Complexity of geometric three-manifolds and Dehn surgery on the three-rings chain-link

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April 19, 2002

Abstract

We compute for all orientable irreducible 3-manifolds having trivial JSJ decomposition certain complexity functions that approximate from above Matveev's natural complexity, known to be equal to the minimal number of tetrahedra in a triangulation. We can show that the upper bounds on Matveev's complexity implied by our computations are sharp for thousands of manifolds, and we conjecture they are for infinitely many, including almost all Seifert manifolds. Our computations and estimates apply to all the Dehn fillings of $M6^3_1$ (the complement of the three-component chain-link, conjectured to be the smallest triply cusped hyperbolic manifold), whence to infinitely many among the smallest closed hyperbolic manifolds. One of the key ingredients of our results is a complete classification of all Dehn fillings of $M6^3_1$ not admitting a metric of negative sectional curvature. Both the computations of complexity and the classification of the fillings of $M6^3_1$ are based on the machinery of the decomposition into 'bricks' of irreducible manifolds, developed in a previous paper.

MSC (2000): 57M27 (primary), 57M20, 57M50 (secondary).

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Introduction

The complexity $c(M)$ of a closed orientable 3-manifold M was defined in [13] as the minimal number of vertices of a simple spine of M. In the same paper it was shown that c is additive under connected sum and that, if M is irreducible, $c(M)$ equals the minimal number of tetrahedra in a triangulation of M, unless M is S^3, $\mathbb{RP}^3$, or $L_{3,1}$. This means that indeed $c(M)$ is a very natural measure of how complicated M is. Despite this fact, $c(M)$ is only known for a finite number of M's. In this paper we give upper bounds for the complexity of a wide class of M's, including many of the smallest known hyperbolic manifolds, all Seifert manifolds, and all torus bundles over the circle. To be more precise we will introduce a sequence $(c_n)_{n=0}^{\infty}$ of functions with values in $\mathbb{N} \cup \{\infty\}$ such that, given M, the sequence $(c_n(M))_{n=0}^{\infty}$ is non-increasing and has constant value $c(M)$ for $n \gg 0$. We will then explicitly compute c_n for $n \leqslant 9$ on torus bundles and on all manifolds having trivial JSJ decomposition. These manifolds may be collectively described as the 'geometric' ones, because they are those conjectured by Thurston to carry one of the eight locally homogeneous geometries [16]. The upper estimate on $c(M)$ given by the knowledge of $c_9(M)$ is known to be sharp for many M's, and we conjecture it to be sharp for all torus bundles and all Seifert manifolds, except for certain very special ones fibred over the sphere with three exceptional fibres. Our estimates also allow us to compute the complexity of 16 of the smallest known closed orientable hyperbolic manifolds.

This paper is based on the theory developed in [11] and reviewed below in Section 2. We roughly sketch this theory here to give the reader an idea of what c_n is and what it means to compute it. If M is a compact orientable 3-manifold bounded by tori $T_1, \ldots, T_k$, we will call *manifold with triods* a pair (M, X) where $X = \{\theta_i\}_{i=1}^{k}$

and θ_i is a triod (the trivalent graph with two vertices and no closed edge, also called θ-curve by some authors) embedded in T_i so that $T_i \setminus \theta_i$ is a disc. Using simple polyhedra we have defined in [11] a complexity $c(M, X)$ for all such pairs, we have shown that $c(M, \emptyset) = c(M)$ for closed M, and that c is additive under connected sum. Moreover, we have introduced an operation $(M_1, X_1) \oplus (M_2, X_2)$, called *assembling*, which consists in gluing $T_1 \subset \partial M_1$ to $T_2 \subset \partial M_2$ via a map $\psi : T_1 \to T_2$ such that $\psi(\theta_1) = \theta_2$ (for the sake of simplicity we do not mention here the operation of self-assembling, see Section 2). The complexity c turns out to be subadditive under $\oplus$, so we have called *sharp* an assembling under which c is additive. Finally, we have called *brick* a pair that cannot be split as a sharp assembling, and we have shown that every irreducible pair can be sharply split into bricks. Using a computer, we have found that up to complexity 9 there are precisely 11 bricks with boundary, and 19 closed ones (that cannot be assembled at all).

We define now $c_n(M)$ as the smallest sum of complexities over the expressions of M as an assembling of bricks having complexity up to n, with $c_n(M) = \infty$ if no such expression exists. So, to compute $c_n(M)$, it is not sufficient to exhibit one expression: an argument must be given to show that a certain expression is the best possible one with the given set of bricks. The precise computation of c_n for $n \leqslant 9$ on Seifert and hyperbolic manifolds is too complicated to be reproduced here (see Theorems 1.11 and 1.16 below), but there is one aspect that makes it remarkable and we can state now. Of course $c_9(M)$ gives a better or equal upper estimate for $c(M)$ than $c_0(M), \ldots, c_8(M)$, but from the computation one sees that, at least for most Seifert manifolds and all torus bundles, the non-increasing sequence $(c_n(M))_{n=0}^{9}$ actually stabilizes very early (namely at $n = 1$ for lens spaces and torus bundles, and at $n = 3$ for all other Seifert manifolds, except some atoroidal ones). This seems to suggest that this stabilized value actually equals $c(M)$ for these M's.

One of the key ingredients of our results is the complete classification (see Theorem 1.13) of all the manifolds obtained by Dehn surgery on the chain-link with three components (denoted by 6_1^3 in [15] and shown in Fig. 1) that do not admit a metric of negative sectional curvature. These manifolds are relevant to the theory because the complement of 6_1^3, itself denoted by $M6_1^3$ in [7], is the basis of two of our bricks. We believe that our classification theorem for Dehn fillings of $M6_1^3$ could have an independent interest, because quite some attention has been devoted in the literature to the understanding of non-hyperbolike fillings of cusped hyperbolic manifolds (the topic was initiated by Thurston himself in [18] and extensively investigated later on, see for instance [9]). Note also that $M6_1^3$ is a rather interesting manifold, being the smallest known triply cusped orientable hyperbolic one. The proof of our classification result uses the technology of simple skeleta [11] for manifolds with triods, the Gromov-Thurston 2π-theorem (see [5]), and the software SnapPea [19] for a long

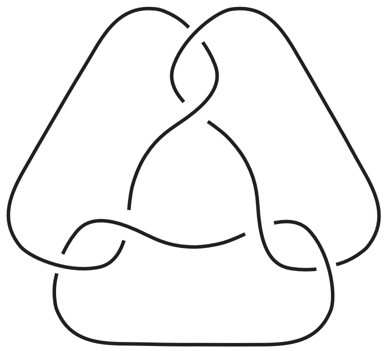

Figure 1: The three-rings chain-link.

but finite discussion of sporadic cases.

This paper is structured as follows. Since the precise definition of c_n requires several details from [11], we have deferred it to Section 2, collecting in Section 1 all our statements, organized so to be readable even without a complete understanding of what c_n is. Section 1 also contains the definition of two complexity functions (one for coprime pairs of integers and one for invertible 2×2 integer matrices) necessary to state our results. Section 2 then contains the exact definitions of brick and c_n, together with the preliminary constructions and results needed for these definitions. In Section 3 we discuss the spaces of slopes and triods on a given torus, and we prove various properties of the auxiliary complexity functions introduced in Section 1. In Section 4 we start setting up the combinatorial machine that we employ to compute c_n, beginning with the atoroidal case. Our main result here is that this case essentially reduces to the understanding of the Dehn fillings of the bricks, rather than more elaborate assemblings. This fact is used in Section 4 itself to compute c_n on atoroidal Seifert manifolds, and later in Section 6 for the Dehn fillings of our hyperbolic bricks, after the proof in Section 5 of the classification theorem for non negatively curved fillings of $M6^3_1$. Section 7 concludes the paper, with the computation of c_n for non-hyperbolic but geometric 3-manifolds. This computation is based on the extension to manifolds with triods of the notion of Seifert fibration and Euler number.

We warmly thank Jeff Weeks for very helpful suggestions on how to use SnapPea for the last step of the proof of Theorem 1.13.

1 Complexity estimates and classification of fillings

As briefly mentioned in the introduction, our theory of decomposition of irreducible 3-manifolds into bricks allows to define a non-increasing sequence $\left(c_n\right)_{n=0}^{\infty}$ of approximated complexity functions such that, for any given M, we have $c_n(M) = c(M)$ for $n \gg 0$. Deferring the precise definition of c_n to Section 2, we state here all the results we can show about it. Namely, we compute c_n on all manifolds having trivial JSJ decomposition, and we state our classification theorem for non negatively curved Dehn fillings of the complement of the 3-component chain-link. We need to begin with some terminology and some preliminaries.

Terminology for 3-manifolds All the manifolds we will consider are compact, connected, 3-dimensional, and orientable (but mostly unoriented). We call *lens space* any genus-1 manifold distinct from $\mathbb{RP}^3$, and *Seifert manifold* any total space of a Seifert fibration over a surface, possibly with boundary. We say that a Seifert manifold is *genuine* when it is not S^3, $\mathbb{RP}^3$, a lens space, $T\times[0,1]$, the solid torus $\boldsymbol{T}$, or a torus bundle over S^1. For $A \in \mathrm{GL}_2(\mathbb{Z})$ we denote by T_A the total space of the torus bundle over S^1 with monodromy A, and we agree that the JSJ decomposition of T_A consists of T_A only (even if T_A is neither Seifert nor atoroidal). We call *graph manifold* one whose JSJ decomposition consists of Seifert blocks only or which is itself a torus bundle. Later we will employ the labels of the Callahan-Hildebrand-Weeks census [7] for cusped hyperbolic manifolds. Namely, we denote by Mi_j^k the j-th manifold among those having an ideal triangulation with i tetrahedra and k cusps, ordered by increasing volume.

We explain now the notation we use for the parameters of a Seifert manifold. Let F be surface, t be an integer, and $\{(p_i, q_i)\}_{i=1}^k$ be coprime pairs of integers with $|p_i| \geqslant 2$. We denote by $\left(F, (p_1, q_1), \ldots, (p_k, q_k), t\right)$ the oriented Seifert manifold obtained from either $F \times S^1$ or $F \widetilde{\times} S^1$ (depending on orientability of F) by drilling $k+1$ trivially fibred solid tori and Dehn filling the resulting boundary components along the slopes $p_1a_1+q_1b_1, \ldots, p_ka_k+q_kb_k, a_{k+1}+tb_{k+1}$, where the a_i's are contained in a section of the bundle, the b_i's are fibres, and each (a_i, b_i) is a positive basis in homology. Recall [8] that for $i \leqslant k$ the i-th filling gives rise to an exceptional fibre with "orbital type" $(|p_i|, r_i)$, where $0 < r_i < |p_i|$ and $q_ir_i \equiv p_i/|p_i| \pmod{|p_i|}$, but we will never use the orbital type, only the "filling type" (p_i, q_i). In addition, the Euler number of the Seifert fibration is given by $t + \sum q_i/p_i$, but we will never use the Euler number itself, only the "twisting number" t.

The Seifert parameters are called *normalized* if $p_i > q_i > 0$ for all i. Any set of Seifert parameters can be promoted to a normalized one by replacing each (p_i, q_i) by $(|p_i|, q_i')$, where $0 < q_i' < |p_i|$, $q_i' \equiv q_ip_i/|p_i| \pmod{|p_i|}$, and t by $t + \sum(q_i/p_i - q_i'/|p_i|)$. The next result, taken from [8] and often needed below, shows that normalized parameters with an extra assumption correspond almost bijectively to genuine Seifert manifolds:

Proposition 1.1. *Let M be a closed and oriented Seifert manifold.*

- *If M is genuine and atoroidal then it can be expressed uniquely as*

$$\big(S^2,(p_1,q_1),(p_2,q_2),(p_3,q_3),t\big)$$

  with $0<q_i<p_i$ for all i, and $t\in\mathbb{Z}$. The only such expressions that do not define atoroidal genuine Seifert manifolds are the following torus bundles:

$$\begin{aligned}
\big(S^2,(2,1),(3,1),(6,1),-1\big) &= T_{\left(\begin{smallmatrix}0 & 1\\ -1 & 1\end{smallmatrix}\right)}\\
\big(S^2,(2,1),(4,1),(4,1),-1\big) &= T_{\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)}\\
\big(S^2,(3,1),(3,1),(3,1),-1\big) &= T_{\left(\begin{smallmatrix}0 & 1\\ -1 & -1\end{smallmatrix}\right)}.
\end{aligned}$$

- *If M is genuine and toroidal then it can be expressed uniquely as*

$$\big(F,(p_1,q_1),\ldots,(p_k,q_k),t\big)$$

  with $k-\chi(F)>0$, $p_i>q_i>0$ for all i, and $t\in\mathbb{Z}$. The only such expressions that do not define toroidal genuine Seifert manifolds are those with $F=S^2$ and $k=3$, and the following one:

$$\big(S^2,(2,1),(2,1),(2,1),(2,1),-2\big)=T_{\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)}.$$

- *The expression of $-M$ is obtained from that of M by replacing t by $-t-k$ and each q_i by p_i-q_i.*

Remark 1.2. When a closed Seifert manifold $\big(F,(p_1,q_1),\ldots,(p_k,q_k),t\big)$ is viewed as an orientable but unoriented manifold, we can assume by the last point of the previous proposition that $t\geqslant -k/2$, but this expression can fail to be unique when the value of t is precisely $-k/2$.

In addition to the standard terminology, we define now $\mathcal{M}^*$ to be the class of all genuine atoroidal Seifert manifolds of type $\big(S^2,(2,1),(n,1),(m,1),-1\big)$, that is

$$\mathcal{M}^*=\big\{(S^2,(2,1),(n,1),(m,1),-1):\ n\geqslant m\geqslant 2,\ (n,m)\neq(6,3),(4,4)\big\}. \qquad (1)$$

Coprime pairs and integer matrices Our statements involve notions of complexity for coprime pairs of integers and for invertible 2×2 integer matrices. These notions first appeared in [14] and [2].

Definition 1.3. We recursively define an $\mathbb{N}$-valued function $|\cdot,\cdot|$ on pairs of coprime integers by setting $|1,0| = |0,1| = |0,-1| = |1,1| = 0$ and

$$|p,q| = 1 + \begin{cases} |p, q-p| & \text{if } q > p > 0, \\ |p-q, q| & \text{if } p > q > 0. \end{cases}$$

If $p > 0 > q$ we define $|p,q|$ as $|p,-q|+1$. If $p < 0$ we define $|p,q|$ as $|-p,-q|$.

A direct method for computing $|p,q|$ will be described in Proposition 3.3. The following proposition will also be proved in Section 3. It implies in particular that $|p,q|$ is a well-defined function for a lens space $L_{p,q}$.

