Stochastic Integration with Respect to a Sequence of Semimartingales

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1 Introduction

Motivated by a problem in mathematical finance, which, however, will not be discussed in this note, we propose a theory of stochastic integration with respect to a sequence of semimartingales. The case of stochastic integration with respect to a sequence of square integrable martingales, is, in fact, a special case of a theory of cylindrical stochastic integration, developed by Mikulevicius and Rozovskii [15, 16]: indeed, a sequence of martingales can be viewed as a cylindrical martingale with values in the set of all real-valued sequences.

The approach to the general case essentially relies on a paper by Mémin [10], which, in turn, is based on some results due to Dellacherie [4]. The basic idea is the following: by making use of an appropriate change in probability, it is possible to replace the integral with respect to a semimartingale with an integral with respect to the sum of a square integrable martingale and a predictable process with integrable variation; analogously, the integral with respect to a sequence of semimartingales can be replaced with the sum of an integral with respect to a sequence of square integrable martingales and an integral with respect to a sequence of predictable processes with integrable variation. It should be pointed out that the new probability is not "universal", but depends on the particular integral we are calculating.

We show that, with our definition, the stochastic integral keeps some good properties of the integral with respect to a finite-dimensional semimartingale, such as invariance with respect to a change in probability and the so-called "Mémin's theorem", which states that the limit of a sequence of stochastic integrals is still a stochastic integral. Yet, there are also some differences with the finite-dimensional case, and some "bad properties", which will be pointed out by some examples.

2 Definitions and preliminary results

Let be given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$, which fulfills the usual assumptions, and denote by $\mathcal{P}$ the predictable σ-field on $\Omega \times [0,T]$. Let $\mathbf{X} = (X^n)_{n\geqslant 1}$ be a sequence of semimartingales: $\mathbf{X}$ can be viewed as a stochastic process with values in the set of all real sequences $\mathbb{R}^{\mathbb{N}}$.

We call *simple integrand* a finite sequence of predictable bounded processes, that is, a process H of the form

$$H = \sum_{i\leqslant n} h^i e_i, \tag{1}$$

where $\{e_i\}_{i\geqslant 1}$ is the canonical basis in $\mathbb{R}^{\mathbb{N}}$, and h^i are predictable bounded processes. The stochastic integral of a simple integrand with respect to $\mathbf{X}$ is naturally defined: if H has the form (1), then

$$\int H\,\mathrm{d}\mathbf{X} = H \cdot \mathbf{X} = \int \sum_{i\leqslant n} h^i\,\mathrm{d}X^i.$$

The purpose of this paper is to extend the stochastic integral to an appropriate class of processes, which will be called *generalized integrands*. The construction which we propose keeps some good properties of the classical stochastic integral for finite-dimensional semimartingales: in particular, the stochastic integral is *isometric* in some proper sense and independent of the probability. For the general theory of stochastic integration, we mainly refer to [12]; a good description of this theory can be found also in [8], [9].

We denote by E the set $\mathbb{R}^{\mathbb{N}}$, provided with the product topology: E is a locally convex space. The dual set E' is the space of linear combinations of Dirac measures.

A simple integrand H can be represented as a process with values in E' of the form $H = \sum_{i\leqslant n} h^i\delta_i$, where, as usual, δ_i denotes the Dirac delta at point i (henceforth we will use this notation for H). It is easy to check that a E'-valued process H is *weakly predictable*, that is, He_i is predictable for all i (or, equivalently, all its components are predictable processes) if and only if H is *strongly predictable*, that is, there exists a sequence of simple integrands $(H^n)_{n\geqslant 1}$, such that for all $e \in E$

$$H_{\omega,t}\, e = \lim_{n\to\infty} H^n_{\omega,t}\, e.$$

Métivier [11] constructed an *isometric* integral for the case of a square integrable martingale with values in a Hilbert space: he proved that, in this case, it is necessary to include in the space of integrands some processes with values in the set of not necessarily bounded (or continuous) operators on E, provided with a proper measurability condition with respect to the predictable σ-algebra $\mathcal{P}$.

Following this idea, we denote by $\mathcal{U}$ the set of not necessarily bounded operators on E ($\mathcal{U} \supset E'$) and, for all $h \in \mathcal{U}$, we denote by $\mathcal{D}(h)$ the domain of h ($\mathcal{D}(h) \subset E$). We say that a sequence $(h^n) \in E'$ converges to $h \in \mathcal{U}$ if $\lim_n h^n(x) = h(x)$, for all $x \in \mathcal{D}(h)$.

Analogously to the notion of predictable E'-valued process, we introduce the following definition:

Definition 1. *A process* $\mathbf{H}$ *with values in* $\mathcal{U}$ *is said to be* predictable *if there exists a sequence* (H^n) *of* E'*-valued predictable processes, such that*

$$\mathbf{H} = \lim_{n\to\infty} H^n,$$

in the sense that for all (ω, t)*, and for all* $x \in \mathcal{D}(H_{\omega,t})$*, the sequence* $H^n_{\omega,t}(x)$ *converges to* $H_{\omega,t}(x)$*, as* n *tends to* ∞*.*

Remark 1. For a given sequence (h^n) in E', it always makes sense to define the limit operator $h = \lim_{n\to\infty} h^n$, where $\mathcal{D}(h) = \{x \in E : \lim_{n\to\infty} h^n x \text{ exists}\}$; possibly, $\mathcal{D}(h)$ can be the trivial set $\{0\}$. Hence, for any sequence (H^n) of E'-valued processes, there always exists the limit $\mathbf{H} = \lim_n H^n$, which is a process with values in $\mathcal{U}$.

Besides the usual Banach spaces of semimartingales $\mathbb{H}^p(\mathbf{P})$ (see, for instance, [14]), we will consider the Banach space of special semimartingales $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$, introduced by Mémin in [10]: a special semimartingale X belongs to $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$ if its canonical decomposition $X = M + B$ is such that $M \in \mathcal{M}^2(\mathbf{P})$ and $B \in \mathcal{A}(\mathbf{P})$. As usual, $\mathcal{M}^2(\mathbf{P})$ denotes the set of square integrable martingales, while $\mathcal{A}(\mathbf{P})$ denotes the set of predictable processes B with integrable variation, such that $B_0 = 0$. The norm on $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$ is defined by the formula:

$$\|X\|_{\mathcal{M}^2 \oplus \mathcal{A}} = \|M\|_{\mathcal{M}^2} + \|B\|_{\mathcal{A}}.$$

The importance of this space is evident in the following result, due to Dellacherie. We state it as formulated by Mémin:

Lemma 1 ([4], Theorem 5; [10], Lemma I.3). *Let be given a sequence* $(X^i)_{i\geqslant 1}$ *of semimartingales. There exists a probability measure* $\mathbf{Q}$*, equivalent to* $\mathbf{P}$*, with* $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P} \in L^\infty(\mathbf{P})$*, such that under* $\mathbf{Q}$*,*

