_t^i, we have $\dfrac{X_t^i}{B_t} = \dfrac{X_0^i}{B_0} + M_t^i + A_t^i$, where M_t^i is a square integrable martingale and A_t^i is a predictable process with finite variation such that the total variation $|A_t|$ is square integrable. Moreover, the d-dimensional predictable process ξ_t is such that

$$E\left[\int_0^T \xi_s d\langle M\rangle_s \xi_s^{'} + \sum_{i=1}^{d}\left(\int_0^T \xi_s^i d|A_s^i|\right)^2\right] < +\infty$$

where $\xi_t^{'}$ is the transposed vector.

The cost process under a numéraire B_t is given by $C_t^B = \dfrac{V_t}{B_t} - \displaystyle\int_0^t \xi_s \, d\left(\frac{X_s}{B_s}\right)$.

Recall that an *option* H is a positive $\mathcal{F}_T$-measurable random variable. The *locally risk minimizing* strategies have been introduced in the general case by Schweizer: roughly speaking, the risk is minimal under all infinitesimal pertubations of the strategy. This definition is made precise in [11] and it is shown to be essentially equivalent to the following:

Definition 2.4. *Given a contingent claim* H *such that* $\dfrac{H}{B_t} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$*, an hedging strategy* (ξ_t, V_t) *is said to be* locally risk minimizing *(shortly, l.r.m.) with respect to the numéraire* B_t *if the following conditions hold:*

1. (ξ_t, V_t) *is an admissible strategy under* B_t
2. $V_T = H$
3. $\dfrac{V_t}{B_t} = \displaystyle\int_0^t \xi_s \, d\left(\frac{X_s}{B_s}\right) + C_t^B$*, where* C_t^B *is a square integrable martingale orthogonal to the martingale part of* $\dfrac{X_t}{B_t}$.

Note that if the optimal strategy exists, it is unique, as it is shown in [5].

Definition 2.5. *Let* B_t *be a numéraire such that* $\dfrac{X_t}{B_t}$ *is a semimartingale of the class* $\mathcal{S}^2$*: an equivalent measure* $\hat{\mathbb{P}}^B \sim \mathbb{P}$ *is called* minimal *(under a numéraire* B_t*) if:*

1. $\hat{\mathbb{P}}^B \equiv \mathbb{P}$ *on* $\mathcal{F}_0$
2. $\dfrac{X_t}{B_t}$ *is a square integrable martingale under* $\hat{\mathbb{P}}^B$
3. *Any square integrable martingale which is orthogonal to the martingale part of* $\dfrac{X_t}{B_t}$ *under* $\mathbb{P}$ *remains a martingale under* $\hat{\mathbb{P}}^B$.

If the minimal martingale measure exists, it is unique and the optimal strategy can be computed in terms of it ([4]). In fact, the value of the l.r.m. portfolio is given by $\frac{V_t}{B_t} = \hat{E}^B \left[\frac{H}{B_T} | \mathcal{F}_t \right]$. The l.r.m. components $(\xi_t^1, \dots, \xi_t^d)$ can be computed by choosing a numéraire B_t and applying the Kunita-Watanabe decomposition to $\frac{V_t}{B_t}$ with respect to the $\hat{\mathbb{P}}^B$-martingale $\frac{X_t}{B_t}$ (see [4] for further details); theorem 3.1 will ensure us that this procedure is well-defined because it is independent from the chosen numéraire. Finally, we recall that a self-financing portfolio remains self-financing after a change of numéraire ([6], pag.445): therefore if S_t is another numéraire, the process $\frac{S_t}{B_t}$ is a continuous local martingale under the minimal probability $\hat{\mathbb{P}}^B$.

3 Invariance under a change of numéraire

In this section, we consider two numéraires B_t and S_t : given a strategy (ξ_t, V_t), we implicitely assume that the two stochastic integrals $\int_0^t \xi_s d(\frac{X_s}{B_s})$ and $\int_0^t \xi_s d(\frac{X_s}{S_s})$ exist.

Lemma 3.1. *If C_t^B and C_t^S are the costs of the strategy (ξ_t, V_t), then*

$$dC_t^S = \frac{B_t}{S_t} dC_t^B + d\langle C_t^B, \frac{B_t}{S_t} \rangle$$

Proof. The process $\frac{B_t}{S_t}$ is a continuous semimartingale, so the “Itô’s multiplication rule” gives:

$$d(\frac{V_t}{S_t}) = d(\frac{V_t}{B_t} \cdot \frac{B_t}{S_t}) = \frac{V_{t-}}{B_t} \, d(\frac{B_t}{S_t}) \; + \; \frac{B_t}{S_t} \, d(\frac{V_t}{B_t}) \; + \; d\langle \frac{V_t}{B_t}, \frac{B_t}{S_t} \rangle =$$

$$= \xi_t \frac{X_t}{B_t} \, d(\frac{B_t}{S_t}) \; + \; \frac{B_t}{S_t} \, d(\frac{V_t}{B_t}) \; + \; d\langle \frac{V_t}{B_t}, \frac{B_t}{S_t} \rangle$$