Proposition 1.4. *1. $|p,q| = |q,p|$ for all coprime p,q;*

2. If $0 < q < p$ then $|p, p-q| = |p,q|$;

3. If $0 < q, q' < p$ and $q \cdot q' \equiv \pm 1 \pmod{p}$ then $|p,q| = |p,q'|$.

Let us now consider the following matrices:

$$S_1 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The following proposition will be proved in Section 3.

Proposition 1.5. *Every $A \in \mathrm{GL}_2(\mathbb{Z})$ can be written in a unique way as*

$$A = \varepsilon \cdot S_{i_0} \cdot J \cdot S_{i_1} \cdot J \cdots S_{i_{n-1}} \cdot J \cdot S_{i_n} \cdot S_1^m$$

with $n \geqslant 0$, $\varepsilon \in \{\pm 1\}$, $i_0, i_n \in \{1,2,3\}$, $i_1, \ldots, i_{n-1} \in \{1,2\}$, and $m \in \{0,1\}$.

Remark 1.6. Since $\det(A) = (-1)^{m+1}$, uniqueness of m in this expression is obvious. Existence, and uniqueness of the other indices, will be harder to show.

Definition 1.7. For $A \in \mathrm{GL}_2(\mathbb{Z})$ we define $|A| \in \mathbb{N}$ as the number n of J's appearing in the expression of A just described. Moreover we define $||A||$ to be the minimum of $|BAB^{-1}|$ as B varies in $\mathrm{GL}_2(\mathbb{Z})$.

Remark 1.8. Since $J^{-1} = J$ and $S_i^{-1} = S_i$ for $i = 1, 2, 3$, one readily sees from the definition that $|A^{-1}| = |A|$ when $\det(A) = -1$. The same equality can actually be checked also for $\det(A) = +1$, using also the relations

$$S_1 \cdot S_i = S_{(1\,3)(i)} \cdot S_3, \quad S_3 \cdot J = -J \cdot S_3, \quad S_3 \cdot S_i = S_{(1\,2)(i)} \cdot S_3, \quad S_3 \cdot S_i = S_{(1\,3)(i)} \cdot S_1.$$

In particular we have $||A^{-1}|| = ||A||$ for all A.

Remark 1.9. Given a certain $A \in \mathrm{GL}_2(\mathbb{Z})$, it is in general very difficult to compute $|A|$ and $\|A\|$ using the definition directly. A practical method to compute them will be given in Corollary 3.5.

Finiteness of c_n We now state our main results, starting from finiteness of our approximated complexity functions. Recall that a special list $\mathcal{M}^*$ of Seifert manifolds was introduced above in (1).

Theorem 1.10. *Let M be a closed, orientable, and irreducible 3-manifold. If $M \notin \mathcal{M}^*$ we have*

$$c_0(M) \geqslant c_1(M) = c_2(M) \geqslant c_3(M) = \ldots = c_7(M) \geqslant c_8(M) \geqslant c_9(M).$$

Moreover:

- $c_0(M) < +\infty$ *if and only if* $M \in \{S^3, \mathbb{RP}^3, L_{3,1}\}$ *or* $M = T_A$ *with* $||A|| \leqslant 1$;
- $c_1(M) < +\infty$ *if and only if M is either S^3, or $\mathbb{RP}^3$, or a lens space, or a torus bundle over S^1;*
- $c_3(M) < +\infty$ *if and only if M is a graph manifold;*
- $c_8(M) < +\infty$ *if and only if either M is a torus bundle, or M is a Dehn filling of $M2_2^1$, or the JSJ decomposition of M consists of some Seifert blocks and some copies of $M2_2^1$;*
- $c_9(M) < +\infty$ *if and only if M is a torus bundle or splits along a family of incompressible tori into a union of Seifert manifolds and (partial) Dehn fillings of $M6_1^3$.*

Computation of c_n: Seifert case We will now be much more precise on the values of the c_n's on torus bundles, on Seifert manifolds, and on atoroidal manifolds. More results could be given for manifolds having an arbitrary JSJ decomposition, but we defer the discussion of the general case to a subsequent paper. We only mention here that the general computation of c_n would make an essential use of the notions of complexity $|A|$ and $||A||$ introduced above for a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$. The latter appears in the present paper only in the expression of c_n for torus bundles, and the former is hidden in the proof of the formula for hyperbolic manifolds (see Lemma 6.2).

We start with torus bundles and Seifert manifolds, using the "almost unique" normalized parameters described in Proposition 1.1 and Remark 1.2, and leaving to the reader to check that the formula for c_n is unchanged when a Seifert manifold admits two normalized expressions.

Theorem 1.11. *Let M be a closed, orientable, and irreducible 3-manifold.*

- *If $M \in \{S^3, \mathbb{RP}^3, L_{3,1}\}$ then $c_n(M) = 0$ for $n = 0, \ldots, 9$;*
- *If $M = T_A$ then:*
  - *If $\|A\| \leqslant 1$ then $c_n(M) = 6$ for $n = 0, \ldots, 9$;*
  - *If $\|A\| \geqslant 2$ then $c_0(M) = +\infty$ and $c_n(M) = \|A\| + 5$ for $n = 1, \ldots, 9$;*
- *If M is a lens space $L_{p,q}$ different from $L_{3,1}$ then $c_0(M) = +\infty$ and $c_n(M) = |p, q| - 2$ for $n = 1, \ldots, 9$;*
- *If M is a genuine Seifert manifold $\big(F, (p_1, q_1), \ldots, (p_k, q_k), t\big)$ with $k - \chi(F) > 0$, $p_i > q_i > 0$ for all i and $t \geqslant -k/2$, but M is neither a member of $\mathcal{M}^*$ nor of the form $(S^2, (2, 1), (3, 1), (p, q), -1)$ with $p/q > 5$, then $c_n(M) = +\infty$ for $n = 0, 1, 2$ and*

$$c_n(M) = \max\{0, t - 1 + \chi(F)\} - 6(\chi(F) - 1) + \sum_{i=1}^{k} (|p_i, q_i| + 2)$$

  for $n = 3, \ldots, 9$;
- *If $M = (S^2, (2, 1), (3, 1), (p, q), -1)$ with $p/q > 5$ and $p/q \notin \mathbb{Z}$, then $c_n(M) = +\infty$ for $n = 0, 1, 2$, $c_n(M) = |p, q| + 3$ for $n = 3, \ldots, 7$, and $c_n(M) = |p, q| + 2$ for $n = 8, 9$.*

Remark 1.12. The only Seifert manifolds to which the previous result does not apply are the members of $\mathcal{M}^*$. These manifolds play a special rôle in our theory, and the results we have about them, together with various speculations on their meaning, are stated at the end of Section 2. We also note that a precise list of the values of $c_n(M)$ for all $n = 0, \ldots, 9$ and $M \in \mathcal{M}^*$ would be possible but rather cumbersome. In addition, as we will explain in Section 2, this list would lead to the exact value of $c(M)$ only for the 19 "smallest" members of $\mathcal{M}^*$. For this reason we have decided to omit the list, giving only the value of c_9.

Classification of fillings By Theorem 1.10, an atoroidal M with $c_9(M) < +\infty$ is either a small Seifert manifold or a Dehn filling of $M6_1^3$ (since $M2_2^1$ is itself a filling of $M6_1^3$). This leads to the study of the Dehn fillings of $M6_1^3$. Our techniques allow us to classify all fillings that are not negatively curved. To give our statement, we first make the following conventions:

Homology bases: Since $M6_1^3$ is a link complement, on each boundary component we use the meridian-longitude homology basis as customary for knot complements, and we express the filling coefficients as rational numbers or ∞ using these bases;

Symmetry: Since the three boundary components of $M6_1^3$ are completely symmetric, we regard the filling coefficients up to permutation;

Lens spaces: For simplicity here (and nowhere else) we employ a more flexible notation for lens spaces than the usual one: we write $L(2, q)$ for $\mathbb{RP}^3$, $L(1, q)$ for S^3, $L(0, q)$ for $S^2 \times S^1$, and $L(p, q)$ for $L_{|p|,q'}$ with $q \equiv q' (\mathrm{mod}\ p)$ and $0 \leqslant q' < |p|$, for any coprime p, q;

Seifert manifolds: We allow non-normalized parameters for genuine Seifert manifolds (but in the statement of Theorem 1.13 we will always make sure that there are as many exceptional fibres as the notation suggests);

Graph manifolds: Since our statement will involve certain manifolds having a non-trivial JSJ decomposition consisting either of two blocks based on the disc or of one block based on the annulus, we need to make more conventions. We consider the disc with i holes D_i and we choose homology bases (a_j, b_j) for $j = 0, \ldots, i$ on the components of $\partial(D_i \times S^1)$, where the a_j's are the components of $(\partial D_i) \times \{\mathrm{pt}\}$, oriented as boundary components of ∂D_i, the b_j's have the form $\{\mathrm{pt}\} \times S^1$, and the bases are positive. We use these bases to describe the filling coefficients on $D_i \times S^1$ as rational numbers or ∞. We also agree that if a partial filling is performed, on the non-filled components the bases are left untouched. Now we note that the Seifert manifold $(D,(p,q),(r,s))$ based on the disc D can be identified to the $(p/q, r/s)$-filling of $(D_2 \times S^1)$. The agreement just made then gives a preferred homology basis on the boundary component of any $(D,(p,q),(r,s))$, so for $A \in \mathrm{GL}_2(\mathbb{Z})$ we have a well-defined way to glue two such manifolds together. Hence a manifold

$$\big(D,(p,q),(r,s)\big) \bigcup\nolimits_A \big(D,(p',q'),(r',s')\big)$$

is well-defined, and its JSJ decomposition typically consists of the two obvious blocks. As a matter of fact, some exceptions occur (when one of the blocks is non-genuine or A matches a fibration of one block with a fibration of the other block) but in the statement of Theorem 1.13 we will always make sure to use the symbol when indeed the JSJ decomposition is the obvious one.

Similarly, noting that D_1 (a disc with one hole) is the annulus, and that $(D_1,(p,q))$ is the p/q-filling of $(D_2 \times S^1)$, we get preferred homology bases on the two boundary components of $(D_1,(p,q))$. We can then unambiguously glue together these components along a given $A \in \mathrm{GL}_2(\mathbb{Z})$, and denote by

$$\big(D_1,(p,q)\big)\big/_A$$

the result. This is typically a graph manifold with graph (the only exceptions occur for $|p| \leqslant 1$, but in Theorem 1.13 we will only use the symbol making sure that $|p| > 1$).

Theorem 1.13. *Let M be a closed Dehn filling with coefficients $\{p/q, r/s, t/u\}$ of the complement $M6^3_1$ of the three-component chain-link in S^3. Then M admits a metric with negative sectional curvature except if one the following cases occurs up to permutation:*

- *$p/q = \infty$, in which case $M = L(tr - us, tr' - us')$, where $r', s' \in \mathbb{Z}$ are such that $rs' - sr' = \pm 1$;*
- *$p/q \in \{-3, -2, -1, 0\}$, in which case M is as described in Table 1 (with n and m standing for generic integers);*
- *$(p/q, r/s) \in \{(1,1), (-4, -1/2), (-3/2, -5/2)\}$, in which case M is as described in Table 2;*
- *$(p/q, r/s, t/u)$ is one of the 14 triples listed in Table 3, together with a description of the corresponding M.*

p/q	r/s	t/u	M
-3	1 or $-5/3$	r/s	$T_{\begin{pmatrix}-1&0\\-1&-1\end{pmatrix}}$
	-1	any	$\mathbb{RP}^3\#L(t+3u,\,u)$
	-2	any	$L(5t+7u,\,2t+3u)$
	$-1+\frac{1}{n}$	$-1+\frac{1}{m}$	$L(4nm+2n+2m-3,\,2mn+m-2)$
	$\neq -2$	$\neq -1,\ \neq -1+\frac{1}{m}$	$(S^2,\,(2,1),\,(2n+1,-2),\,(t+u,u))$
	-3	$\neq -1,\ \neq -1+\frac{1}{m}$	$(\mathbb{RP}^2,\,(2,1),\,(t+u,u))$
	$\neq -1,\ \neq -3$ $\neq -1+\frac{1}{n}$	$\neq -1,\ \neq -3$ $\neq -1+\frac{1}{m}$ $\neq \frac{r}{s}$ if $\frac{r}{s}\in\{1,-\frac{5}{3}\}$	$(D,\,(2,1),\,(r+s,r+2s))\bigcup_{\begin{pmatrix}1&1\\0&-1\end{pmatrix}}$ $(D,\,(2,1),\,(t+u,t+2u))$
-2	-2	any	$L(3,1)\#L(t+2u,\,u)$
	$-2+\frac{1}{n}$	any	$L(3n(t+2u)-2t-u,\,n(t+2u)-t-u)$
	$\neq -2$ $\neq -2+\frac{1}{n}$	$\neq -2$ $\neq -2+\frac{1}{m}$	$(S^2,\,(3,2),\,(r+2s,-s),\,(t+2u,-u))$
-1	-3	any	$\mathbb{RP}^3\#L(t+3u,\,u)$
	$-3+\frac{1}{n}$	any	$L(2n(t+3u)-t-u,\,n(t+3u)+u)$
	$\neq -3$ $\neq -3+\frac{1}{n}$	$\neq -3$ $\neq -3+\frac{1}{m}$	$(S^2,\,(2,1),\,(r+3s,-s),\,(t+3u,-u))$
0	n	$-4-n$	$\mathbb{RP}^3\#L(3,1)$
		$-4-n+\frac{1}{m}$	$L(6m-1,\,2m-1)$
		$\neq -4-n$ $\neq -4-n+\frac{1}{m}$	$(S^2,\,(2,-1),\,(3,1),\,(t+(n+4)u,u))$
	$\frac{1}{2}+n$	$-\frac{9}{2}-n$	$(\mathbb{RP}^2,\,(2,1),\,(3,1),\,-1)$
		$\neq -\frac{9}{2}-n$ $\notin \mathbb{Z}$	$(D,\,(2,2n+5),\,(u,t+2u))\bigcup_{\begin{pmatrix}0&-1\\1&1\end{pmatrix}}$ $(D,\,(2,1),\,(3,1))$
	$\notin \frac{1}{2}\mathbb{Z}$	$\notin \frac{1}{2}\mathbb{Z}$	$(D,\,(s,r+2s),\,(u,t+2u))\bigcup_{\begin{pmatrix}0&-1\\1&1\end{pmatrix}}$ $(D,\,(2,1),\,(3,1))$