(i) X^i *is a special semimartingale, with canonical decomposition*

$$X^i = M^i + B^i;$$

(ii) $M^i \in \mathcal{M}^2(\mathbf{Q})$*;*

(iii) $B^i \in \mathcal{A}(\mathbf{Q})$*.*

In other words: $X^i \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$*, for all* i*.*

The next lemma is an easy consequence of the proof given by Dellacherie to Lemma 1:

Lemma 2. *Let* $X \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$*. If* $\mathbf{Q}$ *is a probability measure equivalent to* $\mathbf{P}$*, such that* $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P} \in L^\infty(\mathbf{P})$*, then,* $X \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$ *and*

$$\|X\|_{\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})} \leqslant C\,\|X\|_{\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})}, \tag{2}$$

where C *is a constant which depends on* $\|\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}\|_{L^\infty}$*.*

Proof. Let $X = M + B$ be the canonical decomposition of X under $\mathbf{P}$. Denote by Z the density of $\mathbf{Q}$ with respect to $\mathbf{P}$ and by $(Z_t)_{t\leqslant T}$ the right-continuous, bounded, positive martingale $(\mathbb{E}_{\mathbf{P}}[Z \mid \mathcal{F}_t])_{t\leqslant T}$. Then, by the Girsanov theorem (see, for instance, [9], Theorem III.3.11), X is still a semimartingale under $\mathbf{Q}$: the canonical decomposition under the new probability measure is given by $X = N + D$, where

$$N = M - Z_-^{-1} \cdot \langle M,\, Z\rangle, \qquad D = B + Z_-^{-1} \cdot \langle M,\, Z\rangle.$$

Clearly, both random variables $V(B)_T$ (where $V(B)$ denotes the variation of B) and $[M,\, M]_T$ are in $L^1(\mathbf{Q})$: in particular,

$$\begin{aligned}
\mathbb{E}_{\mathbf{Q}}[V(B)_T] &\leqslant \|Z\|_{L^\infty}\,\mathbb{E}_{\mathbf{P}}[V(B)_T] = \|Z\|_{L^\infty}\,\|B\|_{\mathcal{A}(\mathbf{P})},\\
\mathbb{E}_{\mathbf{Q}}\big[\,[M,\, M]_T\,\big] &\leqslant \|Z\|_{L^\infty}\,\mathbb{E}_{\mathbf{P}}\big[\,[M,\, M]_T\,\big] = \|Z\|_{L^\infty}\,\|M\|^2_{\mathcal{M}^2(\mathbf{P})}.
\end{aligned}$$

As in the proof of Theorem 5 in [4], one can show that:

$$\begin{aligned}
\mathbb{E}_{\mathbf{Q}}\left[V\left(\frac{1}{Z_-} \cdot \langle M,\, Z\rangle\right)_T\right] &= \mathbb{E}_{\mathbf{Q}}\left[\int_0^T \frac{|\mathrm{d}\langle M,\, Z\rangle_s|}{Z_{s-}}\right]\\
&= \mathbb{E}_{\mathbf{P}}\left[\int_0^T \frac{Z\,|\mathrm{d}\langle M,\, Z\rangle_s|}{Z_{s-}}\right]\\
&= \mathbb{E}_{\mathbf{P}}\left[\int_0^T |\mathrm{d}\langle M,\, Z\rangle_s|\right]\\
&\leqslant \|M\|_{\mathcal{M}^2(\mathbf{P})}\,\|Z\|_{\mathcal{M}^2(\mathbf{P})}.
\end{aligned}$$

Then, D is a predictable process such that $V(D)_T \in L^1(\mathbf{Q})$ and

$$\|D\|_{\mathcal{A}(\mathbf{Q})} \leqslant K_1(\|M\|_{\mathcal{M}^2(\mathbf{P})} + \|B\|_{\mathcal{A}(\mathbf{P})}),$$

where K_1 is a proper constant. Moreover, $\mathbb{E}_{\mathbf{Q}}\big[[N,\, N]_T\big] \leqslant K_2\mathbb{E}_{\mathbf{Q}}\big[[M,\, M]_T\big]$ (see, for instance, [10], Lemma I.1). Hence, inequality (2) holds. □

Lemma 1 shows that, possibly by taking an appropriate equivalent probability, we can always assume that X^i is a special semimartingale, which is the sum of a square integrable martingale and a process with integrable variation. In this case, the following result can be proved:

Lemma 3. *Let $(X^i)_{i\geqslant 1}$ be a sequence in $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$, with canonical decomposition $X^i = M^i + B^i$. Then, there exist:*

(i) *an increasing predictable process A_t, such that $\mathbb{E}[A_T] < \infty$,*

(ii) *a family $Q = (Q^{ij})_{i,j\geqslant 1}$ of predictable processes, such that Q is symmetric and non-negative, in the sense that $Q^{ij} = Q^{ji}$ and $\sum_{i,j\leqslant d} x_i Q^{ij} x_j \geqslant 0$, for all $d \in \mathbb{N}$, for all $x \in \mathbb{R}^d$, $\mathrm{d}\mathbf{P}\,\mathrm{d}A$ a.s.,*

(iii) *a sequence $b = (b^i)_{i\geqslant 1}$ of predictable processes,*

such that

$$\langle M^i, M^j \rangle_t(\omega) = \int_0^t Q^{ij}_{s,\omega}\,\mathrm{d}A_s(\omega), \qquad B^i_t(\omega) = \int_0^t b^i_s(\omega)\,\mathrm{d}A_s(\omega). \tag{3}$$

Proof. Let $(c_i)_{i\geqslant 1}$ be a sequence of strictly positive numbers, such that

$$\sum_{i\geqslant 1} c_i\, \mathbb{E}\big[\langle M^i, M^i \rangle_T + V(B^i)_T\big] < \infty\,,$$

and define the process

$$A_t = \sum_{i\geqslant 1} c_i \big(\langle M^i, M^i \rangle_t + V(B^i)_t\big).$$