Since $d(\frac{V_t}{B_t}) = \xi_t d(\frac{X_t}{B_t}) + dC_t^B$, $d\langle \frac{V_t}{B_t}, \frac{B_t}{S_t} \rangle = \xi_t d\langle \frac{X_t}{B_t}, \frac{B_t}{S_t} \rangle + d\langle C_t^B, \frac{B_t}{S_t} \rangle$ and we obtain:

$$d(\frac{V_t}{S_t}) = \xi_t \left(\frac{X_t}{B_t}\ d(\frac{B_t}{S_t})\ +\ \frac{B_t}{S_t}\ d(\frac{X_t}{B_t})\ +\ d\langle \frac{X_t}{B_t}, \frac{B_t}{S_t} \rangle \right)\ +\ \frac{B_t}{S_t}\ dC_t^B\ +\ d\langle C_t^B, \frac{B_t}{S_t} \rangle =$$

$$= \xi_t\ d(\frac{X_t}{S_t})\ +\ \frac{B_t}{S_t}\ dC_t^B\ +\ d\langle C_t^B, \frac{B_t}{S_t} \rangle$$

□

Proposition 3.1. *Under the same hypothesis of the previous lemma, if the process* C_t^B *is a local martingale such that* $d\langle C_t^B, \frac{X_t}{B_t} \rangle = 0$*, then the process* C_t^S *is a local martingale such that* $d\langle C_t^S, \frac{X_t}{S_t} \rangle = 0$*.*

Proof. Recall that S_t is a self-financing portfolio: therefore $d(\frac{S_t}{B_t}) = \eta_t d(\frac{X_t}{B_t})$ for a suitable predictable process η_t. ¿From Itô's formula, we have that

$$d\left(\frac{B_t}{S_t}\right) = d\left[\left(\frac{S_t}{B_t}\right)^{-1}\right] = -\frac{B_t^2}{S_t^2} d(\frac{S_t}{B_t}) + \frac{B_t^3}{S_t^3} d\langle \frac{S_t}{B_t}, \frac{S_t}{B_t} \rangle$$

Since $d\langle C_t^B, \frac{X_t}{S_t} \rangle = 0$, $d\langle C_t^B, \frac{B_t}{S_t} \rangle = 0$ and from lemma 3.1, $dC_t^S = \frac{B_t}{S_t} dC_t^B$: consequently C_t^S is a local martingale. Again from lemma 3.1

$$d\langle C_t^S, \frac{X_t}{S_t} \rangle = \frac{B_t}{S_t}\ d\langle C_t^B, \frac{X_t}{B_t} \cdot \frac{B_t}{S_t} \rangle =$$

$$= \frac{X_t}{S_t}\ d\langle C_t^B, \frac{B_t}{S_t} \rangle\ +\ \frac{B_t^2}{S_t^2}\ d\langle C_t^B, \frac{X_t}{B_t} \rangle = \frac{X_t}{S_t}\ d\langle C_t^B, \frac{B_t}{S_t} \rangle$$

□

Theorem 3.1. *Let* (ξ_t, V_t) *be an admissible strategy with respect to numéraires* B_t *and* S_t*. If* (ξ_t, V_t) *is locally risk minimizing under the numéraire* B_t*, then* (ξ_t, V_t) *is l.r.m. also with respect to the numéraire* S_t*.*

Proof. The proof is an immediate consequence of proposition 3.1: the cost C_t^S is a local martingale orthogonal to the martingale part of $\frac{X_t}{S_t}$. But since the strategy is admissible with respect to S_t, C_t^S is actually a square integrable martingale.

□

Before showing how the minimal probability varies under a change of numéraire, we prove the following characterization of minimal probabilities. In the following lemma, we consider a numéraire S_t such that $\frac{X_t}{S_t}$ is a $\mathcal{S}^2$-semimartingale, and an equivalent probability $\mathbb{Q} \sim \mathbb{P}$ such that $\frac{X_t}{S_t}$ is a square integrable martingale under $\mathbb{Q}$.

Lemma 3.2. *Suppose that the density process $L_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}$ is a continuous martingale and that $\mathbb{Q}$ has the following property: every contingent claim H such that $\frac{H}{S_T} \in L^2(\mathbb{Q})$ has a l.r.m. strategy and the value of the l.r.m. portfolio V_t is given by*

$$E^{\mathbb{Q}}\left[\frac{H}{S_T} \;\Big|\; \mathcal{F}_t\right] = \frac{V_t}{S_t}$$

Then the minimal martingale measure $\hat{\mathbb{P}}^S$ exists and coincides with $\mathbb{Q}$.

Proof. ¿From definition 2.5, one obtains that the equivalent martingale measure $\mathbb{Q}$ is the minimal measure if every $\mathbb{P}$-square integrable martingale M_t orthogonal to the martingale part of $\frac{X_t}{S_t}$ is orthogonal to $L_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}$.
Consider the decomposition $M_t = M_t^d + M_t^c$, where M_t^d is the purely discontinuous part of M_t and M_t^c the continuous one. It is clear that M_t^d is orthogonal to L_t, so it is sufficient to prove the assertion only for continuous martingales.
Besides, if M_t is a continuous martingale, we can suppose it bounded unless of using stopping times. Therefore there exists $a \in \mathbb{R}$ such that $M_T + a \geq 0$, $\mathbb{P}$-a.e. Consider the option H such that $\frac{H}{S_T} = M_T + a$. The portfolio $\frac{V_t}{S_t} = a + M_t = a + \int_0^t 0 \; d\left(\frac{X_s}{S_s}\right) + M_t$ gives the optimal strategy under S_t. Moreover

$$E^{\mathbb{Q}}\left[a + M_T | \mathcal{F}_t\right] = E^{\mathbb{Q}}\left[\frac{H}{S_T} \;\Big|\; \mathcal{F}_t\right] = \frac{V_t}{S_t} = a + M_t$$

so M_t is a martingale under the probability $\mathbb{Q}$.