Table 1: Classification of non negatively curved fillings

p/q	r/s	t/u	M
1	1	n	$T_{\begin{pmatrix} n+1 & -1 \\ 1 & 0 \end{pmatrix}}$
		$\notin \mathbb{Z}$	$\big(D_1, (u, -t-2u)\big)/_{\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}}$
$-3/2$	$-5/2$	-2	$\mathbb{RP}^3$
		-1	$L(13, 5)$
		0	$\big(\mathbb{RP}^2, (2, 1), (3, 1), -1\big)$
		$-2 + \frac{1}{n}$ $\neq -1$	$\big(S^2, (2, -1), (3, 1), (2n-1, -2)\big)$
		$\neq 0$ $\neq -2$ $\neq -2 + \frac{1}{n}$	$\big(D, (t+2u, -u), (2, 1)\big) \bigcup_{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}$ $\big(D, (2, 1), (3, 1)\big)$
-4	$-1/2$	-1	$L(11, 3)$
		$-\frac{1}{2}$	$\big(\mathbb{RP}^2, (2, 1), (3, 1), -1\big)$
		0	$L(13, 5)$
		n $\neq -1$ $\neq 0$	$\big(S^2, (2, -1), (3, 1), (2n+1, -2)\big)$
		$\notin \mathbb{Z} \cup \{-\frac{1}{2}\}$	$\big(D, (u, -t-u), (2, 1)\big) \bigcup_{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}$ $\big(D, (2, 1), (3, 1)\big)$

Table 2: Classification of non negatively curved fillings (continued)

p/q	r/s	t/u	M
-5	-5	$-1/2$	$\big(D_1,(2,1)\big)/_{\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}}$
-4	-4	$-2/3$	$\big(D_1,(2,1)\big)/_{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$
-4	$-3/2$	$-3/2$	$T_{\begin{pmatrix} -3 & -1 \\ 1 & 0 \end{pmatrix}}$
-4	$-1/3$	1	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$-8/3$	$-3/2$	$-3/2$	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$-5/2$	$-5/2$	$-4/3$	$\big(D_1,(2,1)\big)/_{\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}}$
$-5/2$	$-5/3$	$-5/3$	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$-7/3$	$-7/3$	$-3/2$	$\big(D_1,(2,1)\big)/_{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$
1	2	2	$\big(S^2,(2,1),(3,1),(7,1),-1\big)$
1	2	3	$\big(S^2,(2,1),(4,1),(5,1),-1\big)$
1	2	4	$\big(S^2,(3,1),(3,1),(4,1),-1\big)$
1	2	5	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\big(D,(2,1),(3,1)\big)$
1	3	3	$\big(D,(2,1),(3,1)\big)\bigcup_{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}}\big(D,(2,1),(2,1)\big)$
2	2	2	$\big(D,(2,1),(3,1)\big)\bigcup_{\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}}\big(D,(2,1),(2,1)\big)$

Table 3: Classification of non negatively curved fillings (continued)

The tables always provide the decomposition of M into prime connected summands and the JSJ decomposition of these summands, except when a connected sum involves a lens space and the parameters are such that the lens space is S^3.

To appreciate this result, recall that a closed manifold admitting a metric of negative scalar curvature is irreducible, atoroidal and not Seifert fibred, and has infinite, word-hyperbolic fundamental group. Such a manifold is conjectured to be hyperbolic. Since all the exceptions to negative curvature in our statement are well-understood and decompose into geometric blocks, our result implies that the geometrization conjecture reduces to the hyperbolization conjecture for Dehn fillings of $M6_1^3$. In particular, no Dehn filling of $M6_1^3$ is a counterexample to the Poincaré conjecture. We do not think the hyperbolization conjecture is proved for fillings of $M6_1^3$, but it is probably known to experts for fillings of the form $1, -4, p/q$, which correspond to p/q-fillings of $M2_2^1$. Our theorem anyway easily implies the following:

Corollary 1.14. *A p/q-Dehn filling of $M2_2^1$ is negatively curved unless*

$$p/q \in \{\infty, -3, -2, -1, -1/2, -1/3, 0, 1\}.$$

The next remark is also important to understand the statement of Theorem 1.13.

Remark 1.15. We have compiled Tables 1 to 3 trying to keep them concise but easily usable. In particular:

- We have always excluded the case where one of the filling coefficients is ∞, because the first point of Theorem 1.13 is already sufficient in this case to determine which lens space the result is;
- Whenever a certain value $p/q \in \{-3, -2, -1, 0\}$ is chosen, the corresponding set of lines in Table 1 contains a complete classification of the resulting M, but one may have to permute r/s and t/u to get the result;
- The triples listed in Table 3 do not appear in Tables 1 and 2;
- There is some overlapping between the cases covered in Tables 1 and 2. The recognition of M is of course coherent, either evidently or by Proposition 1.1-(1);
- In Table 3 the elements of each triple are arranged in increasing order, and the triples themselves are arranged in a lexicographically increasing order.

Computation of c_n: hyperbolic case Now we turn to the exact computation of c_8 and c_9 on the Dehn fillings of $M6_1^3$ which do not fall in the special categories listed in the previous theorem. To do so we need to introduce a certain numerical complexity h for triples of rational numbers. Namely, we define:

$$h(p/q, r/s, t/u) = g + |p+2q, q| + |r+2s, s| + |t+2u, u|$$

where

$$g = \begin{cases} 6 & \text{if } p/q, r/s, t/u \text{ are all different from 1 and greater than } -2, \\ 4 & \text{if } p/q = 1,\ r/s \in \{-4,-5\}, \text{ and } t/u < -1 \text{ up to permutation,} \\ 2 & \text{if } p/q = 1,\ r/s \in \{-4,-5\}, \text{ and } t/u > -1 \text{ up to permutation,} \\ 5 & \text{in all other cases.} \end{cases}$$

Theorem 1.16. • *If M is a Dehn filling of $M2_2^1$ but not one of those listed in Corollary 1.14 then $c_8(M)$ is the minimum of $7 + |p, -q|$ over all realizations of M as a p/q-filling of $M2_2^1$;*

- *If M is a Dehn filling of $M6_1^3$ but not one of those listed in Theorem 1.13 then $c_9(M)$ is the minimum of $h(p/q, r/s, t/u)$ over all realizations of M as a $(p/q, r/s, t/u)$-filling of $M6_1^3$.*

The somewhat implicit form of the previous statement is due to the fact that very little is known about manifolds arising as different Dehn fillings of the same cusped hyperbolic manifold (see [6] for a careful discussion of the one-cusped case). We can however show that the following repetitions occur among the fillings of $M6_1^3$ and, after extensive experiments carried out with SnapPea, we put forward the conjecture that this is actually the complete list of repetitions (up to permutation of the filling coefficients):

Proposition 1.17. *Let $N_{p/q,r/s,t/u}$ be the closed Dehn filling of $M6_1^3$ with coefficients $\{p/q, r/s, t/u\}$. Then:*

$$N_{-\frac{3}{2},\frac{p}{q},\frac{r}{s}} = N_{-4,\frac{q}{p+2q}-1,-\frac{r}{s}-3} \tag{2}$$

$$N_{-\frac{3}{2},\frac{p}{q},\frac{r}{s}} = N_{-\frac{3}{2},-\frac{q}{p+2q}-2,-\frac{s}{r+2s}-2} \tag{3}$$

$$N_{-\frac{5}{2},\frac{p}{q},\frac{r}{s}} = N_{-\frac{5}{2},-\frac{q}{p+2q}-1,-\frac{s}{r+s}-2} \tag{4}$$

$$N_{-\frac{1}{2},\frac{p}{q},\frac{r}{s}} = N_{-\frac{1}{2},-\frac{p}{q}-4,-\frac{r}{s}-4} \tag{5}$$

$$N_{1,2,\frac{p}{q}} = N_{1,2,-\frac{p}{q}+2} \tag{6}$$

$$N_{1,-4,\frac{p}{q}} = N_{1,-4,\frac{q}{p}} \tag{7}$$

for all p/q and r/s in $\mathbb{Q} \cup \infty$.

Slopes	Volume	Num	Homology
$\{-4,-3/2,1\}$	0.942707362777	1	$\mathbb{Z}_5+\mathbb{Z}_5$
$\{-4,1,2\}$	0.981368828892	2	$\mathbb{Z}_5$
$\{-5,-1/2,1\}$	1.01494160641	3	$\mathbb{Z}_3+\mathbb{Z}_6$
$\{-3/2,-3/2,1\}$	1.26370923866	4	$\mathbb{Z}_5+\mathbb{Z}_5$

Table 4: The closed orientable hyperbolic manifolds of complexity up to 9.

Hyperbolic manifolds up to complexity 10 Combining Theorems 1.13 and 1.16 it is now quite easy to give the complete list of all closed hyperbolic manifolds of complexity up to 9, and a partial list of those of complexity 10. The first list is obtained by enumerating the triples $\{p/q, r/s, t/u\}$ such that $h(p/q, r/s, t/u) \leqslant 9$, discarding those listed in Theorem 1.13, and taking into account relations (2) to (7) to eliminate duplicates. This process leads to the following four triples:

$$\{-4,-3/2,1\}, \qquad \{-4,1,2\}, \qquad \{-5,-1/2,1\}, \qquad \{-3/2,-3/2,1\},$$

which indeed give distinct manifolds, all having complexity 9. More importantly, these manifolds are precisely the four smallest known closed orientable hyperbolic manifolds, listed in Table 4. (Both in this table and in the next one, Num is the position in the list of known closed orientable hyperbolic manifolds, ordered by increasing volume: the first 10 are in [10], a longer list is distributed with SnapPea).

The partial list of closed hyperbolic manifolds having complexity 10 is obtained by the same method, except that we must consider the triples $\{p/q, r/s, t/u\}$ such that $h(p/q, r/s, t/u) = 10$, and discard those giving one of the manifolds of Table 4. The result is described in Table 5.

Remark 1.18.

- In Tables 4 and 5 we have listed only one triple of slopes for each manifold, but the number of different triples $\{\alpha, \beta, \gamma\}$ giving a certain M so that $c(M) = h(\alpha, \beta, \gamma)$ is actually very small, and appears to decrease as the volume of M increases. More precisely, two alternative realizations exist for the smallest manifold, with $\{-3/2,-3/2,-3/2\}$ and with $\{-4,1,-2/3\}$, and one for the second, ninth, thirteenth, and fifteenth smallest manifolds, respectively with $\{-4,1/2,1\}$, $\{-5/3,-3/2,-3/2\}$, $\{-3/2,-3/2,-1/2\}$, and $\{-3/2,-3/2,-4/3\}$. All other manifolds have a unique realization such that h equals the complexity (but other realizations exist with larger values of h). We also mention that distinct realizations such that h equals the complexity correspond to distinct sharp decompositions into bricks (see Theorem 2.4);
- The fact that all the triples in the tables contain slope 1 implies that all the manifolds listed are actually obtained by Dehn surgery on the Whitehead link.

Slopes	Volume	Num	Homology
$\{-5, 1, 2\}$	1.28448530047	5	$\mathbb{Z}_6$
$\{-5, 1/2, 1\}$	1.41406104417	7	$\mathbb{Z}_6$
$\{-4, 1, 3\}$	1.41406104417	8	$\mathbb{Z}_{10}$
$\{-4, -4/3, 1\}$	1.42361190029	9	$\mathbb{Z}_{35}$
$\{-4, -5/2, 1\}$	1.54356891147	13	$\mathbb{Z}_{35}$
$\{-4, -5/3, 1\}$	1.58316666062	15	$\mathbb{Z}_{40}$
$\{-5, -3/2, 1\}$	1.5886466393	17	$\mathbb{Z}_{30}$
$\{-3/2, -1/2, 1\}$	1.64960971581	19	$\mathbb{Z}_{15}$
$\{-5, -1/3, 1\}$	1.83193118835	23	$\mathbb{Z}_2 + \mathbb{Z}_{12}$
$\{-5/3, -3/2, 1\}$	1.88541472555	31	$\mathbb{Z}_{40}$
$\{-5, -2/3, 1\}$	1.91084379309	34	$\mathbb{Z}_{30}$
$\{-3/2, -4/3, 1\}$	1.95370831542	39	$\mathbb{Z}_{35}$

Table 5: A list of closed orientable hyperbolic manifolds of complexity up to 10.

2 Decomposition of manifolds into bricks

In this section we provide the details needed below of the theory developed in [11] (see also [12] for the non-orientable version). We will not repeat what already stated in the introduction, but we will need to rephrase the definition of *brick* in a more precise way. In the whole paper our manifolds will be 3-dimensional, compact, orientable, and connected by default, and they will be viewed up to homeomorphism. We will mostly use only the term *pair* to refer to a manifold with triods (M, X).

Skeleta and complexity A compact polyhedron P is called *simple* if the link of every point of P embeds in the space given by a circle with three radii. The set of points having the whole of this space as a link is denoted by $V(P)$, and its members are called *vertices*. They are isolated, so $\#V(P) < \infty$.

Given a pair (M, X), a polyhedron P embedded in M is called a *skeleton* of (M, X) if $P \cup \partial M$ is simple, $M \setminus (P \cup \partial M)$ is an open 3-ball, and $P \cap \partial M = X$. We define now $c(M, X)$ as the minimum of $\#V(P)$ over all skeleta P of (M, X), and we call it the *complexity* of (M, X).

A skeleton of (M, X) is called *minimal* if it has $c(M, X)$ vertices and cannot be collapsed onto a smaller skeleton of (M, X). We will see below that, when M is irreducible, a minimal skeleton P of (M, X) must have rather special properties, that we now introduce. We say P is *quasi-standard* if every point has a neighborhood of one of the types (1)-(5) shown in Fig. 2. Note that $V(P)$ is the set of points of type

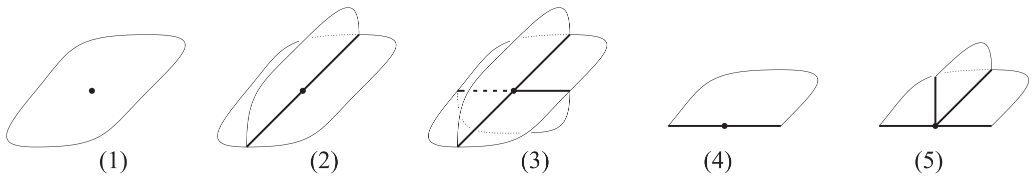

Figure 2: Neighborhoods of points in a quasi-standard polyhedron.

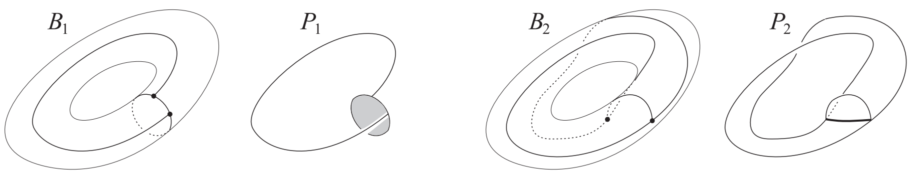

Figure 3: The pairs B_1 and B_2 and skeleta for them.