This process satisfies the condition $\mathbb{E}[A_T] < \infty$; moreover, $\mathrm{d}\langle M^i, M^i \rangle_t$ and $\mathrm{d}V(B^i)$ are absolutely continuous with respect to $\mathrm{d}A_t$ by definition. Finally, for $i \neq j$, the measure $\mathrm{d}\langle M^i, M^j \rangle_t$ is absolutely continuous with respect to $\mathrm{d}\langle M^i, M^i \rangle_t$ and $\mathrm{d}\langle M^j, M^j \rangle_t$ by the Kunita–Watanabe inequality (see, e.g., [12]). Define $(Q^{i,j}_t)_{i,j\geqslant 1}$ and $(b^i)_{i\geqslant 1}$ as follows

$$Q^{ij}_t(\omega) = \begin{cases} \dfrac{\mathrm{d}\langle M^i, M^j \rangle_t(\omega)}{\mathrm{d}A_t(\omega)} & \text{if } \mathrm{d}A_t(\omega) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$b^i_t = \begin{cases} \dfrac{\mathrm{d}B_t(\omega)}{\mathrm{d}A_t(\omega)} & \text{if } \mathrm{d}A_t(\omega) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The processes Q^{ij} and b^i are well-defined: they are predictable and fulfill condition (3). Jacod and Shiryaev have proved, in the case of a finite number of martingales, that Q can be chosen so that it is symmetric and non-negative $\mathrm{d}\mathbf{P}\,\mathrm{d}A$-a.s. ([9], Theorem II.2.9): it is rather easy to adapt their proof to the case of a sequence of martingales. □

Remark 2. The process A is minimal in the following sense: if D is a predictable process such that both measures $\mathrm{d}\mathbf{P}\,\mathrm{d}\langle M^i, M^i \rangle_t$ and $\mathrm{d}\mathbf{P}\,\mathrm{d}V(B^i)_t$ are absolutely continuous with respect to $\mathrm{d}\mathbf{P}\,\mathrm{d}D_t$, then the measure $\mathrm{d}\mathbf{P}\,\mathrm{d}A_t$ is also absolutely continuous with respect to $\mathrm{d}\mathbf{P}\,\mathrm{d}D_t$.

We call *negligible* a predictable set C which is $\mathrm{d}\mathbf{P}\,\mathrm{d}A$-negligible, that is,

$$\mathbb{E}\left[\int_0^T \mathbf{1}_C \,\mathrm{d}A_t\right] = 0.$$

This notion does not depend on the probability $\mathbf{P}$: indeed, it is not difficult to prove that C is $\mathrm{d}\mathbf{P}\,\mathrm{d}A$-negligible if and only $\int h\,\mathrm{d}X^i = 0$ for every i and for every bounded predictable process h, which is zero on the complement of C.

We denote by $\mathcal{S}(\mathbf{P})$ the space of real semimartingales, endowed with the semimartingale topology, which was introduced by Émery [6]. We refer to [6] for general definition and main properties of this topology: it is important to recall that $\mathcal{S}(\mathbf{P})$ is a complete metric space. But we will mainly use this result, which is due to Mémin:

Theorem 1 ([10], Theorem II.3). *Let $(X^i)_{i\geqslant 1}$ be a Cauchy sequence in $\mathcal{S}(\mathbf{P})$. Then, there exist a subsequence (which we still denote by X^i) and a probability measure $\mathbf{Q}$, equivalent to $\mathbf{P}$, such that $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P} \in L^\infty(\mathbf{P})$ and (X^i) is a Cauchy sequence in $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$.*

Remark 3. Lemma 1 and Theorem 1 hold only when the time set is a compact interval $[0,T]$: this explains why we work in this framework. However if the stochastic integral can be defined on a finite interval, then it can be defined on the whole set $[0,+\infty[$ by localization.

Finally, we introduce our definition of integrable process:

Definition 2. *Let $\mathbf{H}$ be a predictable $\mathcal{U}$-valued process. We say that $\mathbf{H}$ is* integrable *with respect to $\mathbf{X}$ if there exists a sequence (H^n) of simple integrands such that:*

(i) *H^n converges to $\mathbf{H}$, a.s.;*
(ii) *$(H^n \cdot \mathbf{X})$ converges to a semimartingale Y in $\mathcal{S}(\mathbf{P})$.*

We call $\mathbf{H}$ a generalized integrand *and define $\int \mathbf{H}\,\mathrm{d}\mathbf{X} = \mathbf{H}\cdot\mathbf{X} = Y$.*

We denote by $L(\mathbf{X},\mathcal{U})$ the set of generalized integrands.

Remark 4. Of course, Definition 2 makes sense if we prove that Y is uniquely defined, in the sense that $\mathbf{H}\cdot\mathbf{X}$ is independent of the sequence H^n which approximates $\mathbf{H}$. This result, with all its consequences and applications, will be the object of section 5.

We wish also to point out that our definition of integrable process is very similar to the notion of integrable function with respect to a vector-valued measure ([5], section IV.10.7).

3 Stochastic integration with respect to a sequence of square integrable martingales

For the case of a sequence of square integrable martingales, we refer to the theory on cylindrical integration recently developed by Mikulevicius and Rozovskii [15], [16]: indeed, a sequence of martingales can be viewed as a cylindrical martingale with values in E. In this section, we briefly recall Mikulevicius and Rozovskii's main results, adapted to our setting.

We assume that $X^i = M^i \in \mathcal{M}^2(\mathbf{P})$ for all i and denote by $\mathbf{M}$ the sequence $(M^i)_{i\geqslant 1}$. The aim in this section is to define the stochastic integral $\int H\,\mathrm{d}\mathbf{M}$ on a proper class of processes, so that the integral is a square integrable martingale. We recall that, even in the one-dimensional case, it may happen that the integral with respect to a square integrable martingale is not even a local martingale (see, for instance, [7]).

For a simple integrand $H_t = \sum_{i\leqslant n} h_t^i \delta_i$, the Ito isometry holds:

$$\mathbb{E}\Big[\Big(\int_0^T H_s\,\mathrm{d}\mathbf{M}_s\Big)^2\Big] = \mathbb{E}\Big[\int_0^T \sum_{i,j\leqslant n} h_s^i h_s^j \,\mathrm{d}\langle M^i, M^j\rangle_s\Big]$$
$$= \mathbb{E}\Big[\int_0^T \sum_{i,j\leqslant n} h_s^i h_s^j Q_s^{ij}\,\mathrm{d}A_s\Big], \tag{5}$$

where Q and A are defined as in Lemma 3. The question is how to complete the set of integrands with respect to the norm induced by the Ito isometry. Intuition may suggest that, to this aim, it is sufficient to consider, as value set of the integrands, the space of the sequences $(h^i)_{i\geqslant 1}$ such that $\mathbb{E}\big[\int \sum_{i,j} h^i h^j Q^{ij}\,\mathrm{d}A\big] < \infty$. In fact, this is not sufficient, as we will show in Example 1 below.