□

Theorem 3.2. *If there exists the minimal martingale measure $\hat{\mathbb{P}}^B$ and the process $\frac{S_t}{B_t}$ is a uniformly integrable martingale under $\hat{\mathbb{P}}^B$, then there exists the minimal martingale measure $\hat{\mathbb{P}}^S$ and the equation*

$$\frac{d\hat{\mathbb{P}}^S}{d\hat{\mathbb{P}}^B} = \frac{S_T}{B_T} \cdot \frac{B_0}{S_0}$$

is satisfied.

Proof. Consider the equivalent probability $\mathbb{Q} \sim \hat{\mathbb{P}}^B$ such that $\dfrac{d\mathbb{Q}}{d\hat{\mathbb{P}}^B} = \dfrac{S_T}{B_T} \cdot \dfrac{B_0}{S_0}$: it is easy to verify that $\dfrac{X_t}{S_t}$ is a $\mathbb{Q}$-martingale and from Bayes formula one obtains that

$$E^{\mathbb{Q}}\left[\frac{H}{S_T} \;\Big|\; \mathcal{F}_t\right] = \frac{\hat{E}^B\left[\dfrac{H}{B_T} \;\Big|\; \mathcal{F}_t\right]}{\dfrac{S_t}{B_t}} = \frac{V_t}{S_t}$$

where V_t is the value of the l.r.m. portfolio both under B_t and S_t. Lemma 3.2 ensures that the probability $\mathbb{Q}$ is actually the minimal probability under S_t. □

4 Some counterexamples

If the stochastic process X_t is a right-continuous semimartingale, definition 2.1 has to be slightly modified: a *strategy* is a pair (ξ_t, V_t) where ξ_t is a d-dimensional predictable process integrable with respect to X_t and $V_{t-} = \xi_t \cdot X_{t-}$. It is known that the minimal martingale measure (if it exists) is not necessarily a true probability, but only a signed probability.
We exhibit here two counterexamples in which both probabilities $\hat{\mathbb{P}}^B$ and $\hat{\mathbb{P}}^S$ exist, but the equality $\dfrac{d\hat{\mathbb{P}}^S}{d\hat{\mathbb{P}}^B} = \dfrac{S_T}{B_T} \cdot \dfrac{B_0}{S_0}$ is false, so theorem 3.2 and and "a fortiori" theorem 3.1 don't hold for right-continuous processes.
Note that in example 4.2 the filtration is quasi-left continuous (i.e. for every predictable stopping time τ one has $\mathcal{F}_\tau = \mathcal{F}_{\tau-}$, see [2] or [8]): therefore this good property of the filtration doesn't guarantee the validity of theorem 3.1.

Example 4.1. Consider a discrete time model $(\Omega, \mathcal{F}, \mathbb{P}), (t = 0, t = 1)$, with two assets S_t and B_t: we assume $S_0 \equiv 1$, $B_0 \equiv B_1 \equiv 1$ and $\mathcal{F}_0 = (\emptyset, \Omega)$.
It is easy to calculate the densities of $\hat{\mathbb{P}}^B$, $\hat{\mathbb{P}}^S$ with respect to the given probability $\mathbb{P}$:

1. $\dfrac{d\hat{\mathbb{P}}^B}{d\mathbb{P}} = 1 - \dfrac{E\left[\Delta S_1 | \mathcal{F}_0\right]}{Var\left[\Delta S_1 | \mathcal{F}_0\right]} \left(\Delta S_1 - E\left[\Delta S_1 | \mathcal{F}_0\right]\right)$

2. $\dfrac{d\hat{\mathbb{P}}^S}{d\mathbb{P}} = 1 - \dfrac{E\left[\Delta\left(\frac{1}{S_1}\right)|\mathcal{F}_0\right]}{Var\left[\Delta\left(\frac{1}{S_1}\right)|\mathcal{F}_0\right]}\left(\Delta\left(\dfrac{1}{S_1}\right) - E\left[\Delta\left(\dfrac{1}{S_1}\right)\Big|\mathcal{F}_0\right]\right)$

Note that $Z^B = \dfrac{d\hat{\mathbb{P}}^B}{d\mathbb{P}}$ is a random variable such that $E[Z^B] = 1$ and $E[Z^B \Delta S_1] = 0$; moreover, if $E[\Delta N_1] = 0$ and $E[\Delta N_1 \Delta S_1] = 0$, then $E[Z^B \Delta N_1] = 0$. Similar computations can be made for $\dfrac{d\hat{\mathbb{P}}^S}{d\mathbb{P}}$.