(3). We further define $S(P)$ as the set of points of type (2), (3), or (5), ∂P as the set of points of type (4) or (5), and we say that P is *standard* if the components of $P \setminus (S(P) \cup \partial P)$ are open discs and the components of $S(P) \setminus V(P)$ are open segments. We also say that P is *properly embedded* in (M, X) if $\partial P = X$.

Our definition of c implies subadditivity under assembling quite clearly: Suppose for $i = 1, 2$ that P_i is a skeleton of a pair (M_i, X_i), and we glue $(T_1, \theta_1) \subset \partial(M_1, X_1)$ to $(T_2, \theta_2) \subset \partial(M_2, X_2)$ along a map ψ, getting a pair $(M_1, X_1) \oplus (M_2, X_2)$; Then we get a skeleton of the new pair by gluing P_1 to P_2 along the restriction of ψ.

Examples and standard skeleta When M is closed, $(M, \emptyset)$ is a manifold with triods, and a skeleton of $(M, \emptyset)$ is just a spine of M minus a point. So $c(M, \emptyset)$ is Matveev's $c(M)$, see [13]. We describe now more examples to be used below.

If T is the torus, it is not hard to see that up to *homeomorphism* there exists only one embedding in T of the triod θ such that $T \setminus \theta$ is a disc (but there are infinitely many embeddings up to *isotopy*, see Section 3). So we can define a pair $B_0 = (T \times [0, 1], \{\theta \times \{0\}, \theta \times \{1\}\})$, and $c(B_0) = 0$, because B_0 has the skeleton $\theta \times [0, 1]$.

Now let $\boldsymbol{T}$ be the solid torus. We define two pairs B_1 and B_2 based on $\boldsymbol{T}$ as shown in Fig. 3. Again they have zero complexity, because they have the skeleta P_1 and P_2 also shown in the same figure. Note that P_1 has a 1-dimensional portion,

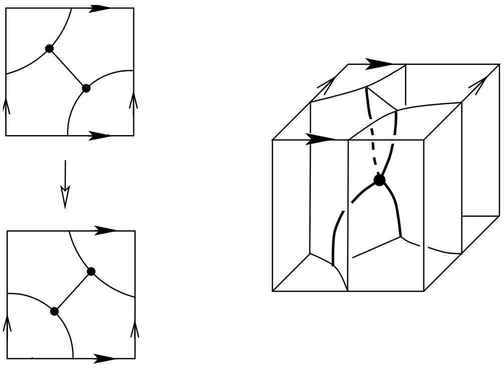

Figure 4: A flip on a triod in the torus, and the skeleton of B_3.

so it is not standard, while P_2 is a Möbius strip with a half-disc attached, and it is standard.

Before proceeding with the examples we note that the operation of connected sum $\#$ is obviously well-defined for manifolds with triods, so it is natural to call (M, X) irreducible if M is. As already mentioned, we have proved in [11] that c is additive under $\#$. The argument requires, among other things, the next result (also taken from [11] and needed below):

Theorem 2.1. *If M is irreducible and P is a minimal skeleton of (M, X) then P is standard and properly embedded, unless (M, X) is B_1, S^3, $L_{3,1}$, or $\mathbb{RP}^3$.*

Our next example, denoted by B_3, is based on $T \times [0, 1]$ as B_0, but in this case we choose $\theta_0 \subset T \times \{0\}$ and $\theta_1 \subset T \times \{1\}$ to be related by a *flip*, as shown in Fig. 4-left. Even if three different flips can be applied to a given θ_0, the symmetries of θ_0 in T easily imply that $B_3 = (T \times [0, 1], \{\theta_0, \theta_1\})$ is indeed well-defined. Moreover it has the 1-vertex skeleton shown in Fig. 4-right.

Our last example B_4 is based on $D_2 \times S^1$, where D_2 denotes a disc with two holes (this notation will be used again below). To describe the triods on ∂B_4, we note that on each of the three tori we have a natural figure-eight graph, obtained by choosing basepoints in the factors of the product $(\partial D_2) \times S^1$. There are two possible resolutions of the tetravalent vertex of the graph, yielding triods related by a flip. We then define the triods on ∂B_4 by performing one type of resolution on two components, and the other type on the third component, as suggested in Fig. 5-left. On the right in the same figure we show a 3-vertex skeleton of B_4.

Sharp assembling The single most important result of [11], on which the rest of the theory is based, is that irreducibility is preserved under sharp assembling and

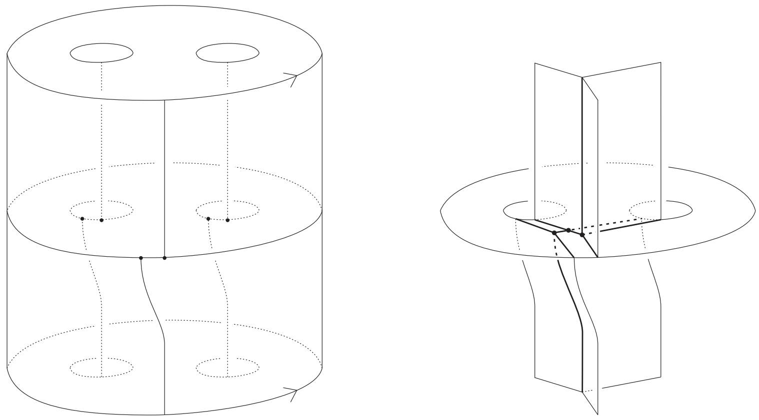

Figure 5: The pair B_4 and a skeleton for it.

deassembling. To achieve this property, however, we need to give a slightly more accurate notion of sharpness than anticipated in the introduction, by ruling out certain *trivial* assemblings.

We start by noting that the way to assemble two given (M_1, X_1) and (M_2, X_2) along $(T_1, \theta_1) \subset \partial(M_1, X_1)$ and $(T_2, \theta_2) \subset \partial(M_2, X_2)$ is not unique, because a gluing map $\psi : T_1 \to T_2$ has to be chosen. However, the restriction $\psi(\theta_1) = \theta_2$ easily implies that only a few different results can be obtained up to equivalence. So when we write $(M_1, X_1) \oplus (M_2, X_2)$ we actually mean one of the finitely many possible assemblings, as customary with connected sum. Now one easily sees that $(M, X) \oplus B_0 = (M, X)$ whatever (M, X), so we call *trivial* any assembling involving B_0.

The other type of assembling that we need to consider *trivial* arises in a more subtle way. The reader can show quite easily that there exist assemblings $B_1 \oplus B_1$ and $B_1 \oplus B_2$ giving S^3 as a result (but, for instance, $B_2 \oplus B_2$ is never S^3). Now we call trivial an assembling $(M_1, X_1) \oplus (M_2, X_2)$ if, up to permutation, we have that:

- $(M_1, X_1) = B_i$ for $i \in \{1, 2\}$;
- (M_2, X_2) splits as $(N, Y)\#B_j$ with $j \in \{1, 2\}$;
- B_i is assembled to B_j so to give S^3.

Note that $(M_1, X_1) \oplus (M_2, X_2)$ would be (N, Y) in this case, so the operation of assembling with B_i would be an "inverse" of the operation of connected sum with B_j. This explains why we are dismissing this type of assembling. We now call *sharp* a non-trivial assembling on which complexity is additive, and we quote the following from [11]:

Theorem 2.2. *Let $(M,X) = (M_1,X_1) \oplus (M_2,X_2)$ be a sharp assembling with $M \neq S^2 \times S^1$. Then (M,X) is irreducible if and only if both (M_1,X_1) and (M_2,X_2) are.*

Self-assembling Besides $\oplus$, our theory uses another operation $\odot$, called self-assembling, that is again multi-valued, but takes one argument only. Given (M,X) and distinct components (T_i,θ_i) of $\partial(M,X)$ for $i=1,2$, we define $\odot(M,X)$ as a pair obtained from (M,X) by dropping θ_1 and θ_2 and gluing T_1 to T_2 along an orientation-reversing homeomorphism $\psi : T_1 \to T_2$ such that $\psi(\theta_1)$ intersects θ_2 transversely in two points. The next fact was not explicitly stated in [11], so we single it out here.

Remark 2.3. Let θ be a triod in the torus T so that $T \setminus \theta$ is a disc. Then there are four isotopy classes of triods θ' such that θ' intersects θ transversely in two points and $T \setminus \theta'$ is again a disc, namely the class of θ itself and those of the three triods obtained from θ by a flip.

If P is a skeleton of (M,X), and a self-assembling $\odot(M,X)$ identifies two tori to a single one T, then $T \cup P$ is a skeleton of $\odot(M,X)$, which easily implies that $c(\odot(M,X)) \leqslant c(M,X)+6$. The self-assembling is called *sharp* if equality holds. Since T is a non-separating torus in $\odot(M,X)$, we deduce (even without assuming sharpness) that $\odot(M,X)$ is irreducible if and only if (M,X) is.

Bricks and decomposition We define a pair to be a *brick* if it is irreducible and cannot be expressed as a sharp assembling or self-assembling. We now extend our definition and call (sharp) assembling a finite combination of (sharp) assemblings and self-assemblings (including the empty combination). Using Theorem 2.2 and certain finiteness properties not stated here, we have proved in [11] the following:

Theorem 2.4. *Every irreducible manifold with triods is a sharp assembling of bricks.*

The practical relevance of this result depends on the fact that the set $\mathcal{B}_{\leqslant n}$ of all bricks having complexity up to n is much smaller and easier to determine than the set of all manifolds up to the same complexity. We will soon describe $\mathcal{B}_{\leqslant 9}$, but we first note that closed bricks play a special (and marginal) rôle in the theory, because they cannot be assembled at all. So we split $\mathcal{B}_{\leqslant n}$ as $\mathcal{B}^0_{\leqslant n} \sqcup \mathcal{B}^1_{\leqslant n}$, with $\mathcal{B}^0_{\leqslant n}$ consisting of closed bricks.

Approximated complexity We can now formally introduce the functions c_n to the study of which the present paper is mainly devoted. Let (M,X) be a manifold with triods. If M is prime we define $c_n(M,X)$ as the minimum of $\sum_{i=1}^h c(B^{(i)})+6m$ over all realizations of (M,X) as an assembling of bricks $B^{(1)},\ldots,B^{(h)} \in \mathcal{B}_{\leqslant n}$

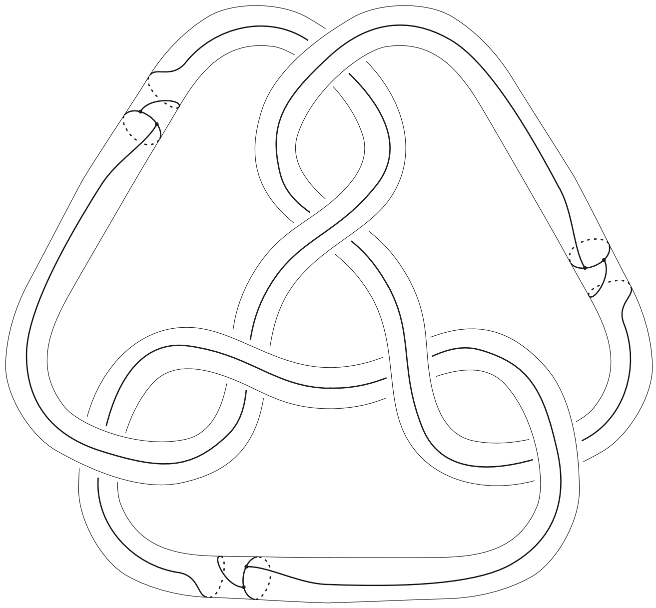

Figure 6: The pair B_9.

followed by m self-assemblings. As usual, the minimum over an empty list is taken to be $+\infty$. If M is not prime we define $c_n(M,X)$ as the sum of c_n over the prime connected summands of (M,X).

The next result is a direct consequence of Theorem 2.4 and additivity of c under connected sum [11]:

Proposition 2.5. *For any pair (M,X), the sequence $\{c_n(M,X)\}_{n=0}^{\infty}$ is monotonic non-increasing. Moreover $c_n(M,X)=c(M,X)$ for $n\geqslant c(M,X)$.*

Small bricks with boundary We have shown in [11] that $\mathcal{B}^1_{\leqslant 9}$ consists of precisely 11 bricks, namely the pairs $B_0,\ldots,B_4$ introduced above, and certain other $B_5,\ldots,B_{10}$. A complete description of these last bricks requires some preliminary work on triods, so we defer it to Sections 4 and 5, but we mention at least the manifolds they are based on. The brick B_5 is the last Seifert one in our list, being based on $(D,(2,1),(3,1))$. The other bricks $B_6,\ldots,B_{10}$ are based on cusped hyperbolic manifolds. Using as above the labels of [7], we have that B_6, B_7, and B_8 are based on $M2^1_2$, $M3^1_4$, and $M4^2_1$ respectively. Both B_9 and B_{10} are based on $M6^3_1$, with different triods on the boundary, and B_9 is shown in Fig. 6. Note that $M6^3_1$ is the complement of the three-component chain-link, already considered by Thurston [18], and that $M2^1_2, M3^1_4$, and $M4^2_1$ can be obtained from $M6^3_1$ via appropriate Dehn fillings.

Concerning complexity, we already know that $c(B_i) = 0$ for $i = 0, 1, 2$. Moreover it is shown in [11] that $c(B_3) = 1$ and $c(B_4) = 3$, so the skeleta shown above are minimal ones. In addition, $c(B_i) = 8$ for $i = 5, 6$ and $c(B_i) = 9$ for $i = 7, \ldots, 10$. The next result is used below:

Theorem 2.6. *Let B be a brick with $c(B) \geqslant 4$. Assume B has k boundary components and P is a minimal skeleton of B. Then the two edges of P leaving the i-th boundary component are distinct, and they terminate at vertices v^i_1 and v^i_2 of P respectively. Moreover, the $2k$ vertices $\{v^i_j\}$ are all distinct from each other.*

This theoretical result was crucial for the computer program by means of which we have checked [11] that $\mathcal{B}^1_{\leqslant 9}$ consists precisely of $B_0, \ldots, B_{10}$.

Small closed bricks Given integers $i > 0$, $j > 0$, $k \geqslant 0$ we employ the normalized parameters for Seifert manifolds (Proposition 1.1) to define

$$\begin{aligned} E_k &= (S^2, (2,1), (3,1), (5+k,1), -1), \\ C_{i,j} &= (S^2, (2,1), (1+i,1), (1+j,1), -1). \end{aligned}$$

In [11] we have described a systematic way to construct skeleta of E_k and $C_{i,j}$, with $5+k$ and $i+j$ vertices respectively. The following topological result is easy:

Proposition 2.7. *The set $\mathcal{M}^*$ of manifolds listed in equation (1) coincides with the following one:*

$$\begin{aligned} &\{E_k : \ k \geqslant 0, \ k \neq 1\} \\ &\quad \bigcup \{C_{i,j} : \ 1 \leqslant i \leqslant j, \ (i,j) \neq (3,3), \ \textit{and } j \in \{2,3\} \textit{ if } i = 2\}. \end{aligned} \tag{8}$$

Both lists (1) and (8) contain no repetitions.