Consider Q for fixed (ω, t) (which we omit, for simplicity). We can define on E' a linear mapping with values in E, which we still denote by Q, in the following way: for $h = \sum_{i\leqslant n} h^i \delta_i \in E'$, we define Qh as the sequence whose i-th component is $(Qh)_i = \sum_j Q^{ij} h^j$. It is easy to check that Q is symmetric and non-negative, namely $\langle h, Qk\rangle_{E',E} = \langle k, Qh\rangle_{E',E}$ and $\langle h, Qh\rangle_{E',E} \geqslant 0$ for all h, $k \in E'$, where $\langle\ ,\ \rangle_{E',E}$ denotes the duality. The mapping Q induces a seminorm on E', by the formula:

$$|h|_Q^2 = \langle h, Qh\rangle_{E',E} = \sum_{i,j\geqslant 1} h^i Q^{ij} h^j. \tag{6}$$

Thus, the main problem is to find a completion of E' with respect to this seminorm. Following the approach by Mikulevicius and Rozovskii [15, 16], we consider on the set QE' the scalar product:

$$(Qh, Qk)_{QE'} = \langle h, Qk\rangle_{E',E} = \langle k, Qh\rangle_{E',E}.$$

This scalar product induces a norm, with respect to which QE' is a pre-Hilbert space. Its completion K is a Hilbert space and can be continuously embedded in E. Denote by K' the topological dual of K: the set K' contains E' and coincides with the completion of $E'/\ker Q$, with respect to the norm induced by (6). Thus, if $h \in E'$, then $|h|^2_{K'} = \langle h, Qh \rangle_{E',E}$. We recall that K and K' depend on (ω, t): so we have defined a family of Hilbert spaces, depending on Q. Note that, if H is a simple integrand, the isometry (5) can be rewritten in the form:

$$\mathbb{E}\left[\left(\int_0^T H_s \, \mathrm{d}\mathbf{M}_s\right)^2\right] = \mathbb{E}\left[\int_0^T |H_s|^2_{K'_s} \, \mathrm{d}A_s\right].$$

Now, it seems natural to take as generalized integrand a predictable process $\mathbf{H}$ with values in $\mathcal{U}$, such that for all (ω, t), the domain of $\mathbf{H}_{\omega,t}$ contains $K_{\omega,t}$, the restriction of $\mathbf{H}_{\omega,t}$ to $K_{t,\omega}$ is an element of $K'_{\omega,t}$, and $\mathbb{E}\big[\int |\mathbf{H}|^2_{K'} \, \mathrm{d}A\big] < \infty$. We observe that if $\mathbf{H}$ is predictable, then $|\mathbf{H}|^2_{K'}$ is also predictable: this is a consequence of the argument below.

Consider now the canonical basis in E' and take the sequence $\eta_n = Q\delta_n$. By a standard orthogonalization procedure, we can construct an orthonormal basis $\{k^i\}_{i \geqslant 1}$ in K (once again, for simplicity of notations, we omit (ω, t)): every k^i is an element of $\mathrm{span}\{\eta_1, \ldots, \eta_i\}$. Hence, $k^i = Qh^i$, where h^i is an element of K', such that $h^i \in \mathrm{span}(\delta_1, \ldots, \delta_i)$ and $\{h^i\}_{i \geqslant 1}$ is an orthonormal basis in K' (see [15] pag. 141 for details).

If we consider h^i as a function of (ω, t), it follows that $h^i_t = \sum_{j \leqslant i} \alpha^{ij}_t \delta_j$ where α^{ij}_t are real predictable. If $\mathbf{H}$ is predictable, in the sense of Definition 1, $\mathbf{H}_t(Q_t \delta_j)$ is predictable and so is $\mathbf{H}(k^j_t)$.

Every process $\mathbf{H}$, such that $\mathbf{H}_{\omega,t} \in K'_{\omega,t}$, can be written in the form:

$$\mathbf{H}_t = \sum_{i \geqslant 1} \lambda^i_t h^i_t,$$

where $\lambda^i_t = (\mathbf{H}_t, h^i_t)_{K'_t} = \mathbf{H}_t(k^i_t)$; then $|\mathbf{H}_t|^2_{K'_t} = \sum_i (\lambda^i_t)^2$ is predictable. Furthermore, the process $\mathbf{H}$ can be approximated by the sequence

$$H^n_t = \sum_{i \leqslant n} \lambda^i_t h^i_t = \sum_{i \leqslant n} \sum_{j \leqslant i} \lambda^i_t \alpha^{ij}_t \delta_j = \sum_{j \leqslant n} r^{nj}_t \delta_j \tag{7}$$

(a proof of this fact can be found in [16], Corollaries 2.2 and 2.3).

The following theorem is essentially due to Mikulevicius and Rozovskii: we give a formulation which is slightly different from the original one, but better fits into our context. We also recall the main steps of their proof.

Theorem 2. *Let* $\mathbf{H}$ *be a* $\mathcal{U}$*-valued process such that:*

(i) $\mathcal{D}(\mathbf{H}_{\omega,t}) \supset K_{\omega,t}$ *for all* (ω, t)*;*

(ii) $\mathbf{H}_{\omega,t}|_{K_{\omega,t}} \in K'_{\omega,t}$*;*

(iii) $\mathbf{H}_t(Q_t\delta_n)$ *is predictable for all* n*;*

(iv) $\mathbb{E}\big[\int_0^T |\mathbf{H}_t|^2_{K'_t}\,\mathrm{d}A_t\big] < \infty$.

Then, there exists a sequence (H^n) *of simple integrands, such that* $H^n_{\omega,t}$ *converges to* $\mathbf{H}_{\omega,t}$ *in* $K'_{\omega,t}$ *for all* (ω, t) *and* $(H^n \cdot \mathbf{M})$ *is a Cauchy sequence in* $\mathcal{M}^2(\mathbf{P})$*.*

As a consequence, we can define the stochastic integral $\mathbf{H}\cdot\mathbf{M}$ *as the limit of the sequence* $(H^n \cdot \mathbf{M})$*.*

Proof. It is easy to check that the approximating sequence H^n defined by (7) converges to $\mathbf{H}$ in K' and it is such that $(H^n \cdot \mathbf{M})$ is a Cauchy sequence in $\mathcal{M}^2(\mathbf{P})$. However, every H^n may not be a simple integrand. Fix n: as we have already observed, the process H^n is of the form $\sum_{i\leqslant n} r^{ni}\delta_i$, where r^{ni} are predictable. We define the sequence

$$H^{n,m}_t = \sum_{i\leqslant n} r^{ni}_t \delta_i \,\mathbf{1}_{\{\max_{i\leqslant n}|r^{ni}_t|\leqslant m\}},$$

for every $m \in \mathbb{N}$. Then, $(H^{n,m})_{m\geqslant 1}$ is a sequence of simple integrands, which converges to H^n in K' (as $m \to \infty$), and it is such that $(H^{n,m}\cdot\mathbf{M})_{m\geqslant 1}$ is a Cauchy sequence in $\mathcal{M}^2(\mathbf{P})$ (see, for instance, [15], Proposition 9). Thus, by a standard diagonalization procedure, we can build a sequence $\widetilde{H}^n = H^{n,m_n}$ of simple integrands, which satisfies the required properties. □