If the equality $\dfrac{d\hat{\mathbb{P}}^S}{d\mathbb{P}} = S_1 \dfrac{d\hat{\mathbb{P}}^B}{d\mathbb{P}}$ were true, the following equation should hold:

$$\alpha S_1^3 - (\alpha\beta + 1)S_1^2 + (1 + \gamma\delta)S_1 - \gamma \;=\; 0 \tag{1}$$

where $\alpha = \dfrac{E\left[\Delta S_1\right]}{Var(\Delta S_1)}$, $\beta = E\left[S_1\right]$, $\gamma = \dfrac{E\left[\Delta\left(\frac{1}{S_1}\right)\right]}{Var[\Delta\left(\frac{1}{S_1}\right)]}$, $\delta = E\left[\left(\frac{1}{S_1}\right)\right]$. It is immediate to find a discrete random variable which assumes at time $t = 1$ at least four distinct values and doesn't satisfy equation (1).

Example 4.2. Consider a foreign exchange market: let D_t be the dollar cash bond with $D_t = e^{\rho t}$, $\rho \in \mathbb{R}^+$, S_t its sterling counterpart such that $S_t = e^{rt}$, $r \in \mathbb{R}^+$, and C_t the exchange rate. We suppose that C_t follows Merton's model:

$$dC_t = C_{t-}\left(\mu dt + \sigma dW_t + \beta dN_t\right)$$

for some brownian motion W_t, Poisson process N_t with intensity λ and constants $\beta \geq -1, \mu, \sigma \in \mathbb{R}^+$. We consider as assets D_t and $X_t = S_t C_t$, which is a dollar tradable. Assuming D_t as numéraire, the minimal measure $\hat{\mathbb{P}}^D$ has been calculated in [13], where it is shown that, if $\eta = \dfrac{\mu - \lambda\beta + r - \rho}{\sigma^2 + \beta^2\lambda}$ belongs to the interval $[-1, 0]$, then $\hat{\mathbb{P}}^D$ exists and

$$\frac{d\hat{\mathbb{P}}^D}{d\mathbb{P}} = (1 - \eta\beta)^{N_T}\, exp\left[-\sigma\eta W_T + \left(\lambda\beta\eta - \frac{1}{2}\sigma^2\eta^2\right)T\right]$$

Viceversa, the sterling investor is concerned about the sterling worth $\dfrac{D_t}{C_t}$ of 1 dollar and S_t, so he uses S_t as basic unit of account. This corresponds to assume $X_t = C_t S_t$ as numéraire in the dollar market. The minimal martingale measure $\hat{\mathbb{P}}^X$ with respect to X_t is given by the following expression

$$\frac{d\hat{\mathbb{P}}^X}{d\mathbb{P}} = exp\left[\sigma\gamma W_T + \left(\alpha\lambda\gamma - \frac{1}{2}\sigma^2\gamma^2\right)T\right] \cdot (1-\alpha\gamma)^{N_T}$$

where $\alpha = -\dfrac{\beta}{1+\beta}$ and $\gamma = \dfrac{\rho - r - \mu + \sigma^2 + \alpha^2\lambda - \alpha\lambda}{\sigma^2 + \alpha^2\lambda}$.

If the equality $\dfrac{d\hat{\mathbb{P}}^X}{d\hat{\mathbb{P}}^D} = \dfrac{X_T}{D_T} \cdot \dfrac{D_0}{X_0}$ were true, we would have

$$\frac{d\hat{\mathbb{P}}^X}{d\mathbb{P}} = \frac{X_T}{D_T}\frac{D_0}{X_0}\frac{d\hat{\mathbb{P}}^D}{d\mathbb{P}} =$$

$$= \left[(1-\eta\beta)(1+\beta)\right]^{N_T} exp\left[\sigma(1-\eta)W_T + \left(\lambda\beta\eta - \tfrac{1}{2}\sigma^2\eta^2 + r - \rho + \mu - \tfrac{1}{2}\sigma^2\right)T\right]$$

Substituting numerical values to the parameters, it can easily be seen with long but not complicated calculus that the desired equality is false.

5 A generalization of Merton's formula

We recall Merton's formula following closely the approach of [6] (see also [1] and [9]). Consider a "call" option $H = (X_T - K)^+$ on a risky asset X_t under the presence of a stochastic interest rate (we suppose H square-integrable). Besides the risky asset X_t, we consider a zero-coupon bond $B(t,T)$ of maturity T as tradable asset. If the process $Z_t = \dfrac{X_t}{B(t,T)}$ satisfies the equation

$$\frac{dZ_t}{Z_t} = \mu_t dt + \sigma_t \cdot dW_t$$

where $W_t = (W_t^1, \ldots, W_t^d)$ is a d-dimensional brownian motion and the volatility σ_t is a deterministic function, then option H is attainable even if the market is not necessarily complete. The value V_t of the replicating portfolio at time t is given by Merton's Formula:

$$V_t = X_t N(d_1(X_t, B(t,T), t)) - KB(t,T)N(d_2(X_t, B(t,T), t))$$

where $N(x)$ is the distribution function of a standard gaussian random variable and

$$d_{1,2} = \frac{\ln X_t - \ln KB(t,T) \pm \frac{1}{2}\int_t^T |\sigma(s)|^2 ds}{\left(\int_t^T |\sigma(s)|^2 ds\right)^{\frac{1}{2}}}$$