For $M \in \mathcal{M}^*$ define now $c^*(M)$ to be $5+k$ or $i+j$ depending on the type of M. In [11] we have shown that $\mathcal{B}^0_{\leqslant 9}$ consists precisely of the 19 elements of $\mathcal{M}^*$ such that $c^*(M) \leqslant 9$.

Results on $\mathcal{M}^*$ and closed bricks By definition if (M, X) is a brick then $c_n(M, X)$ is strictly bigger than $c(M, X)$ for all $n < c(M, X)$. In particular, the upper estimate $c_9(M)$ will never give $c(M)$ on a closed brick $(M, \emptyset)$, except on the first 19 ones (those such that $c(M) \leqslant 9$). A complete understanding of all closed bricks would therefore be very desirable, and we will now state some conjectures we have, starting from the result which motivates them. (Notation is as above, points 1 and 2 were proved in [11], and point 3 is proved below.)

Theorem 2.8. *1. $c(M) \leqslant c^*(M)$ for all $M \in \mathcal{M}^*$;*

2. If $c^*(M) \leqslant 9$ then $c(M) = c^*(M)$;

3. if $c^*(M) > 9$ then $c_9(M) = c^*(M) + 1$.

This result suggests that all the elements of $\mathcal{M}^*$ are bricks, and that $c^*(M)$ is precisely the complexity of M for $M \in \mathcal{M}^*$, and indeed we believe this is true. With a slightly smaller degree of confidence, we also believe that the only closed Seifert bricks are those in $\mathcal{M}^*$. However, we do not risk a prediction on whether geometrically toroidal or closed hyperbolic bricks exist in complexity greater than 9. Our experience with high complexity is too limited to even make a conjecture. We are planning to investigate manifolds and bricks of complexity 10 in the close future, and we hope that the experimental data will provide some insight to these matters.

3 Slopes and triods on the torus

Our computation of c_n will be based, among other things, on a convenient and unified geometric encoding for simple closed curves and triods on the torus. Our encoding uses the so-called Farey tessellation and its dual graph, and it was partially inspired by [1]. This approach leads to a very direct proof of the fact, previously established in [2], that any two triods on the torus can be transformed into each other by a unique minimal sequence of flips. Quite surprisingly, it also leads to geometric proofs of Propositions 1.4 and 1.5 (algebraic proofs are also possible but quite awkward, so we have opted for the geometric approach).

Slopes as points at infinity in hyperbolic plane Let us fix for the rest of the section a torus T, and let us denote by $\mathcal{S}(T)$ the set of *slopes* on T, *i.e.* the isotopy classes of non-contractible simple closed curves (without orientation) on T. It is well-known that, given a basis $\mathfrak{b} = (a, b)$ of $H_1(T)$, we have a bijection $\Phi_{\mathfrak{b}} : \mathcal{S}(T) \to \mathbb{Q} \cup \{\infty\}$, where $\Phi_{\mathfrak{b}}(\gamma) = p/q$ if the homology class of γ is $\pm(p \cdot a + q \cdot b)$. We note now that $\mathbb{Q} \cup \{\infty\}$ is a subset of the boundary $\mathbb{R} \cup \{\infty\}$ of the upper half-plane model of hyperbolic plane $\mathbb{H}^2$, and for this reason we will denote it by $\partial_{\mathbb{Q}}\mathbb{H}^2$. In the sequel it will be convenient to freely switch from the half-plane to the Poincaré disc model of $\mathbb{H}^2$ without changing notation. We do this by fixing once and forever an isometry from the half-plane to the disc, namely the orientation-preserving isometry that maps ∞ to i and $\{\infty, 0, 1\}$ to the vertices of a (Euclidean) equilateral triangle.

It is now appropriate to recall that $\mathrm{GL}_2(\mathbb{Z})$ acts on $\mathbb{H}^2$, using the half-plane model, by fractional linear or anti-linear transformations (depending on the sign of

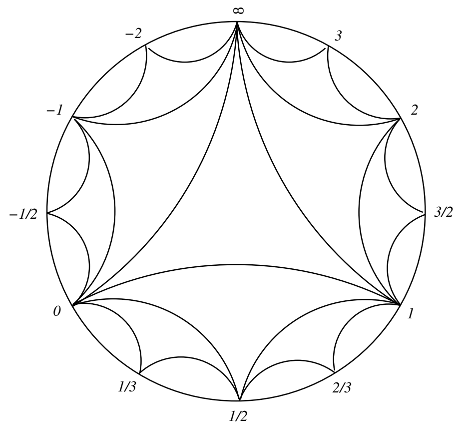

Figure 7: The Farey tessellation: an ideal triangulation of the Poincaré disc.

the determinant). This action extends to $\partial\mathbb{H}^2$ and leaves $\partial_{\mathbb{Q}}\mathbb{H}^2$ invariant. On the other hand, given a basis $\mathfrak{b}$ of $H_1(T)$, we also have an action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathcal{S}(T)$, because every $A \in \mathrm{GL}_2(\mathbb{Z})$ determines up to isotopy precisely one automorphism f of T such that $f_* : H_1(T) \to H_1(T)$ has matrix A with respect to $\mathfrak{b}$. It is now a routine matter to show that these actions of $\mathrm{GL}_2(\mathbb{Z})$ are equivariant with respect to the bijection $\Phi_{\mathfrak{b}} : \mathcal{S}(T) \to \partial_{\mathbb{Q}}\mathbb{H}^2$.

The Farey tessellation We denote now by $\mathcal{L}$ the set of lines in $\mathbb{H}^2$ joining two points $\Phi_{\mathfrak{b}}(\gamma), \Phi_{\mathfrak{b}}(\gamma') \in \partial_{\mathbb{Q}}\mathbb{H}^2$ whenever γ and γ' intersect each other transversely in only one point (up to isotopy). Using the $\mathbb{Q} \cup \{\infty\}$ model of $\partial_{\mathbb{Q}}\mathbb{H}^2$ we see that there is a line from p/q to s/t precisely if $|p \cdot t - s \cdot q| = 1$, so indeed $\mathcal{L}$ is independent of $\mathfrak{b}$. Moreover $\mathcal{L}$ is invariant under the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{H}^2$.

Lemma 3.1. *1. The lines in $\mathcal{L}$ are pairwise disjoint (except at their ends);*

2. For all $\ell \in \mathcal{L}$ there exist precisely two points of $\partial_{\mathbb{Q}}\mathbb{H}^2$ which are joined by lines of $\mathcal{L}$ to both the ends of ℓ;

3. $\mathcal{L}$ is the 1-skeleton of an ideal triangulation of $\mathbb{H}^2$.

The ideal triangulation whose 1-skeleton is $\mathcal{L}$ is usually called the *Farey tessellation* [4, 17], and it is shown in Fig. 7. As customary in the literature our pictures of $\mathcal{L}$ will always be only combinatorially correct, because metrically correct pictures would be harder to understand.

Proof of 3.1. For point 1, let $\ell, \ell' \in \mathcal{L}$. If ℓ and ℓ' have a common end then they do not meet in $\mathbb{H}^2$. So we can assume they have distinct ends. Using the fact that

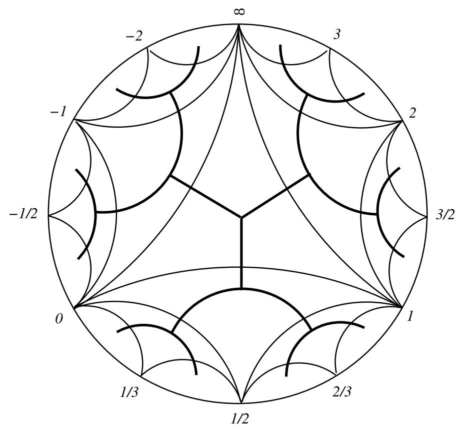

Figure 8: The trivalent tree dual to the Farey triangulation of $\mathbb{H}^2$.

$\mathrm{GL}_2(\mathbb{Z})$ acts in a triply transitive way on $\partial_{\mathbb{Q}}\mathbb{H}^2$, we can now assume in the half-space model that ℓ joins 0 to 1 and ℓ' has one end at ∞. The other end of ℓ' is then an integer, and point 1 follows. Point 2 is particularly easy to show using the bijection between $\partial_{\mathbb{Q}}\mathbb{H}^2$ and $\mathcal{S}(T)$: the ends of ℓ represent two slopes a and b having geometric intersection 1, and we must show that there are precisely two other slopes having intersection 1 with both a and b. If we choose orientations on a and b we get a basis of $H_1(T)$, and the two other slopes we are looking for are $\pm(a+b)$ and $\pm(a-b)$. Point 3 now follows from points 1 and 2, and the density of $\partial_{\mathbb{Q}}\mathbb{H}^2$ in $\partial\mathbb{H}^2$. $\square$

Triods as vertices of the dual tree Let us denote now by $\Theta(T)$ the set of isotopy classes of triods in T having a disc as a complement. It is not hard to see that the elements of $\Theta(T)$ correspond bijectively to the unordered triples of elements of $\mathcal{S}(T)$ having pairwise geometric intersection 1, the three slopes corresponding to a given $\theta \in \Theta(T)$ being those contained in θ (up to isotopy). So a triple $\{\gamma, \gamma', \gamma''\}$ defines a triod if and only if $\Phi_{\mathfrak{b}}(\gamma), \Phi_{\mathfrak{b}}(\gamma'), \Phi_{\mathfrak{b}}(\gamma'')$ are the vertices of a triangle of the Farey triangulation of $\mathbb{H}^2$. Therefore, if we denote by $\mathcal{T}$ the graph dual to the Farey triangulation, we get a bijection $\Psi_{\mathfrak{b}}$ from $\Theta(T)$ to the set $\mathcal{T}^{(0)}$ of vertices of $\mathcal{T}$. Note that $\mathcal{T}$ is a tree embedded in $\mathbb{H}^2$, see Fig. 8.

The correspondence between $\Theta(T)$ and $\mathcal{T}^{(0)}$ has a geometric consequence. Note first that $\mathcal{T}$ is trivalent, *i.e.* there are precisely three triods joined to a given $\theta \in \Theta(T)$ by a single edge of $\mathcal{T}$. Now one sees quite easily that these three triods are precisely those obtained from θ by the elementary transformation called *flip* and described above in Fig. 4-left. So we deduce:

Proposition 3.2. *Given distinct triods $\theta, \theta' \in \Theta(T)$ there exists a unique sequence $\{\theta_i\}_{i=0}^n \subset \Theta(T)$ of pairwise distinct triods such that $\theta_0 = \theta$, $\theta_n = \theta'$, and θ_i is obtained from θ_{i-1} by a flip, for $i = 1, \ldots, n$.*

Distance between slopes and triods Let us endow $\mathcal{T}^{(0)}$ with the distance defined as usual for the vertices of a graph. Namely, given $v, v' \in \mathcal{T}^{(0)}$, their distance $d(v, v')$ is the number of edges of the only simple path in $\mathcal{T}$ that joins v to v' (this path is unique because $\mathcal{T}$ is a tree). Using $\Psi_{\mathfrak{b}}$ we deduce a distance on $\Theta(T)$, again denoted by d. This distance is independent of $\mathfrak{b}$, because $d(\theta, \theta')$ is intrinsically interpreted as the number of flips required to transform θ into θ'.

We will need in the sequel a notion of "distance" between a slope $\gamma \in \mathcal{S}(T)$ and a triod $\theta \in \Theta(T)$. We define $d(\gamma, \theta)$ to be $n - 1$, where n is the number of lines in $\mathcal{L}$ intersected by the half-line in $\mathbb{H}^2$ that joins $\Psi_{\mathfrak{b}}(\theta)$ to $\Phi_{\mathfrak{b}}(\gamma)$. Independence of $\mathfrak{b}$ is easy and left to the reader. Of course our function d is not quite a distance, because $d(\gamma, \theta) = -1$ if γ is contained in θ up to isotopy.

Action of $\mathrm{GL}_2(\mathbb{Z})$ on triods As above for slopes, we have now two different ways to define an action of $\mathrm{GL}_2(\mathbb{Z})$ on triods. On one hand, we can use the basis $\mathfrak{b}$ to associate an automorphism of T to a given matrix, and then we can let this automorphism act on $\Theta(T)$. On the other hand, the matrix acts by fractional linear or anti-linear transformation on $\mathbb{H}^2$ leaving $\mathcal{T}^{(0)}$ invariant, so we have an action defined through the bijection $\Psi_{\mathfrak{b}} : \Theta(T) \to \mathcal{T}^{(0)}$. The reader is invited to check that these actions of $\mathrm{GL}_2(\mathbb{Z})$ actually coincide.

Our next results concern a relation between the distance function on the Farey tessellation and the complexity $|p, q|$ for a coprime pair (p, q) defined in Section 1. To state the relation we introduce a notation repeatedly used below, namely for $i \in \mathbb{Z}$ we denote by $\theta^{(i)} \in \mathcal{T}^{(0)}$ the centre of the triangle with vertices at i, $i+1$, and ∞. Note that $\theta^{(0)}$ is the centre of the disc model of $\mathbb{H}^2$.

Proposition 3.3. *Let (p, q) be a pair of coprime integers. Consider the point corresponding to p/q in $\partial_{\mathbb{Q}}\mathbb{H}^2$. Then $|p, q|$ equals the number of lines in $\mathcal{L}$ met by the half-line that starts at $\theta^{(0)}$ and ends at p/q. Any other path from $\theta^{(0)}$ to p/q meets at least $|p, q|$ lines of $\mathcal{L}$.*

Proof. For $\alpha, \alpha' \in \mathcal{T}^{(0)} \cup \partial_{\mathbb{Q}}\mathbb{H}^2$ let $\ell(\alpha, \alpha')$ be the geodesic segment, line, or half-line with ends at α and α', and let $n(\alpha, \alpha')$ be the number of intersections of $\ell(\alpha, \alpha')$ with the lines of $\mathcal{L}$.

We must show that $|p, q| = n(p/q, \theta^{(0)})$. We proceed by induction on $|p, q|$ and note that the equality is obvious when $|p, q| = 0$, namely for $p/q \in \{0, 1, \infty\}$. Let us then consider the case $|p, q| > 0$. Suppose first that $p > q > 0$. Let f be the automorphism of $\mathbb{H}^2$ associated to $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $f(\mathcal{L}) = \mathcal{L}$, $f(p/q) = (p-q)/q$, and $f(\theta^{(0)}) = \theta^{(-1)}$, so $n(p/q, \theta^{(0)}) = n((p-q)/q, \theta^{(-1)})$. Now we note that $\ell(\theta^{(0)}, \theta^{(-1)})$ only meets $\ell(0,\infty)$, and $\ell(0,\infty)$ cuts the closed disc in two parts, leaving $(p-q)/q$ and $\theta^{(0)}$ on one side and $\theta^{(-1)}$ on the other side. So

$$n((p-q)/q, \theta^{(-1)}) = n((p-q)/q, \theta^{(0)}) + 1.$$

Since $|p-q,q| = |p,q| - 1 < |p,q|$, the induction assumption now implies that $n((p-q)/q, \theta^{(0)}) = |p,q| - 1$, whence the conclusion.