Remark 5. It may seem sufficient, in order to define the integral $\mathbf{H}\cdot\mathbf{M}$, to know the restriction of $\mathbf{H}_{\omega,t}$ to $K_{\omega,t}$. In fact, this is not exactly true: indeed, let $\mathbf{H}$ be a process which fulfills conditions (i)–(iv), and (H^n) be the sequence defined as in the theorem. If we set $\mathbf{J} = \lim_n H^n$, we can say that $\mathbf{J}$, but not $\mathbf{H}$, is $\mathbf{M}$-integrable, according to Definition 2. This will be evident in Example 1. However, an immediate consequence of Theorem 2 is the following: assume that (H^n) and (J^n) are two sequences of simple integrands, such that both sequences $(H^n\cdot\mathbf{M})$, $(J^n\cdot\mathbf{M})$ are Cauchy sequences in $\mathcal{M}^2(\mathbf{P})$. Define $\mathbf{H} = \lim_n H^n$ and $\mathbf{J} = \lim_n J^n$. If $\mathbf{H}|_K = \mathbf{J}|_K$ a.s., then $\lim_n(H^n\cdot\mathbf{M}) = \lim_n(J^n\cdot\mathbf{M})$; hence, $\mathbf{H}\cdot\mathbf{M} = \mathbf{J}\cdot\mathbf{M}$.

Remark 6. The condition of measurability given by Mikulevicius and Rozovskii (condition (iii) in Theorem 2) is strictly related to the probability measure $\mathbf{P}$: under an equivalent measure $\mathbf{Q}$, the processes M^i may not be martingales and, in any case, the process $\mathrm{d}\langle M^i, M^j\rangle$ is no longer the same. For this reason, we have decided to give a different notion of predictable process, which does not depend on the probability.

Remark 7. Consider the stable subspace generated by the sequence $\mathbf{M} = (M^i)_{i\geqslant 1}$ of square integrable martingales (for the definition of stable subspace, we refer to [12]). It is natural to expect that a characterization of this space holds in terms of the stochastic integral with respect to $\mathbf{M}$. Indeed, the following result holds: the set of all the stochastic integrals $\mathbf{H} \cdot \mathbf{M}$, with $\mathbf{H}$ fulfilling conditions (i)–(iv) of Theorem 2, is a closed set in $\mathcal{M}^2(\mathbf{P})$ and coincides with the stable subspace generated by $\mathbf{M}$ in $\mathcal{M}^2(\mathbf{P})$. A proof of this can be found in [16]; it is, in fact, a simple extension of the analogous result in the finite-dimensional case.

Remark 8. One could think that the above mentioned construction can be adapted to the case of a sequence of local martingales, just replacing the predictable quadratic covariation $\langle M^i, M^j \rangle$, with the quadratic covariation $[M^i, M^j]$. In fact, this does not work. The reason is that the process $[M^i, M^j]$ is optional, not predictable: a factorization as in Lemma 3 can still be found, but Q and A are optional. So, it is not possible to repeat the construction of the approximating sequence and the proof of Theorem 2.

4 Stochastic integration with respect to a sequence of predictable processes with finite variation

In this section we assume that $X^i = B^i \in \mathcal{A}(\mathbf{P})$ for all i and denote by $\mathbf{B}$ the sequence $(B^i)_{i \geqslant 1}$. The property of finite variation is invariant with respect to a change in probability. However, if $\mathbf{Q}$ is an equivalent probability measure, it may occur that the variation of B is in $L^1(\mathbf{P})$ but not in $L^1(\mathbf{Q})$.

Assume that B^i has integrable variation for all i. Then, as we have proved in Lemma 3, there exists a factorization $dB^i_t = b^i_t \, \mathrm{d}A_t$ where A is an increasing predictable process and $(b^i)_{i \geqslant 1}$ is a sequence of predictable processes. In this case, the construction of the stochastic integral $\mathbf{H} \cdot \mathbf{B}$ is easier than in the previous case, since the set of predictable processes with finite variation is closed in $\mathcal{S}(\mathbf{P})$ (see, for instance, [10], Theorem IV.7).

So, if $\mathbf{H}$ is a $\mathcal{U}$-valued process such that $\mathbf{H} = \lim_n H^n$, where H^n are simple integrands, then it is easy to verify that $\mathbf{H}$ is $\mathbf{B}$-integrable if and only if $b_{\omega,t}$ belongs to the domain of $\mathbf{H}_{\omega,t}$ for all (ω, t) and the random variable

$$\int_0^T |\mathbf{H}_t b_t| \, \mathrm{d}A_t \tag{8}$$

is finite $\mathbf{P}$-a.s. Note that $\mathbf{H}b = \lim_n H^n b$ is a predictable process. Moreover, assume that the random varable defined in (9) belongs to $L^1(\mathbf{P})$, and H^n is a sequence of simple integrands which converges to $\mathbf{H}$; then, the sequence

$$J^n = H^n \mathbf{1}_{\{|H^n b| \leqslant 2 |\mathbf{H}b|\}}$$

is such that $(J^n \cdot \mathbf{B})$ converges to $\mathbf{H} \cdot \mathbf{B}$ in $\mathcal{A}(\mathbf{P})$.

For this case, the result corresponding to Theorem 2 is much simpler: assume that $\mathbf{H}$ is a $\mathcal{U}$-valued process such that:

(i) $b_{\omega,t} \in \mathcal{D}(\mathbf{H}_{\omega,t})$ for all (ω, t);

(ii) the process $\mathbf{H}b$ is predictable;

(iii) $\mathbb{E}\big[\int_0^T |\mathbf{H}_t b_t| \, \mathrm{d}A_t\big] < \infty$.

Then, there exists a sequence (H^n) of simple integrands such that $H^n b$ converges to $\mathbf{H}b$ and $(H^n \cdot \mathbf{B})$ is a Cauchy sequence in $\mathcal{A}(\mathbf{P})$.

As a consequence, we can define the stochastic integral $\mathbf{H} \cdot \mathbf{B}$ as the limit of the sequence $(H^n \cdot \mathbf{B})$.

In fact, it is sufficient to find an E'-valued process H such that $Hb = \mathbf{H}b$: assume, for simplicity that $b^1_{\omega,t} \neq 0$ for all (ω, t); then, we can define $H_{\omega,t} = \mathbf{H}_{\omega,t}\delta_1/b^1_{\omega,t}$. If the predictable set $\{b^1 = 0\}$ is not negligible, it is clear how to modify such a construction. Finally, H can be easily approximated by a sequence of simple integrands.

5 The general case

The construction of a stochastic integral with respect to a sequence of semimartingales relies on the two previous cases: an appropriate change of probability allows us to reduce to the case of a stochastic integral with respect to the sum of a sequence of square integrable martingales and a sequence of predictable processes with finite variation.