We remark that, whatever is the equivalent probability measure $\mathbb{Q}$ under the numéraire $B(t,T)$, one has that $\dfrac{V_t}{B(t,T)} = E^{\mathbb{Q}}[H|\mathcal{F}_t]$. Besides, it is shown in [6] that:

$$V_t = X_t E^X\left[I_A|\mathcal{F}_t\right] - KB(t,T)E^T\left[I_A|\mathcal{F}_t\right]$$

where $A = \{X_T \geq K\}$, $E^X\left[I_A|\mathcal{F}_t\right]$ is the conditional expectation of I_A under an equivalent martingale measure $\mathbb{P}^X$ with respect to the numéraire X_t and $E^T\left[I_A|\mathcal{F}_t\right]$ is the conditional expectation under an equivalent martingale measure $\mathbb{P}^T$ for the numéraire $B(t,T)$. In particular, the two predictable stochastic processes $\xi_t^1 = E^X\left[I_A|\mathcal{F}_t\right]$ and $\xi_t^2 = -KE^T\left[I_A|\mathcal{F}_t\right]$ (or better the the two continuous versions of the martingales $E^X\left[I_A|\mathcal{F}_t\right]$ and $E^T\left[I_A|\mathcal{F}_t\right]$) are the components of the unique replicating strategy.

We suppose now that the volatility σ_t is “stochastic” (more precisely, affected by an exterior source of randomness), closely following the approach given by Föllmer and Schweizer in [4], where the randomness of the volatility is seen as a problem of *“incomplete information”*. The additional source of randomness is given by a probability space $(S,\mathcal{S},\nu)$: more precisely, we work on a product space $\bar{\Omega} = \Omega \times S$ and suppose that, letting $Z_t = \dfrac{X_t}{B(t,T)}$, the conditional law of Z_t given $\eta \in S$ is the law of the solution of equation

$$\frac{dZ_t}{Z_t} = \mu(t,\eta)dt + \sigma(t,\eta)\cdot dW_t$$

where $W_t = (W_t^1,\ldots,W_t^n)$ is a d-dimensional Wiener process.

The probability on $\bar{\Omega} = \Omega \times S$ is given by $\int_S \nu(d\eta)\ d\mathbb{P}^\eta$ (see [4] for further details). We remark that the law of Z_t under $\mathbb{P}^\eta$ is the law of

$$\frac{dZ_t}{Z_t} = \mu(t,\eta)dt + |\sigma(t,\eta)|dW_t^\sigma$$

where $|\sigma(t,\eta)| = (\sigma_1(t,\eta)^2 + \cdots + \sigma_n(t,\eta)^2)^{\frac{1}{2}}$ and W_t^σ is a one-dimensional Wiener process.

The natural filtration for option H is actually the right-continuous filtration $\mathcal{F}_t$ generated by $(X_s, B(s,T),\ 0 \leq s \leq t)$. Note that $|\sigma(t,\eta)|$ is $\mathcal{F}_t$-adapted since

$$\int_0^t |\sigma(s,\eta)|^2 Z_s^2 ds := \lim_{\sup_i |t_{i+1}-t_i| \to 0} \sum_{i=1}^n |Z_{t_{i+1}} - Z_{t_i}|^2$$

$\mathbb{P}$- a.e. We suppose that $\mu(t,\eta)$ is $\mathcal{F}_t$-adapted, so W_t^σ results to be a $\mathcal{F}_t$-Wiener process.

Consider now the larger filtration $\tilde{\mathcal{F}}$ obtained by adding to $\mathcal{F}$ the full information about η since the initial instant $t = 0$: it follows that $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$, $0 \leq t < T$. We suppose that $\mathcal{F}_T = \tilde{\mathcal{F}}_T$ and that W_t^σ is a $\tilde{\mathcal{F}}$-Wiener process. Assuming $B(t, T)$ as numéraire, the minimal probability $\hat{\mathbb{P}}^T$ exists if and only if

$$L_t = exp\left[-\int_0^t \frac{\mu(s,\eta)}{|\sigma(t,\eta)|} dW_t^\sigma - \frac{1}{2}\int_0^t \left(\frac{\mu(s,\eta)}{|\sigma(t,\eta)|}\right)^2 ds\right]$$

is a uniformly integrable martingale (see [4] for details) and under $\hat{\mathbb{P}}^T$ the process Z_t satisfies the following stochastic equation:

$$\frac{dZ_t}{Z_t} = |\sigma(t,\eta)| d\hat{W}_t^\sigma$$

where $\hat{W}_t^\sigma$ is a $\hat{\mathbb{P}}^T$-brownian motion both for $\tilde{\mathcal{F}}_t$ and $\mathcal{F}_t$. Note that the density process $L_t = \left.\frac{d\hat{\mathbb{P}}^T}{d\mathbb{P}}\right|_{\tilde{\mathcal{F}}_t}$ is $\mathcal{F}_t$-adapted and continuous; besides, the minimal probability under the numéraire X_t satisfies $\left.\frac{d\hat{\mathbb{P}}^X}{d\hat{\mathbb{P}}^T}\right|_{\tilde{\mathcal{F}}_t} = \frac{X_t}{B(t,T)} \cdot \frac{B(0,T)}{X_0}$ since $\frac{X_t}{B(t,T)}$ is a $\tilde{\mathcal{F}}_t$-martingale under $\hat{\mathbb{P}}^T$. With respect to the larger filtration $\tilde{\mathcal{F}}_t$, option H is attainable because the volatility $\sigma(t, \eta)$ results to be deterministic (see also [4]) and the replicating portfolio $\tilde{V}_t$ is given by

$$\tilde{V}_t = X_t N(d_1(X_t, B(t,T), t, \eta)) - KB(t,T)N(d_2(X_t, B(t,T), t, \eta))$$

where

$$d_{1,2}(X_t, B(t,T), t, \eta) = \frac{\ln X_t - \ln KB(t,T) \pm \frac{1}{2}\int_t^T |\sigma(s,\eta)|^2 ds}{\left(\int_t^T |\sigma(s,\eta)|^2 ds\right)^{\frac{1}{2}}}$$