If $q > p > 0$ we proceed similarly, using the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. If $p > 0 > q$ we use the automorphism $\left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$, which fixes $\theta^{(0)}$ and maps p/q to $(p-q)/(-q)$. So $n(p/q, \theta^{(0)}) = n((p-q)/(-q), \theta^{(0)})$, and $n((p-q)/(-q), \theta^{(0)}) = |p-q, -q|$ by the cases already discussed. By definition of $|\cdot,\cdot|$ we now have $|p-q,-q| = |p,-q|+1 = |p,q|$, whence the conclusion also in this case. For $p < 0$ we simply note that $p/q = (-p)/(-q)$ and $|p,q| = |-p,-q|$ by definition. The first assertion of the statement is now proved, and the second one is easy, because if a line ℓ of $\mathcal{L}$ meets $\ell(p/q, \theta^{(0)})$ then it leaves $\theta^{(0)}$ and p/q on opposite sides, so any path joining $\theta^{(0)}$ to p/q must meet ℓ. $\square$

Corollary 3.4. *If p and q are coprime then $d(p/q, \theta^{(i)}) = |p - iq, q| - 1$.*

Proof. The case $i = 0$ follows from Proposition 3.3. The other cases follow using the automorphism $\left(\begin{smallmatrix} 1 & -i \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathbb{H}^2$ that sends $\theta^{(i)}$ to $\theta^{(0)}$ and p/q to $(p-iq)/q$. $\square$

With the geometric tools we have developed we can now establish the two technical results stated without proof in Section 1.

Proof of 1.4. If $p, q \geqslant 1$ then $|p,q|$ is the number of operations required to reach (p,q) from $(1,1)$ by either adding the left coordinate to the right one, or the right coordinate to the left one. Since in $(1,1)$ left and right are the same, point 1 follows for $p,q \geqslant 1$. The same assertion for $p \leqslant 0$ or $q \leqslant 0$ is now easy and left to the reader.

Using point 1 we can now very easily establish point 2: $|p,q| = |p-q,q| + 1 = |q, p-q| + 1 = |p, p-q|$.

Let us prove point 3. By point 2 we can suppose $qq' \equiv 1 \pmod p$. By Proposition 3.3 and point 1, to prove that $|p,q| = |p,q'|$ it is now sufficient to show that

$$d(\theta^{(0)}, p/q) = d(\theta^{(0)}, q'/p). \tag{9}$$

We also note that the desired equality is obvious if $q = 1$, so we proceed assuming p/q is not an integer.

For $r \in \partial_{\mathbb{Q}}\mathbb{H}^2$ we denote now by θ_r the triod closest to $\theta^{(0)}$ among those that contain r. So $d(r, \theta^{(0)}) = d(\theta_r, \theta^{(0)}) - 1$. Moreover, if r is neither an integer nor ∞, one sees that θ_r can be characterized as the only triod which contains r and two other slopes r' and r'' such that $r' < r < r''$ with respect to the linear order of $\mathbb{Q}$.

Now let $s > 0$ be such that $qq' = sp + 1$ and consider the following triples of slopes:

$$\left(\frac{p-q'}{q-s}, \frac{p}{q}, \frac{q'}{s}\right), \qquad \left(\frac{s}{q}, \frac{q'}{p}, \frac{q'-s}{p-q}\right).$$

Using the relation $qq' = sp+1$ one readily sees that in both triples the slopes intersect pairwise in a single point, so the triples define triods. The same relation and its immediate consequence that $q > s$ also easily imply that in both triples the slopes are arranged in increasing order. This proves that the triples define $\theta_{p/q}$ and $\theta_{q'/p}$ respectively. Consider now the automorphism f corresponding to $\left(\begin{smallmatrix} p & -q' \\ q & -s \end{smallmatrix}\right)$. Then $f(\theta^{(0)}) = \theta_{p/q}$ and $f(\theta_{q'/p}) = \theta^{(0)}$. Since f acts isometrically on $\mathcal{T}^{(0)}$, relation (9) follows immediately, and the proof is complete. $\square$

Proof of 1.5. Our argument is based on the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathcal{T}^{(0)}$, that we analyze having in mind the correspondence between $\mathcal{T}^{(0)}$ and the set of triods on a torus with fixed homology basis. We begin by providing an alternative description of $\mathcal{T}^{(0)}$ that allows to understand the action of $\mathrm{GL}_2(\mathbb{Z})$ better. We do this by noting that a triod can be viewed as a triple of slopes intersecting pairwise once. So $\mathcal{T}^{(0)}$ can be identified to the quotient of the set of 2×3 integer matrices such that all three 2×2 submatrices are invertible over the integers, where two matrices are identified if they are obtained from each other by reordering and sign-switching of columns. With this description of $\mathcal{T}^{(0)}$, a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ acts on $[B] \in \mathcal{T}^{(0)}$ giving just $[A \cdot B]$. In addition, $[B]$ and $[B']$ are adjacent in $\mathcal{T}$ (*i.e.* they are joined by an edge) if and only if they share two columns (up to sign). We prove now a series of claims which eventually will lead to the conclusion of the proof.

Claim 1: The matrices S_1, S_2, S_3 generate a subgroup of $\mathrm{GL}_2(\mathbb{Z})$ *isomorphic to $\mathfrak{S}_3$, in which they play the rôle of the generating transpositions.* The relations $S_i^2 = I$ for $i = 1, 2, 3$ and $S_{i_1} S_{i_2} S_{i_1} = S_{i_3}$ for $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ are proved by direct inspection and imply the conclusion.

Claim 2: The stabilizer of $\theta^{(0)}$ in $\mathrm{GL}_2(\mathbb{Z})$ *is generated by $-I, S_1, S_2, S_3$ and isomorphic to $\left(\mathbb{Z}/_2\right) \times \mathfrak{S}_3$.* This is established very easily noting that

$$\theta^{(0)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Claim 3: If $A \in$ $\mathrm{GL}_2(\mathbb{Z})$ *and $A(\theta^{(0)}) = \theta^{(0)}$ then A can be written in a unique way as $A = \varepsilon \cdot S_i \cdot S_1^m$ with $\varepsilon \in \{\pm 1\}$, $m \in \{0, 1\}$, and $i \in \{1, 2, 3\}$.* This is a direct consequence of the previous claims.

Claim 4: Given $A \in \mathrm{GL}_2(\mathbb{Z})$, the three triods in $\mathcal{T}^{(0)}$ adjacent to $A(\theta^{(0)})$ are the following ones:

$$(A \cdot S_1 \cdot J)(\theta^{(0)}), \qquad (A \cdot S_2 \cdot J)(\theta^{(0)}), \qquad (A \cdot S_3 \cdot J)(\theta^{(0)}) = (A \cdot J)(\theta^{(0)}).$$

The action of A on $\mathcal{T}^{(0)}$, being the restriction of one on $\mathcal{T}$, preserves adjacency of triods. So it is sufficient to check the claim with $A = I$. A straight-forward computation shows that

$$(S_1 \cdot J)(\theta^{(0)}) = \left[\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right], \qquad (S_2 \cdot J)(\theta^{(0)}) = \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right],$$

$$(S_3 \cdot J)(\theta^{(0)}) = J(\theta^{(0)}) = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

whence the conclusion.

Claim 5: Given $n \geqslant 1$, $i_0 \in \{1,2,3\}$, and $i_1, \ldots, i_{n-1} \in \{1,2\}$, the sequence of triods

$$\begin{array}{rcl} \theta_0 & = & \theta^{(0)} \\ \theta_1 & = & (S_{i_0} \cdot J)(\theta^{(0)}) \\ \theta_2 & = & ((S_{i_0} \cdot J) \cdot (S_{i_1} \cdot J))(\theta^{(0)}) \\ & \cdots & \\ \theta_n & = & ((S_{i_0} \cdot J) \cdot (S_{i_1} \cdot J) \cdots (S_{i_{n-1}} \cdot J))(\theta^{(0)}) \end{array}$$

is the sequence of vertices of a simple simplicial path in $\mathcal{T}$. Using Claim 4 with $A = I$, this is obvious for $n = 1$. Using Claim 4 again we only have to show for $n > 1$ that $\theta_n \neq \theta_{n-2}$. Define A so that $\theta_{n-2} = A(\theta^{(0)})$, whence $\theta_{n-1} = (A \cdot (S_{i_{n-2}} \cdot J))(\theta^{(0)})$ and $\theta_n = (A \cdot (S_{i_{n-2}} \cdot J) \cdot (S_{i_{n-1}} \cdot J))(\theta^{(0)})$. We know that there will exist precisely one $i \in \{1,2,3\}$ such that $(A \cdot (S_{i_{n-2}} \cdot J) \cdot (S_i \cdot J))(\theta^{(0)}) = \theta_{n-2}$. If we show that this is the case for $i = 3$, knowing that $i_{n-1} \in \{1,2\}$, we get the desired conclusion. And, indeed:

$$\begin{array}{rcl} (A \cdot (S_{i_{n-2}} \cdot J) \cdot (S_3 \cdot J))(\theta^{(0)}) & = & (A \cdot (S_{i_{n-2}} \cdot J) \cdot J)(\theta^{(0)}) \\ & = & (A \cdot S_{i_{n-2}})(\theta^{(0)}) = A(\theta^{(0)}) = \theta_{n-2}. \end{array}$$

Claim 6: Given a length-n simple simplicial path in $\mathcal{T}$ starting from $\theta^{(0)}$, the sequence of its vertices arises as in Claim 5 from a unique choice of $i_0 \in \{1,2,3\}$ and $i_1, \ldots, i_{n-1} \in \{1,2\}$. This is a by-product of the previous argument.

Conclusion. Now let $A \in \mathrm{GL}_2(\mathbb{Z})$ and choose $i_0 \in \{1,2,3\}$ and $i_1, \ldots, i_{n-1} \in \{1,2\}$ to describe the only simple path in $\mathcal{T}$ from $\theta^{(0)}$ to $A(\theta^{(0)})$. So $A = S_{i_0} \cdot J \cdot S_{i_1} \cdot$

$J \cdots J \cdot S_{i_{n-1}} \cdot A'$ for some A' in the stabilizer of $\theta^{(0)}$. Claims 3 and 6 now readily imply the conclusion. $\square$

Using the notation of the proof just completed, we note that the same argument combined with the transitivity of the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathcal{T}^{(0)}$ shows the following:

Corollary 3.5. *Given $A \in \mathrm{GL}_2(\mathbb{Z})$, if $|A|$ and $||A||$ are the functions defined in Section 1 and d is the graph distance on $\mathcal{T}^{(0)}$, we have*

$$|A| = d(\theta^{(0)}, A(\theta^{(0)})), \qquad ||A|| = \min_{\theta \in \mathcal{T}^{(0)}} d(\theta, A(\theta)).$$

4 Complexity of atoroidal manifolds: generalities and the Seifert case

The computation of c_n for pairs based on (geometrically) atoroidal manifolds is considerably easier than for general ones. One reason is that we do not have to consider self-assemblings. More reasons will be clear soon.

Effect of small bricks We begin by describing the effect on a given pair of the assembling with one of the smallest bricks $B_0, \ldots, B_3$. The proof is immediate.

Proposition 4.1. *Let a pair (M, X) be given. Let T be a component of ∂M and let $\theta \in X$ be the triod lying in T. For $0 \leqslant i \leqslant 3$, the effect of assembling B_i to (M, X) along T is as follows:*

- *if $i = 0$ then (M, X) remains unaffected;*
- *if $i = 1$, a Dehn filling is performed on M that kills one of the three slopes contained in θ,* i.e. *a slope having distance -1 from θ; all such fillings can be realized by an assembling of B_1 with an appropriate map;*
- *if $i = 2$, a Dehn filling is performed on M that kills one of the three slopes having distance 0 from θ; all such fillings can be realized by an assembling of B_2 with an appropriate map;*
- *if $i = 3$, the triod θ gets replaced by one of the three triods having distance 1 from θ; all such replacements can be realized by an assembling of B_3 with an appropriate map.*

We note here that the result just stated, together with the fact that $\mathcal{M}^*$ contains $\mathcal{B}^0_{\leqslant 9}$ and with the knowledge of $\mathcal{B}^1_{\leqslant 9}$, readily implies Theorem 1.10.

Assemblings with a centre If B is a brick, let us call *assembling with centre* B an assembling of B (the centre) with some copies of B_2 and B_3. We will prove in this paragraph that $c_n(M, X)$ is realized by an assembling with centre whenever M is atoroidal and $n \leqslant 9$. This result will be crucial in the sequel. Its proof relies on several of the facts reviewed in Section 2, and on the following experimental evidence, that we have checked for $n \leqslant 9$: *the value of c_n equals that of c_1 for pairs based on $\boldsymbol{T}$ or on $T \times [0, 1]$.* We formalize this evidence as follows:

Definition 4.2. An integer $n \geqslant 1$ is said to be *well-behaved on tori* if $c_n(M, X)$ equals $c_1(M, X)$ whenever M is $\boldsymbol{T}$ or $T \times [0, 1]$.

Conjecture 4.3. *All $n \geqslant 1$ are well-behaved on tori.*

Remark 4.4. If n is well-behaved on tori, then all integers smaller than n also are.

We will prove below that $n = 9$ is well-behaved on tori. To show the main result of this paragraph we start with the following:

Lemma 4.5. *Let B be a brick of complexity at most n and let a pair (M, X) be the result of an assembling of B with h copies of B_1. Then*

$$c_n(M, X) \leqslant \max\{c(B) - 2h, 0\}.$$

If $B \notin \{B_0, \ldots, B_4\}$, then $c(B) - 2h \geqslant 0$. If $B \in \{B_1, \ldots, B_4\}$, then either $c_n(M, X) \leqslant c(B) - h$ or M is closed and $c(M) = 0$.

Proof. Of course we may well assume that $c(B) = n$. The assumptions imply that $c(M, X) \leqslant n$, so $c_n(M, X) = c(M, X)$. Therefore it is sufficient to prove the inequalities for $c(M, X)$ instead of $c_n(M, X)$.