Let $\mathbf{H}$ be a $\mathcal{U}$-valued process and assume, as in Definition 2, that there exists a sequence (H^n) of simple integrands, such that $\mathbf{H} = \lim_n H^n$ and $(H^n \cdot \mathbf{X})$ is a Cauchy sequence in $\mathcal{S}(\mathbf{P})$.

By Lemma 1, there exists a probability measure $\mathbf{Q}_1$, equivalent to $\mathbf{P}$, with $\mathrm{d}\mathbf{Q}_1/\mathrm{d}\mathbf{P} \in L^\infty(\mathbf{P})$ and such that $\mathbf{X} \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q}_1)$, that is, every X^i belongs to $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q}_1)$. Furthermore, by Theorem 1, we can find a subsequence, which we still denote by (H^n), and a probability measure $\mathbf{Q}_2$, equivalent to $\mathbf{Q}_1$, such that $\mathrm{d}\mathbf{Q}_2/\mathrm{d}\mathbf{Q}_1 \in L^\infty(\mathbf{Q}_1)$ and $(H^n \cdot \mathbf{X})$ is a Cauchy sequence in $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q}_2)$. Lemma 2 makes sure that $\mathbf{X}$ belongs also to $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q}_2)$: if $\mathbf{X} = \mathbf{M} + \mathbf{B}$ is the canonical decomposition of $\mathbf{X}$ under $\mathbf{Q}_2$, then, for all n, $H^n \cdot \mathbf{M} + H^n \cdot \mathbf{B}$ is the canonical decomposition of $H^n \cdot \mathbf{X}$. Moreover, $(H^n \cdot \mathbf{M})$ is a Cauchy sequence in $\mathcal{M}^2(\mathbf{Q}_2)$, whereas $(H^n \cdot \mathbf{B})$ is a Cauchy sequence in $\mathcal{A}(\mathbf{Q}_2)$. Then $\mathbf{H} \cdot \mathbf{M}$ and $\mathbf{H} \cdot \mathbf{B}$ exist, in the sense shown respectively in sections 3 and 4. The integral $\mathbf{H} \cdot \mathbf{X}$, which is defined as the limit in $\mathcal{S}(\mathbf{P})$ of the sequence $(H^n \cdot \mathbf{X})$, coincides with $\mathbf{H} \cdot \mathbf{M} + \mathbf{H} \cdot \mathbf{B}$.

We are now able to prove what we claimed in Remark 4, namely, that Definition 2 is a good definition:

Proposition 1. *Let (H^n) and (J^n) be two sequences of simple integrands such that:*

(i) *there exists a $\mathcal{U}$-valued process $\mathbf{H}$ such that $\mathbf{H} = \lim_n H^n = \lim_n J^n$;*

(ii) *if $Y^n = H^n \cdot \mathbf{X}$ and $Z^n = J^n \cdot \mathbf{X}$, then (Y^n) and (Z^n) are Cauchy sequences in $\mathcal{S}(\mathbf{P})$.*

Then $\lim_n Y^n = \lim_n Z^n$.

Hence, the process $\mathbf{H} \cdot \mathbf{X} = \lim_n Y^n = \lim_n Z^n$ *is well-defined. In other words, the definition of the integral does not depend on the approximating sequence.*

Proof. If we apply twice Lemma 1 and Theorem 1, we can find a probability measure $\mathbf{Q}$, equivalent to $\mathbf{P}$ and such that $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P} \in L^\infty(\mathbf{P})$ and a subsequence (which we still denote by n) such that $\mathbf{X} \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$ and $(Y^n), (Z^n)$ are both Cauchy sequences in $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$. This means that, if $\mathbf{X} = \mathbf{M} + \mathbf{B}$ is the canonical decomposition under $\mathbf{Q}$, then, $(H^n \cdot \mathbf{M})$ and $(J^n \cdot \mathbf{M})$ are both Cauchy sequences in $\mathcal{M}^2(\mathbf{Q})$, whereas $(H^n \cdot \mathbf{B})$ and $(J^n \cdot \mathbf{B})$ are both Cauchy sequences in $\mathcal{A}(\mathbf{Q})$. From hypothesis (i) and from the results of the previous sections, it follows immediately that it must be

$$\lim_{n\to\infty} H^n \cdot \mathbf{M} = \lim_{n\to\infty} J^n \cdot \mathbf{M}, \qquad \lim_{n\to\infty} H^n \cdot \mathbf{B} = \lim_{n\to\infty} J^n \cdot \mathbf{B}.$$

So, the claim follows. □

Remark 9. We have just proved that, when the stochastic integral exists, then, by an appropriate change of probability, it can be represented as the sum of an integral with respect to a sequence of square integrable martingales and an integral with respect to a sequence of processes with bounded variation.

The converse does not hold true: it may happen that the two abovementioned integrals exist, but the process is not integrable in the sense of Definition 2. Let be given a probability measure $\mathbf{Q}$ such that $\mathbf{X} \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$, with canonical decomposition $\mathbf{X} = \mathbf{M} + \mathbf{B}$, and a process $\mathbf{H}$ which is separately integrable with respect to $\mathbf{M}$ and $\mathbf{B}$ in the sense of section 3 and 4; namely, $\mathbf{H}$ fulfills the conditions of Theorem 2 and condition (i), (ii), (iii) of section 4. Then, Theorem 2 suggests the construction of an approximating sequence H^n such that $H^n|_K$ converges to $\mathbf{H}|_K$ and $(H^n \cdot \mathbf{M})$ is a Cauchy sequence in $\mathcal{M}^2(\mathbf{Q})$. Yet, this does not imply that $H^n b$ converges to $\mathbf{H}b$. So, condition (i) of Definition 2 is not satisfied. However, if (H^n) can be chosen so that $H^n b$ converges to $\mathbf{H}b$, then the stochastic integral does exist in the sense of Definition 2.

The following result is the extension to the infinite-dimensional case of Mémin's theorem ([10], Theorem III.4), which states that the set of stochastic integrals with respect to a semimartingale is closed in $\mathcal{S}(\mathbf{P})$.

Theorem 3. *Let be given a sequence of semimartingales* $\mathbf{X} = (X^i)_{i\geqslant 1}$ *and a sequence* $(\mathbf{H}^n)$ *of generalized integrands: assume that* $(\mathbf{H}^n \cdot \mathbf{X})$ *is a Cauchy sequence in* $\mathcal{S}(\mathbf{P})$. *Then, there exists a generalized integrand* $\mathbf{H}$ *such that* $\lim_{n\to\infty} \mathbf{H}^n \cdot \mathbf{X} = \mathbf{H} \cdot \mathbf{X}$.