It is easy to adapt the argument of [6] pag.451 and find that

$$\tilde{V}_t = X_t \hat{E}^X \left[I_A | \tilde{\mathcal{F}}_t\right] - KB(t,T)\hat{E}^T \left[I_A | \tilde{\mathcal{F}}_t\right]$$

with $A = \{X_T \geq K\}$. The two processes $\tilde{\xi}_t^1 = \hat{E}^X \left[I_A | \tilde{\mathcal{F}}_t\right]$ and $\tilde{\xi}_t^2 = -K\hat{E}^T \left[I_A | \tilde{\mathcal{F}}_t\right]$ represent the components of the replicating portfolio with respect to the filtration $\tilde{\mathcal{F}}_t$.

Let us calculate now the portfolio V_t and the components ξ_t^1 and ξ_t^2 of the l.r.m. strategy with respect to the natural filtration $\mathcal{F}_t$. The value V_t is given by $V_t = B(t,T)\hat{E}^T\left[(X_T - K)^+|\mathcal{F}_t\right]$ and applying theorem 3.2, it follows that

$$V_t = B(t,T)\hat{E}^T\left[(X_T - K)^+|\mathcal{F}_t\right] = X_t\hat{E}^X\left[I_A|\mathcal{F}_t\right] - KB(t,T)\hat{E}^T\left[I_A|\mathcal{F}_t\right]$$

In order to obtain the components ξ_t^1 and ξ_t^2 of the l.r.m strategy, we follow closely the approach of [4](theorem 4.6). Chosen $B(t,T)$ as numéraire, we have that

$$\frac{\tilde{V}_t}{B(t,T)} = \int_0^t \tilde{\xi}_s^1 \, d\left(\frac{X_s}{B(s,T)}\right) + \tilde{V}_0$$

Let η_t be the predictable projection of $\tilde{\xi}_t^1$ with respect to the filtration $\mathcal{F}_t$ and the minimal probability $\hat{\mathbb{P}}^T$: one verifies that

$$\frac{V_t}{B(t,T)} = \int_0^t \eta_s d\left(\frac{X_s}{B(s,T)}\right) + C_t^B$$

where C_t^B is a martingale orthogonal to the martingale part of $\dfrac{X_t}{B(t,T)}$ under $\hat{\mathbb{P}}^T$ (and therefore under $\mathbb{P}$). It follows that η_t coincides with the first component ξ_t^1 of the l.r.m. strategy. Simmetrically, one gets that ξ_t^2 is the $\mathcal{F}_t$-predictable projection of $\tilde{\xi}_t^2$ under $\hat{\mathbb{P}}^X$.

Note that $\tilde{\xi}_t^1$ is the optional projection (with respect to the filtration $\tilde{\mathcal{F}}_t$ and the probability $\hat{\mathbb{P}}^X$) of the measurable process $Y(t,\omega) = I_A(\omega)$.

Proposition 5.1. *The process ξ_t^1 coincides with the $\mathcal{F}_t$-predictable projection of $Y(t,\omega) = I_A(\omega)$ with respect to the probability $\hat{\mathbb{P}}^X$.*

Proof. We remark that $R_t = \left.\dfrac{d\hat{\mathbb{P}}^X}{d\hat{\mathbb{P}}^T}\right|_{\tilde{\mathcal{F}}_t} = \dfrac{X_t}{B(t,T)} \cdot \dfrac{B(0,T)}{X_0}$ is $\mathcal{F}_t$-adapted and continuous; therefore if τ is a $\mathcal{F}_t$-predictable stopping time, it is easy to verify that $\left.\dfrac{d\hat{\mathbb{P}}^X}{d\hat{\mathbb{P}}^T}\right|_{\tilde{\mathcal{F}}_{\tau-}} = R_\tau$. Consequently, one obtains that

$$\xi_\tau^1 = \hat{E}^T\left[\tilde{\xi}_\tau^1|\mathcal{F}_{\tau-}\right] = \hat{E}^T\left[\hat{E}^X\left[I_A|\tilde{\mathcal{F}}_\tau\right]|\mathcal{F}_{\tau-}\right] =$$

$$= \hat{E}^T\left[\frac{1}{R_\tau}\hat{E}^T\left[I_A\ R_T|\tilde{\mathcal{F}}_\tau\right]|\mathcal{F}_{\tau-}\right] = \frac{1}{R_\tau}\hat{E}^T\left[I_A\ R_T|\mathcal{F}_{\tau-}\right] = \hat{E}^X\left[I_A|\mathcal{F}_{\tau-}\right]$$

□

Similarly, ξ_t^2 turns out to be the predictable projection of I_A with respect to the probability $\hat{\mathbb{P}}^T$.
Finally, we remark that if there exists a left-continuous version of the stochastic process $\hat{E}^X\left[I_A|\mathcal{F}_{t-}\right]$, then it coincides with the $\mathcal{F}_t$-predictable projection under the probability $\hat{\mathbb{P}}^X$: from now on, we suppose that the processes $\hat{E}^X\left[I_A|\mathcal{F}_{t-}\right]$ and $\hat{E}^X\left[I_A|\mathcal{F}_t\right]$ have respectively a left-continuous version and a a right-continuous one (and simmetrically for probability $\hat{\mathbb{P}}^T$).
Under these hypotheses, proposition 5.1 allows us to obtain the following results.