Let us assume first that B is distinct from $B_0, \ldots, B_4$. Then Theorem 2.6 implies that B has at most $c(B)/2$ boundary components. Since at most $c(B)/2$ copies of B_1 can be assembled to B, formula $c(B) - 2h \geqslant 0$ is established, and to prove the first assertion we have to show that $c(M, X) \leqslant c(B) - 2h$. Let P be a minimal skeleton of B. Recall that a minimal skeleton for $B_1 = (\boldsymbol{T}, \{\theta\})$ is given by $D \cup \theta$, where D is a disc with $\partial D \subset \theta$ being a meridian of $\boldsymbol{T}$. A skeleton Q for (M, X) is then obtained as follows: for each component T_i of ∂M to which a B_1 is assembled, a copy of $D \cup \theta$ is glued to P along the corresponding $\theta_i \in X$. One of the three faces of P incident to θ_i then becomes a face of Q having a free edge, so it can be collapsed in Q. Since we are assuming that $B \notin \{B_0, \ldots, B_4\}$, Theorem 2.6 implies that collapsing this face one eliminates two vertices from Q, and that the $2h$ vertices thus eliminated are all distinct, whence the conclusion.

Let us prove now the first assertion for $B \in \{B_0, \ldots, B_4\}$. For $B \in \{B_0, B_1, B_2\}$ the conclusion is obvious because $c(M, X) = 0$. For $B = B_3$ and $B = B_4$ one checks by hand, looking at the singular set of the skeleton of B, that at least $\min\{2h, c(B)\}$ distinct vertices get eliminated when collapsing Q.

We are left to prove the last assertion for $B \in \{B_1, \ldots, B_4\}$. Note that B has at most $c(B) + 1$ boundary components. Assuming $c(M, X) > c(B) - h$ we deduce that $c(M, X) = 0$ by the first assertion, so $h > c(B)$. Then $h = c(B) + 1$ and M is closed. □

Proposition 4.6. *1. Let $n \geqslant 1$ be well-behaved on tori and let (M, X) be an irreducible pair with positive complexity. If M is atoroidal and $c_n(M, X)$ is finite then there is an assembling with centre realizing $c_n(M, X)$.*

2. Assume $n \geqslant 2$ is not well-behaved on tori. Then there exists a pair (M, X) based on either $\boldsymbol{T}$ or $T \times [0, 1]$, and an assembling with centre B realizing $c_n(M, X)$, with $c(B) > 1$.

Proof. For the first assertion, we begin by noting that in any assembling $(M, X) = B^{(1)} \oplus \ldots \oplus B^{(h)}$ realizing $c_n(M, X)$ we can dismiss all the $B^{(i)}$'s equal to B_0. We next show that no sub-assembling of $B^{(1)} \oplus \ldots \oplus B^{(h)}$ can give B_1. Assume by contradiction this is the case. Then we can replace the sub-assembling with B_1 itself, still getting a realization of $c_n(M, X)$, and assume that $B^{(i)} = B_1$. Let $B^{(j)}$ be the brick assembled to $B^{(i)}$. If $B^{(j)}$ is B_1 or B_2 then $c(M, X) = 0$, against the assumption. Otherwise $c(B^{(j)}) > 0$, and Lemma 4.5 implies that $c_n(B^{(i)} \oplus B^{(j)}) < c_n(B^{(j)})$. This contradicts the assumption that our original assembling realizes $c_n(M, X)$.

Now we fix one assembling $(M, X) = B^{(1)} \oplus \ldots \oplus B^{(h)}$ realizing $c_n(M, X)$ without B_0's and with a minimal number $p \geqslant 0$ of bricks $B^{(i)}$ different from B_2 and B_3. We must show that $p \leqslant 1$, so we assume by contradiction that both $B^{(1)}$ and $B^{(2)}$ are different from B_2 and B_3. Let us concentrate on the boundary component T of $B^{(1)}$ that leaves $B^{(1)}$ and $B^{(2)}$ on opposite sides. Since M is irreducible and atoroidal, one of the following holds: (i) T bounds a solid torus in M; (ii) T is parallel to a component of ∂M; (iii) T is contained in a ball. Using the fact that each $B^{(i)}$ is irreducible one easily sees that if (iii) holds then (i) also does. This implies that there is a sub-assembling of $B^{(1)} \oplus \ldots \oplus B^{(h)}$ involving either $B^{(1)}$ or $B^{(2)}$ and giving a result (N, Y) based on $\boldsymbol{T}$ or on $T \times [0, 1]$. Since n is well-behaved on tori and (N, Y) is not B_1, we can replace the sub-assembling with an assembling of B_2's and B_3's, contradicting minimality of p.

The proof of the second assertion is similar. By assumption there is a pair (N, Y) based on $\boldsymbol{T}$ or on $T \times [0, 1]$ such that $c_n(N, Y) < c_1(N, Y)$. So in any assembling $(N, Y) = B^{(1)} \oplus \ldots \oplus B^{(h)}$ that realizes $c_n(N, Y)$ we must have $c(B^{(i)}) > 1$ for some

i. Now we consider all pairs (M,X) based on $\boldsymbol{T}$ or on $T\times[0,1]$ and all assemblings $(M,X) = B^{(1)}\oplus\ldots\oplus B^{(h)}$ realizing $c_n(M,X)$ with $p>0$ of the $B^{(i)}$'s having complexity bigger than 1. We then select one such $(M,X)=B^{(i)}\oplus\ldots\oplus B^{(h)}$ which gives the minimal possible value of p. Note that we *do not* require $c_n(M,X) < c_1(M,X)$. The assumption that n is not well-behaved is only used to show that the minimum is taken over a non-empty list of cases.

Since we must have $c(M,X)>0$, as above we can dismiss from $B^{(1)}\oplus\ldots\oplus B^{(h)}$ the copies of B_0 and be sure that no sub-assembling gives B_1. The conclusion that $p=1$ is now proved as above, proceeding by contradiction and showing that the assembling can be split into two sub-assemblings such that both involve a brick different from B_2 and B_3, and one gives a result based on $\boldsymbol{T}$ or $T\times[0,1]$. $\square$

Complexity of Dehn fillings Let N be a manifold with $\partial N = T_1\sqcup\ldots\sqcup T_k$. Consider a k-tuple $\alpha_1,\ldots,\alpha_k$ where $\alpha_i\in\mathcal{S}(T_i)\cup\Theta(T_i)$, namely α_i is either a slope or a triod in T_i. We define a pair $N_{\alpha_1,\ldots,\alpha_k}$ obtained as follows from N: when $\alpha_i\in\Theta(T_i)$, we equip T_i with the triod α_i; when $\alpha_i\in\mathcal{S}(T_i)$, we perform on T_i the Dehn filling that kills α_i.

Recall now that we have defined in Section 3 a distance $d(\alpha,\alpha')$ when α is either a slope or a triod and α' is a triod on a given torus. Given tori $T_1,\ldots,T_k$ and two k-tuples $\alpha=\{\alpha_1,\ldots,\alpha_k\}$ and $\alpha'=\{\alpha'_1,\ldots,\alpha'_k\}$, where α_i is either a slope or a triod and α' is a triod on T_i, we now set $d(\alpha,\alpha')=\sum_{i=1}^k d(\alpha_i,\alpha'_i)$.

Proposition 4.7. *1. Let $n\geqslant 1$ be well-behaved on tori and let an irreducible pair (M,X) with positive complexity be given. If M is atoroidal, then $c_n(M,X)$ equals the minimum of $c(N,Y)+d(\alpha,Y)$ over all realizations of (M,X) as N_α, with $(N,Y)\in\mathcal{B}_{\leqslant n}\setminus\{B_1,B_3\}$ and $(N,Y)\neq B_0$ if $M\neq T\times[0,1]$.*

2. If $n\geqslant 2$ is not well-behaved on tori then there is a brick (N,Y) with $1<c(N,Y)\leqslant n$ and k boundary components, and a k-tuple α of slopes and triods on ∂N such that N_α is based on $\boldsymbol{T}$ or on $T\times[0,1]$ and $c_n(N_\alpha)=c(N,Y)+d(\alpha,Y)$.

Proof. We first prove inequality $\geqslant$, so we suppose $c_n(M,X)<\infty$. Proposition 4.6 implies that there exists an assembling that realizes $c_n(M,X)$ involving one $B\in\mathcal{B}_{\leqslant n}$ together with some B_2's and B_3's. We have $B\neq B_1$ by Lemma 4.5. If $B=B_3$ and $M\neq T\times[0,1]$, at least one B_2 must occur in the assembling, and we redefine B as one of these B_2's. If $B=B_3$ and $M=T\times[0,1]$, we can insert one B_0 and redefine B to be this B_0. So in the assembling realizing $c_n(M,X)$ with centre B we can actually assume that $B\in\mathcal{B}_{\leqslant n}\setminus\{B_1,B_3\}$ and $B=B_0$ only if $M=T\times[0,1]$.

Now let h be the number of copies of B_3 appearing in our assembling, so that $c_n(M,X)=c(B)+h$. Let $B=(N,Y)$, with $Y=\{\theta_1,\ldots,\theta_k\}$. Proposition 4.1

shows that a successive assembling along a component T_i of ∂N of some j copies of B_3 corresponds to replacing θ_i by some $\theta_i' \in \Theta(T_i)$ with $d(\theta_i, \theta_i') \leqslant j$. The same proposition implies that if we first assemble j copies of B_3 and then a copy of B_2 then M gets filled along a slope $\alpha_i \in \mathcal{S}(T_i)$ with $d(\alpha_i, \theta_i) \leqslant j$. This shows that $(M, X) = N_\alpha$ with $d(\alpha, Y) \leqslant h$, whence $c_n(M, X) \geqslant c(B) + d(\alpha, Y)$, and inequality $\geqslant$ follows.

Turning to inequality $\leqslant$, suppose the right-hand side equals $c(N, Y) + d(\alpha, Y)$ for some $(N, Y) \in \mathcal{B}_{\leqslant n} \setminus \{B_1, B_3\}$ and some realization $(M, X) = N_\alpha$. Suppose $\partial N = T_1 \sqcup \ldots \sqcup T_k$, $Y = \{\theta_1, \ldots, \theta_k\}$, $\alpha = \{\alpha_1, \ldots, \alpha_k\}$. We can now associate to the realization N_α of (M, X) an assembling yielding (M, X). When $\alpha_i \in \Theta(T_i)$ we assemble $d(\alpha_i, \theta_i)$ copies of B_3, following the unique simple path in $\Theta(T_i)$ from θ_i to α_i. When $\alpha_i \in \mathcal{S}(T_i)$ and $d(\alpha_i, \theta_i) \geqslant 0$ we assemble $d(\alpha_i, \theta_i)$ copies of B_3 and one of B_2. When $\alpha_i \in \mathcal{S}(T_i)$ and $d(\alpha_i, \theta_i) = -1$ we assemble one copy of B_1. Now we apply the last assertion of Lemma 4.5 (noting that if $(N, Y) = B_0$ then there is no B_1 at all, otherwise the result could not be $T \times [0, 1]$). So the partial assembling of (N, Y) with the B_1's only has complexity c_n at most $c(N, Y) - h$, where h is the number of B_1's. Adding also the B_2's and B_3's we then get $c_n(M, X) \leqslant c(N, Y) + d(\alpha, Y)$, whence the conclusion.

The second assertion is now proved by the very same argument, using the second part of Proposition 4.6 as a starting point. $\square$

Let (N, Y) be a brick of complexity $n \geqslant 1$. From Proposition 4.7 we know that $c_n(N_\alpha) \leqslant n + d(\alpha, Y)$ for any k-tuple α. We then say that α is *degenerate* if $c_n(N_\alpha) < n + d(\alpha, Y)$ or $c(N_\alpha) = 0$. If S is a set of slopes lying on pairwise distinct components of ∂N, we add to S the $(\#Y) - (\#S)$ triods of Y on the tori that S does not intersect, and we call S degenerate if the resulting k-tuple is.

Proposition 4.8. *1. If (N, Y) is a brick of complexity $n \geqslant 1$ then every slope s_i with $d(s_i, \theta_i) = -1$ is degenerate;*

2. If $n \geqslant 2$ then any k-tuple containing a degenerate set of slopes is degenerate.

Proof. The first assertion follows from Lemma 4.5 and Proposition 4.1. To prove the second one, let $\alpha_1, \ldots, \alpha_k$ be a k-tuple where $\{\alpha_1, \ldots, \alpha_j\}$ is a degenerate set of slopes for some $1 \leqslant j \leqslant k$. Namely, one of the following holds:

$$\begin{aligned} c_n(N_{\alpha_1, \ldots, \alpha_j, \theta_{j+1}, \ldots, \theta_k}) &= 0, \qquad \text{or} \\ c_n(N_{\alpha_1, \ldots, \alpha_j, \theta_{j+1}, \ldots, \theta_k}) &< n + \sum_{i=1}^{j} d(\alpha_i, \theta_i). \end{aligned}$$

If the second relation does not hold, so the first does, we deduce that $n \leqslant k$, whence $n \leqslant 3$ by Theorem 2.6. But $n \geqslant 2$, so $(N, Y) = B_4$ and $k = j = 3$, and we have nothing to prove. So we can proceed assuming the second relation holds. Now Lemma 4.5 and Proposition 4.1 again imply that either $c_n(N_{\alpha_1,\ldots,\alpha_k}) = 0$ or

$$c_n(N_{\alpha_1,\ldots,\alpha_k}) \leqslant c_n(N_{\alpha_1,\ldots,\alpha_j,\theta_{j+1},\ldots,\theta_k}) + \sum_{i=j+1}^{k} d(\alpha_i, \theta_i),$$

so $c_n(N_{\alpha_1,\ldots,\alpha_k}) < n + d(\alpha, X)$. $\square$

We note here that the second assertion of the previous proposition does not hold for $B_3 = (T \times [0,1], \{\theta_0, \theta_1\})$, because there is one slope $\alpha_0 \subset \theta_0$ such that $(B_3)_{\alpha_0} = B_2$, so α_0 is degenerate, but it will follow from our computations of c_1 on lens spaces that if $d(\alpha_1, \theta_1) \geqslant 1$ then the pair of slopes $\{\alpha_0, \alpha_1\}$ is not degenerate.

The following result will be used later in this section and in Section 6 to show that n is well-behaved on tori for $n \leqslant 9$.

Proposition 4.9. *Let $n > 1$ and assume the following holds for every brick B with $1 < c(B) \leqslant n$:*

- *B is not based on $\boldsymbol{T}$ or on $T \times [0,1]$;*
- *If α is a non-empty set of* slopes *on ∂B and B_α is based on $\boldsymbol{T}$ or on $T \times [0,1]$ then α is degenerate.*

Then n is well-behaved on tori.