Proof. Without loss of generality, we can assume that $\mathbf{H}^n = H^n$ are simple integrands. We can choose an equivalent probability measure $\mathbf{Q}$ and a subsequence, which we still denote by H^n, such that $X^i \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$ and $(H^n \cdot \mathbf{X})$ is a Cauchy sequence in $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{Q})$. Define $\mathbf{H} = \lim_n H^n$. Since

$$\mathbb{E}\left[\int_0^T |H_t^n - H_t^m|^2_{K_t'}\,\mathrm{d}A_t\right] \longrightarrow 0 \qquad \text{as } n,\, m \to \infty,$$

it is clear that the domain of $\mathbf{H}_{t,\omega}$ must contain $K_{\omega,t}$ for all (ω, t) and

$$\mathbb{E}\left[\int_0^T |H_t^n - \mathbf{H}_t|^2_{K_t'}\,\mathrm{d}A_t\right] \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Analogously, since

$$\mathbb{E}\left[\int_0^T |H_t^n b_t - H_t^m b_t|\,\mathrm{d}A_t\right] \longrightarrow 0 \qquad \text{as } n,\, m \to \infty,$$

it is clear that the domain of $\mathbf{H}_{t,\omega}$ must contain $b_{\omega,t}$ for all (ω, t) and

$$\mathbb{E}\left[\int_0^T |H_t^n b_t - \mathbf{H}_t b_t|\,\mathrm{d}A_t\right] \longrightarrow 0 \qquad \text{as } n \to \infty.$$

So the claimed result is proved. □

6 Examples

In this section, we show some examples to point out the main differences between the stochastic integral with respect to a finite-dimensional semimartingale and with respect to a sequence of semimartingales. In the examples we will consider, the spaces $K_{\omega,t}$ do not depend on (ω, t) and the integrands are constants. Nonetheless, even in these simple cases, some differences are evident.

The first example is taken from [3] (Example 2.1): there it is shown that, even in the case of a sequence of square integrable martingales, the set of E-valued processes is not large enough as set of integrands.

Example 1. Consider the sequence of martingales $\mathbf{M} = (M^i)_{i\geqslant 1}$ defined by:

$$M^i = W + N^i,$$

where W is a Wiener process, (N^i) is a sequence of independent compensated Poisson processes all with the same intensity $\lambda = 1$, such that N^i is independent of W, for all i.

According to Lemma 3, we find a factorization of the quadratic variation of $\mathbf{M}$, by setting $A_t = t$, $Q^{ij} = 1 + \delta_{ij}$. In [3], it has been proved that K is the subset of E of all sequences of the form $(\alpha + y_1, \alpha + y_2, \dots)$, with $\alpha \in \mathbb{R}$ and $(y_i)_{i\geqslant 1} \in l^2$, and the norm of such a sequence in K is $\alpha^2 + \sum_{i\geqslant 1} y_i^2$; the dual set K' is isomorphic to $\mathbb{R} \oplus l^2$.

Consider the sequence of simple integrands $H^n = n^{-1}\sum_{i\leqslant n}\delta_i$. It is clear that H^n converges to the constant $\mathcal{U}$-valued process $\mathbf{H}$, defined by

$$\mathbf{H}(x) = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} .$$

Notice that H^n does not converge componentwise to $\mathbf{H}$: indeed, $\lim_n H_i^n = 0$, for all i. Moreover,

$$H^n \cdot \mathbf{M} = W + \frac{N^1 + \cdots + N^n}{n} ,$$

which converges to W in $\mathcal{M}^2(\mathbf{P})$. Therefore, we have that $\mathbf{H}\cdot\mathbf{M} = W$.

Consider now the operator $\mathbf{J}(x) = \lim_n x_n$; it is not difficult to check that $\mathbf{H}$ coincides with $\mathbf{J}$ on K. However, we observe that there does not exist $\lim_n M^n = \lim_n(W + N^n)$. This proves what we have claimed in Remark 5. The sequence H^n converges to $\mathbf{H}$ and not to $\mathbf{J}$ on E: the sequence $(H^n\cdot\mathbf{M}) = (H^n(\mathbf{M}))$ converges in $\mathcal{M}^2(\mathbf{P})$ to $\mathbf{H}\cdot\mathbf{M} = \mathbf{H}(\mathbf{M})$, while $\mathbf{J}(\mathbf{M})$ is not defined, although $\mathbf{H}|_K = \mathbf{J}|_K$. So, we can say that $\mathbf{H}$ is $\mathbf{M}$-integrable, whereas $\mathbf{J}$ is not $\mathbf{M}$- integrable.

The following example is obtained by a modification of an example due to Émery [6].

Example 2. Let $(T_n)_{n\geqslant 1}$ be a sequence of independent random variables, such that T_n is exponentially distributed with $\mathbb{E}[T_n] = n^2$. We take as filtration the smallest filtration which satisfies the usual conditions and such that T_n are stopping times. We define a sequence of martingales M^n as follows:

$$M_t^n = \frac{t\wedge T_n}{n^2} - \mathbf{1}_{\{t\geqslant T_n\}} = \frac{t}{n^2}\mathbf{1}_{\{t<T_n\}} + \left(\frac{T_n}{n^2} - 1\right)\mathbf{1}_{\{t\geqslant T_n\}}. \tag{9}$$

Consider the simple integrand $H^n = n^{-1}\sum_{i\leqslant n} i^2\delta_i$. Then, with the usual notation $\mathbf{M} = (M^i)_{i\geqslant 1}$,

$$H^n \cdot \mathbf{M} = \frac{M^1 + 2^2M^2 + \cdots + n^2M^n}{n} = \frac{N^1 + \cdots + N^n}{n} ,$$

where $N^i = t\wedge T_i - i^2\mathbf{1}_{\{t\geqslant T_i\}}$. The sequence (H^n) converges (as $n\to\infty$) to the operator $\mathbf{H}$ on E, defined by:

$$\mathbf{H}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i\leqslant n} i^2 x_i, \tag{10}$$

for all $x\in E$ such that this limit does exist, while the sequence $(H^n\cdot\mathbf{M})$ converges to the increasing process $A_t = t$. Consider the stopping times $S_n = \inf_{m\geqslant n} T_m$. Using Borel–Cantelli lemma, it can be proved that S_n tends to infinity (as $n\to\infty$). In particular, the sequence $S_n\wedge T$ converges to T

stationarily, namely, $S_n \equiv T$ definitely, $\mathbf{P}$-a.s. So, for fixed ε, there exists some n such that $\mathbf{P}(S_n \leqslant T) < \varepsilon$. On the stochastic interval $[\![0, S_n \wedge T]\!]$, the martingale N^m coincides with the process A, for $m \geqslant n$. Then, if we stop the processes $H^k \cdot \mathbf{M}$ at time S^n, we have that, for $k > n$

$$(H^k \cdot \mathbf{M})^{S_n} = \frac{(N^1 + \cdots + N^n)^{S_n}}{k} + \frac{(k-n)}{k}\,(t \wedge S_n).$$

It is not difficult to check that the sequence $(H^k \cdot \mathbf{M})^{S_n}$ converges to $t \wedge S_n$ as n tends to ∞: as a consequence, $(H^k \cdot \mathbf{M})$ converges to A_t in $\mathcal{S}(\mathbf{P})$ (see [6] for further details).