Theorem 5.1. *The value of the l.r.m. portfolio V_t is given by*

$$V_t = X_t\hat{E}^X\left[I_A|\tilde{\mathcal{F}}_t\right] - KB(t,T)\hat{E}^T\left[I_A|\tilde{\mathcal{F}}_t\right]$$

The components ξ_t^1 and ξ_t^2 of the l.r.m.strategy are respectively $\xi_t^1 = \hat{E}^X\left[I_A|\mathcal{F}_{t-}\right]$ and $\xi_t^2 = -K\hat{E}^T\left[I_A|\mathcal{F}_{t-}\right]$.

Remark 5.1. *If we add as a tradable asset the money market account $D_t = exp(\int_0^t r(s)ds)$ (or a zero-coupon bond with a different maturity S), the l.r.m. strategy doesn't change and it is based on the two assets X_t and $B(t,T)$. In fact the component $\tilde{\xi}_t^3$ relative to D_t in the $\tilde{\mathcal{F}}_t$-portfolio is zero, so it will be zero its $\mathcal{F}_t$-predictable projection under the minimal probability with D_t as numéraire.*

Example 5.1. We consider a market where the stock X_t follows the equation

$$\frac{dX_t}{X_t} = \mu_1 dt + \sigma_1 dW_t^1$$

and the zero-coupon bond

$$\frac{dB(t,T)}{B(t,T)} = \mu_2 dt + \sigma_2 dW_t^2$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \mathbb{R}^+$, W_t^1 and W_t^2 are brownian motions such that

$$d\langle W_t^1, W_t^2\rangle = \left(\rho_1 I_{[0,\eta[}(t) + \rho_2 I_{[\eta,T]}(t)\right), \quad \rho_i \in [0,1], i = 1,2$$

where η is an independent stopping time on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(\eta = t) = 0$, $\forall t < T$. In this example $S = [0,1]$ and ν is the law of the stopping time η. The volatility of $Z_t = \dfrac{X_t}{B(t,T)}$ is $\sigma(t,\eta) = (\sigma_1^2 + \sigma_2^2 + 2\rho_1\sigma_1\sigma_2)^{\frac{1}{2}} I_{[0,\eta[} + (\sigma_1^2 + \sigma_2^2 + 2\rho_2\sigma_1\sigma_2)^{\frac{1}{2}} I_{[\eta,T]}$. Note that in this particular case the filtration $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$ are given by

1. $\tilde{\mathcal{F}}_t = \sigma(X_s, B(s,T), s \leq t; \{\eta \leq s\}, s \leq T)$
2. $\mathcal{F}_t = \sigma(X_s, B(s,T), s \leq t; \{\eta \leq s\}, s \leq t)$
3. $\mathcal{F}_{t-} = \sigma(X_s, B(s,T), s \leq t; \{\eta \leq s\}, s < t)$

The replicating portfolio with respect to $\tilde{\mathcal{F}}$ is

$$\tilde{V}_t = X_t N(d_1(X_t, B(t,T), t, \eta)) - KB(t,T)N(d_2(X_t, B(t,T), t, \eta))$$

where

$$d_{1,2}(X_t, B(t,T), t, \eta) = \frac{\ln X_t - \ln KB(t,T) \pm \frac{1}{2}\int_t^T \sigma^2(s,\eta)ds}{\left(\int_t^T \sigma^2(s,\eta)ds\right)^{\frac{1}{2}}}$$

¿From theorem 3.2 it follows that the local risk minimizing portfolio is $V_t = X_t \hat{E}^X[I_A|\mathcal{F}_t] - KB(t,T)\hat{E}^T[I_A|\mathcal{F}_t]$, while the components of the optimal strategy are $\xi_t^1 = \hat{E}^X[I_A|\mathcal{F}_{t-}]$ and $\xi_t^2 = -KB(t,T)\hat{E}^T[I_A|\mathcal{F}_{t-}]$. In order to compute ξ_t^1 and ξ_t^2, we introduce the following lemma.

Lemma 5.1. *Consider a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*: let* $\mathcal{B}$ *be a sub* σ*-algebra of* $\mathcal{F}$ *and* $B \in \mathcal{B}$ *, let* $\mathbb{P}^B = \mathbb{P}(\cdot|B)$ *be the conditional probability with respect to* B *and* $\mathcal{B}_B = \sigma(C \cap B, C \in \mathcal{B})$ *the trace* σ*-algebra. If* X *is a random variable in* $L^1(\Omega, \mathcal{F}, \mathbb{P})$*, then*

$$E[X|\mathcal{B}] = E^B[X|\mathcal{B}_B]I_B + E^{B^c}[X|\mathcal{B}_{B^c}]I_{B^c}$$

where B^c *is the complementary set of* B*.*

Proof. For each $C \in \mathcal{B}$ one has that

$$\int_C X\mathbb{P}(d\omega) = \mathbb{P}(B)\int_{C\cap B} X\frac{I_B}{\mathbb{P}(B)}\mathbb{P}(d\omega) + \mathbb{P}(B^c)\int_{C\cap B^c} X\frac{I_{B^c}}{\mathbb{P}(B^c)}\mathbb{P}(d\omega) =$$

$$= \mathbb{P}(B)\int_{C\cap B} X\mathbb{P}^{\mathbb{B}}(d\omega) + \mathbb{P}(B^c)\int_{C\cap B^c} X\mathbb{P}^{B^c}(d\omega) =$$

$$= \mathbb{P}(B)\int_{C\cap B} E^B[X|\mathcal{B}_B]\mathbb{P}^B(d\omega) + \mathbb{P}(B^c)\int_{C\cap B^c} E^{B^c}[X|\mathcal{B}_{B^c}]\mathbb{P}^{B^c}(d\omega) =$$

$$= \int_C \left(E^B[X|\mathcal{B}_B]I_B + E^{B^c}[X|\mathcal{B}_{B^c}]I_{B^c}\right)\mathbb{P}(d\omega)$$

So the thesis follows. □

Consider $B = \{\eta < t\}$. From the lemma, one obtains that the component relative to the asset X_t is given by

$$\xi^1_t = \hat{E}^X\left[I_A|\mathcal{F}_{t-}\right] =$$
$$= \hat{E}^X_B\left[I_A|\mathcal{F}^B_{t-}\right] I_{\{\eta<t\}} \ + \ \hat{E}^X_{B^c}\left[I_A|\mathcal{F}^{B^c}_{t-}\right] I_{\{\eta\geq t\}}$$

It follows that:

1. $\hat{E}^X_B\left[I_A|\mathcal{F}^B_{t-}\right] = N(d_1(X_t, B(t,T), t, \eta))$ because the σ-fields $\mathcal{F}_{t-}$ and $\tilde{\mathcal{F}}_{t-}$ coincide if restricted to the set $\{\eta < t\}$;
2. $\hat{E}^X_{B^c}\left[I_A|\mathcal{F}^{B^c}_{t-}\right] = \hat{E}^X_{B^c}\left[N(d_1(X_t, B(t,T), t, \eta))|\mathcal{F}^{B^c}_{t-}\right]$.

We remark that X_t and $B(t,T)$ are $\mathcal{F}_{t-}$-adapted and η is independent from the trace σ-algebra $\mathcal{F}^{B^c}_{t-}$, $B^c = \{\eta \geq t\}$: we obtain therefore that

$$\hat{E}^X_{B^c}\left[I_A|\mathcal{F}^{B^c}_{t-}\right] = \int_t^T N(d_1(X_s, B(s,T), s, \eta))\nu^t(d\eta)$$

where $\nu^t(\cdot)$ is the conditional law of η with respect to $B^c = \{\eta \geq t\}$.

Finally, we have that

$$\xi^1_t = N(d_1(X_t, B(t,T), t, \eta))I_{\{\eta<t\}} + I_{\{\eta\geq t\}}\int_t^T N(d_1(X_s, B(s,T), s, \eta))\nu^t(d\eta)$$

and simmetrically the amount ξ^2_t of zero-coupon bond to be held in the local risk minimizing portfolio, i.e.:

$$-\frac{1}{K}\xi^2_t = N(d_2(X_t, B(t,T), t, \eta))I_{\{\eta<t\}} + I_{\{\eta\geq t\}}\int_t^T N(d_2(X_s, B(s,T), s, \eta))\nu^t(d\eta)$$

We remark that the event $\{\eta < t\}$ is known at time t. With analogous calculus, one obtains that

$$V_t = X_t[N(d_1(X_t, B(t,T), t, \eta))I_{\{\eta\leq t\}} + I_{\{\eta>t\}}\textstyle\int_t^T N(d_1(X_s, B(s,T), s, \eta))\nu^t(d\eta)] +$$

$$-KB(t,T)[N(d_2(X_t, B(t,T), t, \eta))I_{\{\eta\leq t\}} + I_{\{\eta>t\}}\textstyle\int_t^T N(d_2(X_s, B(s,T), s, \eta))\nu^t(d\eta)]$$

Note that the conditional laws of η given $\{\eta > \ t\}$ and $\{\eta \geq t\}$ coincide since we have supposed that $\{\eta = t\}$ is a negligible set. Recalling that $V_{t-} =$

$X_t\xi_t^1 + B(t,T)\xi_t^2$, it follows that the discontinuities of the cost process C_t are given by

$$\Delta C_t = \Delta V_t =$$

$$= I_{\{\eta=t\}} \left(N(d_1(X_t, B(t,T), t, \eta)) - \int_t^T N(d_1(X_s, B(s,T), s, \eta))\nu^t(d\eta) \right) X_t +$$

$$+ K I_{\{\eta=t\}} \left(\int_t^T N(d_2(X_s, B(s,T), s, \eta))\nu^t(d\eta) - N(d_2(X_t, B(t,T), t, \eta)) \right) B(t,T)$$

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