In the finite-dimensional case, it is well-known that, if $X \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$ with canonical decomposition $X = M + B$, and $H \cdot X$ also belongs to $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$, then, necessarily, the canonical decomposition of the stochastic integral is $H \cdot X = H \cdot M + H \cdot B$. This is no longer the case, when $\mathbf{X}$ is a sequence of semimartingales, as the previous example shows: indeed, $\mathbf{X} \in \mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$ and $\mathbf{X} = \mathbf{M} + \mathbf{0}$, the process $\mathbf{H}$ is $\mathbf{X}$-integrable, the integral $\mathbf{H} \cdot \mathbf{X}$ belongs to $\mathcal{M}^2 \oplus \mathcal{A}(\mathbf{P})$, but the canonical decomposition is $\mathbf{H} \cdot \mathbf{X} = 0 + A$. The reason for this different behaviour will be explained in Remark 10.

Ansel and Stricker [1] have proved that if M is a finite-dimensional local martingale and H an integrable process, such that $H \cdot M$ is bounded from below, then $H \cdot M$ is a local martingale (hence it is a supermartingale). Example 2 shows that, as opposed to the finite-dimensional case, this does not necessarily occur when $\mathbf{M}$ is a sequence of local martingales. Indeed, in the mentioned example, $\mathbf{H}$ is a $\mathbf{M}$-integrable process, such that $\mathbf{H} \cdot \mathbf{M}$ is bounded from below, but the stochastic integral $\mathbf{H} \cdot \mathbf{M}$ is not a local martingale.

However if, for a generalized integrand $\mathbf{H}$, an approximating sequence (H^n) of simple integrands can be found, such that $H^n \cdot \mathbf{M}$ converges to $\mathbf{H} \cdot \mathbf{M}$ and there also exists a random variable $W \in L^1(\mathbf{P})$ such that, for all t, $\int_0^t H^n_s \,\mathrm{d}\mathbf{M}_s \geqslant W$, then it can be proved, using Fatou's lemma, that $\mathbf{H} \cdot \mathbf{M}$ is a supermartingale.

Moreover, we observe that, if every X^i is continuous, the result by Ansel and Stricker still holds even in the case of a sequence of martingales; indeed, the set of continuous local martingales is closed in $\mathcal{S}(\mathbf{P})$ (see [10], Theorem IV.5). This is also a consequence of the following remark: assume that $\mathbf{X}$ is continuous and $\mathbf{H}$ is an $\mathbf{X}$-integrable process such that $\mathbf{H} \cdot \mathbf{X} \geqslant -C$, where C is a positive constant. Then, for all $\varepsilon \geqslant 0$, there exists a sequence of simple integrands (H^n) such that $H^n \cdot \mathbf{X} \geqslant -C - \varepsilon$: indeed, given an approximating sequence H^n such that $H^n \cdot \mathbf{X}$ converges to $\mathbf{H} \cdot \mathbf{X}$, we can define the stopping time

$$T_n = \inf\left\{ t : \int_0^t H^n_s \,\mathrm{d}\mathbf{X}_s < -C - \varepsilon \right\}.$$

Since $\lim_n \mathbf{P}(T_n < T) = 0$, we can find a subsequence (which we still denote by T_n) such that $\sum_n \mathbf{P}(T_n < T) < \infty$. We set $S_n = \inf_{m \geqslant n} T_m$ and $\tilde{H}^n =$

$\mathbf{1}_{[0,S_n]}H^n$. Then, clearly $\tilde{H}^n \cdot \mathbf{X} \geqslant -C - \varepsilon$. Moreover $\tilde{H}^n \cdot \mathbf{X} = H^n \cdot \mathbf{X}$ on the set $[0, S_n]$ and S_n converges to T, $\mathbf{P}$-a.s.; hence $\tilde{H}^n \cdot \mathbf{X}$ converges to $\mathbf{H} \cdot \mathbf{X}$.

Remark 10. When X is a $\mathbb{R}^k$-valued semimartingale, a predictable process H, with values in $\mathbb{R}^k$, is X-integrable if the sequence $H^n = H\mathbf{1}_{\{\|H\| \leqslant n\}}$ is such that $(H^n \cdot X)$ is a Cauchy sequence in $\mathcal{S}(\mathbf{P})$ (see [2]). Moreover, if S is a stopping time, and denoting, as usual, by $\Delta_s X$ the jump $X_s - X_{s-}$, we have that $\Delta_S(H \cdot X) = H_S \Delta_S X$; for fixed t, the sequence $\sum_{s \leqslant t} \Delta_s(H^n \cdot X)^2$ is increasing and converges to $\sum_{s \leqslant t} \Delta_s(H \cdot X)^2$. Hence,

$$\mathbb{E}\left[\sum_{s \leqslant t} \Delta_s(H \cdot X)^2\right] = \lim_{n \to \infty} \mathbb{E}\left[\sum_{s \leqslant t} \Delta_s(H^n \cdot X)^2\right], \tag{11}$$

whether the expectation is finite or not.

Given a sequence of semimartingales $\mathbf{X}$, a generalized integrand $\mathbf{H}$ and an approximating sequence H^n, for all stopping times S, one still has $\Delta_S(H^n \cdot \mathbf{X}) = H^n_S \cdot (\Delta_S \mathbf{X})$; this sequence converges in probability to $\Delta_S(\mathbf{H} \cdot \mathbf{X})$: thus, $\Delta_S \mathbf{X}$ belongs to $\mathcal{D}(\mathbf{H}_S)$ and it is equal to $\mathbf{H}_S \cdot (\Delta_S \mathbf{X})$. However, in this case, (11) does not hold. The sequence $\sum_{s \leqslant t} \Delta_s(H^n \cdot \mathbf{X})^2$ does not necessarily converge: we can only say that

$$\mathbb{E}\left[\sum_{s \leqslant t} \Delta_s(\mathbf{H} \cdot \mathbf{X})^2\right] \leqslant \min \lim_{n \to \infty} \mathbb{E}\left[\sum_{s \leqslant t} \Delta_s(H^n \cdot \mathbf{X})^2\right].$$

In particular, in the finite-dimensional case, if M is a purely discontinuous local martingale such that $\mathbb{E}\left[\sum_{s \leqslant t} H_s^2 (\Delta_s M)^2\right] < \infty$, then $H \cdot M$ is a square integrable martingale. This is no longer true for the case of a sequence of local martingales, as we have seen in Example 2.

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