Proof. Assume n is not well-behaved on tori. By Proposition 4.7-(2) there is a brick B with $1 < c(B) \leqslant n$ and a k-tuple β such that B_β is based on $\boldsymbol{T}$ or on $T \times [0,1]$ and β is non-degenerate. Let α be obtained from β by dismissing the triods and keeping the slopes. Then B_α is also based on $\boldsymbol{T}$ or on $T \times [0,1]$, so α is non-empty and degenerate by assumption. But β contains α, which contradicts Proposition 4.8-(2). $\square$

Remark 4.10. We emphasize that the set α in the previous statement consists of slopes only, not of triods too. We also recall that that none of the bricks of complexity between 2 and 9 is based on $\boldsymbol{T}$ or on $T \times [0,1]$. Moreover, of the non-closed bricks $B_4, \ldots, B_{10}$, we know that B_5, B_6, B_7 have only one boundary component. Therefore, to conclude that $n = 9$ is well-behaved on tori, we will only need to show that all the fillings of B_4, B_8, B_9, B_{10} giving $T \times [0,1]$ or $\boldsymbol{T}$ are degenerate. Moreover B_4 alone already suffices for n up to 8.

Computation of c_1 We start by computing $c_n(M,X)$ for $n=1$ and any irreducible and atoroidal (M,X). For $n \geqslant 3$ we will confine ourselves to the closed case. The bricks B_0 and B_2 are based respectively on $T\times[0,1]$ and on the solid torus $\boldsymbol{T}$. Therefore $c_1(M,X)<\infty$ precisely when M is $T\times[0,1]$, $\boldsymbol{T}$, a lens space, S^3, or $\mathbb{RP}^3$. And of course $c_1(S^3)=c_1(\mathbb{RP}^3)=0$.

Proposition 4.7-(1) implies that if $M=T\times[0,1]$ then $c_1(M,X)$ equals the minimum of $d(\theta_0',\theta_0)+d(\theta_1',\theta_1)$ over all realizations of (M,X) as $(T\times[0,1])_{\theta_0',\theta_1'}$, where $\theta_0\subset T\times\{0\}$ and $\theta_1\subset T\times\{1\}$ are the triods such that $B_0=(T\times[0,1],\{\theta_0,\theta_1\})$. The product structure of $T\times[0,1]$ gives an obvious bijection $f:\Theta(T\times\{0\})\to\Theta(T\times\{1\})$. Since $f(\theta_0)=\theta_1$ we have $d(\theta_0',\theta_0)+d(\theta_1',\theta_1)\geqslant d(f(\theta_0'),\theta_1')$. Moreover $d(f(\theta_0'),\theta_1')$ depends on (M,X) only, and it is possible to realize (M,X) with $\theta_0'=\theta_0$, and in this case we have $d(\theta_0',\theta_0)+d(\theta_1',\theta_1)=d(f(\theta_0'),\theta_1')$. We have proved the following:

Proposition 4.11. *We have* $c_1(T\times[0,1],\{\theta_0',\theta_1'\})=d(f(\theta_0'),\theta_1')$.

Let us now consider the case $M=\boldsymbol{T}$. Choosing a basis $\mathfrak{b}=(\mu,\lambda)$ for $H_1(\partial\boldsymbol{T})$ with μ null in $H_1(\boldsymbol{T})$, we can identify the space of pairs $(\boldsymbol{T},\theta)$ up to orientation-preserving homeomorphism to the quotient of $\Theta(\partial\boldsymbol{T})$ under the action of the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ consisting of upper triangular matrices. Via the map $\Psi_\mathfrak{b}$ of Section 3 we can also identify $\Theta(\partial\boldsymbol{T})$ to the set of vertices of the graph $\mathcal{T}$ dual to the Farey tessellation of $\mathbb{H}^2$. Upper triangular matrices in $\mathrm{GL}_2(\mathbb{Z})$ act on $\mathbb{H}^2$ as parabolic isometries fixing point $\infty\in\partial_\mathbb{Q}\mathbb{H}^2$, and a fundamental domain is delimited by two consecutive lines of $\mathcal{L}$ ending at ∞. In the disc model, one such domain is the non-shadowed region shown in Fig. 9, so the vertices of $\mathcal{T}$ that it contains precisely represent the pairs based on $\boldsymbol{T}$, up to orientation-preserving homeomorphism. In particular, the positions of B_1 and B_2 are shown in the figure. If we also consider orientation-reversing homeomorphisms we must add the reflection in the vertical axis. This implies the fact that B_1 and B_2 admit an orientation-reversing automorphism, while all other pairs based on $\boldsymbol{T}$ do not. Finally, Proposition 4.7-(1) implies that if $c(\boldsymbol{T},\theta)>0$ then $c_1(\boldsymbol{T},\theta)$ equals the distance between $(\boldsymbol{T},\theta)$ and B_2 in Fig. 9. Some values are shown in Fig. 9: as expected, the values are symmetric with respect to the vertical axis, because manifolds with triods are only defined up to homeomorphism.

Turning to the case where M is a lens space, note that $L_{p,q}=\boldsymbol{T}_\alpha$ with a slope α having coordinates (q,p) with respect to a basis $\mathfrak{b}$ as above. Figure 9 and Proposition 3.3 now imply that the distance between α and the triod of B_2 is $|q,p|-2$. Since $|q,p|=|p,q|$ by Proposition 1.4-(2), we have then shown the following result. (Note that by convention we view $L_{p,q}$ as a honest lens space precisely when $|p,q|\geqslant 2$.)

Proposition 4.12. *Let $L_{p,q}$ be a lens space. Then* $c_1(L_{p,q})=|p,q|-2$.

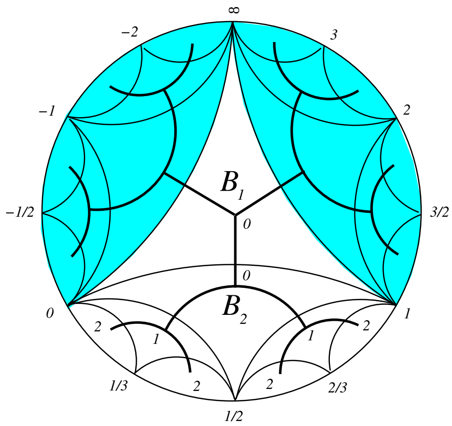

Figure 9: Oriented pairs based on the solid torus and their complexity.

A preview on torus bundles Even if this section is devoted to atoroidal manifolds, to proceed we need to state the (easy) computation of c_1 on torus bundles. The fact that c_n equals c_1 on these manifolds for $n = 2, \ldots, 9$ is much harder (see Proposition 7.11).

Proposition 4.13. *If $A \in \mathrm{GL}_2(\mathbb{Z})$ and T_A is the torus bundle over the circle with monodromy A then:*

- *$c_0(T_A)$ equals 6 if $||A|| \leqslant 1$ and ∞ otherwise;*
- $c_1(T_A) = \max\{||A|| + 5, 6\}$.

Proof. The only way to obtain a torus bundle T_A using bricks having complexity 0 is to do a self-assembling of $B_0 = (T \times [0,1], \{\theta \times \{0\}, \theta \times \{1\}\})$ along a map ψ acting as (a matrix conjugate to) A on homology. Recall that $\psi(\theta \times \{0\})$ has distance at most 1 from $\theta \times \{1\}$. It follows from Corollary 3.5 that we can realize precisely those bundles T_A such that $||A|| \leqslant 1$, and $c_0 = 6$ on them. This shows the first assertion.

If we also allow bricks of complexity 1, we can realize T_A only as a self-assembling along a ψ as above of some $(T \times [0,1], \{\theta_0, \theta_1\})$, which is itself either B_0 or an assembling of copies of B_3. Moreover $c_1(T \times [0,1], \{\theta_0, \theta_1\}) = d(f(\theta_0), \theta_1)$ by Proposition 4.11, where $f : \Theta(T \times \{0\}) \to \Theta(T \times \{1\})$ is the obvious map. So, viewing A as a fixed map from $T \times \{0\}$ to $T \times \{1\}$, we have

$$c_1(T_A) = 6 + \min\big\{d(f(\theta_0), \theta_1) : \theta_i \in \Theta(T \times \{i\}),\ d(A\theta_0, \theta_1) \leqslant 1\big\}.$$

Corollary 3.5 then shows that $c_1(T_A) = 6 + \max\{0, ||A|| - 1\}$, and the second assertion is proved. □

Computation of c_3 The brick B_4 is based on $D_2 \times S^1$, where D_2 is the disc with two holes. Therefore the closed manifolds having finite c_3 are all the total spaces of Seifert fibrations over S^2 with at most 3 singular fibers. Recall that in Section 1 we have fixed a canonical homology basis (a_i, b_i) on the i-th component of $\partial(D_2 \times S^1)$, where a_i is a component of ∂D_2 with induced orientation, b_i is a copy of the fibre S^1, and the basis is positive. Using these bases we will henceforth use rational numbers or ∞ to describe slopes. Recalling also that we are denoting by $\theta^{(i)}$ the triod containing the slopes i, $i+1$, and ∞, it is now easy to see from Fig. 5 that

$$B_4 = (D_2 \times S^1)_{\theta^{(0)},\theta^{(0)},\theta^{(-1)}}.$$

Using Remark 4.10, the next result shows that $n = 8$ is well-behaved on tori.

Proposition 4.14. *1. If α consists of some slopes on ∂B_4 and $(B_4)_\alpha$ is based on $\boldsymbol{T}$ or on $T \times [0,1]$ then α contains a slope 0 or $1/q$ for $q \in \mathbb{Z}$;*

2. *All slopes 0 and $1/q$ for $q \in \mathbb{Z}$ on any component of ∂B_4 are degenerate;*

3. *A filling of B_4 along a non-degenerate triple of slopes is either a genuine closed atoroidal Seifert manifold or $T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)}$;*

4. *If $0 < q_i < p_i$ for $i = 1,2,3$ and $t \geqslant -1$ then*

$$c_3\big(S^2, (p_1,q_1), (p_2,q_2), (p_3,q_3), t\big) = |p_1,q_1| + |p_2,q_2| + |p_3,q_3| + t + 1 \qquad (10)$$

except when $t = -1$, $q_1 = q_2 = q_3 = 1$ and $\{p_1,p_2,p_3\}$ equals either $\{2,2,2\}$, or $\{2,2,3\}$, or $\{2,3,6\}$, or $\{2,4,4\}$. For these cases the value of c_3 is respectively 2, 3, 6, and 6.

Remark 4.15. Let us explain why there are four special manifolds excluded from the range of applicability of (10), but $\big(S^2,(3,1),(3,1),(3,1),-1\big) = T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)}$, which is also quite special, is not excluded. The first two manifolds excluded are the two closed bricks having complexity at most 3 (denoted by $C_{1,1}$ and $C_{1,2}$ in Proposition 2.7). Their complexity is 2 and 3 respectively, and (10) is incorrect for them, because it would give 3 and 4 respectively. Turning to the torus bundles

$$T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)}, \quad T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)}, \quad T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)}$$

(see Proposition 1.1), we know [11, 13] that their complexity is 6, and we have seen in Proposition 4.13 that c_1 already equals 6 on them, while formula (10) gives 8, 7, and 6 respectively. This means that the formula is incorrect for the first two of them, but it is correct for the last one. It also means that $T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)}$ can be expressed both as a sharp self-assembling of one B_0, and as a sharp assembling of one B_4 and three B_2's. We have here one of the (not many) instances we know of non-uniqueness in Theorem 2.4.

Proof of 4.14. For point 1, we note first that α cannot be empty, because B_4 is based on $D_2 \times S^1$. Assume now that in α there are no slopes 0 or $1/q$. Since there are no 0's we see that $(B_4)_\alpha$ has a natural Seifert fibration, and all the members of α contribute with an exceptional fibre, because there are no $1/q$'s. If $(B_4)_\alpha$ is based on $T \times [0,1]$ we get a contradiction because $T \times [0,1]$ has a unique Seifert fibration (up to isomorphism), and it has no exceptional fibres. If $(B_4)_\alpha$ is based on $\boldsymbol{T}$ we also get a contradiction, because α has two members, but $\boldsymbol{T}$ can only be fibred with one exceptional fibre.

Let us turn to point 2. The slope 0 is degenerate on each boundary component by Proposition 4.8-(1), because it is contained in both $\theta^{(0)}$ and $\theta^{(-1)}$. Let us prove that a slope $1/q$ on the j-th boundary component is degenerate. If we perform a $1/q$-Dehn filling on the j-th component we get some $(T \times [0,1], \{\theta_0, \theta_1\})$. If $f : \Theta(T \times \{0\}) \to \Theta(T \times \{1\})$ is the obvious bijection, it is now easy to prove that:

$$\begin{aligned} d(f(\theta_0), \theta_1) &= \begin{cases} |q| & \text{if } j = 1,2 \\ |q-1| & \text{if } j = 3 \end{cases} \\ d(1/q, \theta^{(i)}) &= \begin{cases} \max\{q-2, -q-1\} & \text{if } i = 0 \\ \max\{q-1, -q-2\} & \text{if } i = -1 \end{cases} \end{aligned}$$

(see also Corollary 3.4). In all cases we get

$$c_1(T \times [0,1], \{\theta_0, \theta_1\}) = d(f(\theta_0), \theta_1) < 3 + d(1/q, \theta^{(i)})$$

by Proposition 4.11, and point 2 is proved.

To prove point 3, let us take a triple α of slopes not containing 0's and $1/q$'s. As a byproduct of the proof of point 1 we see that $(B_4)_\alpha$ is Seifert fibred over S^2 with three exceptional fibres, so it is either genuine and atoroidal or one of the three exceptional torus bundles of Proposition 1.1-(1). We will show together with point 4 that α is degenerate when $(B_4)_\alpha$ is $T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)}$ or $T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)}$, and it is not when $(B_4)_\alpha$ is $T_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)}$.

So we turn to point 4, treating first the case of a genuine atoroidal manifold

$$M = \big(S^2, (p_1, q_1), (p_2, q_2), (p_3, q_3), t\big).$$

The shape of $B_0, \ldots, B_3$ and the knowledge of the closed bricks of small complexity show that only for $M = C_{1,1}$ and $M = C_{1,2}$ it is possible to realize M as an assembling of bricks with complexity at most 3 but without B_4's. Of course $c_3(M) = c(M)$ for $M = C_{1,1}$ and $M = C_{1,2}$, so we exclude these two cases. By Proposition 4.7 the conclusion will now follow as soon as we prove that the minimum of

$$3 + d(\alpha_1, \theta^{(0)}) + d(\alpha_2, \theta^{(0)}) + d(\alpha_3, \theta^{(-1)})$$

over all realizations of M as $(D_2 \times S^1)_{\alpha_1,\alpha_2,\alpha_3}$ is $|p_1, q_1| + |p_2, q_2| + |p_3, q_3| + t + 1$. It is well-known that, up to permutation, the choices of $\alpha_1, \alpha_2, \alpha_3$ leading to one such realization are precisely those of the form

$$(\alpha_1, \alpha_2, \alpha_3) = \big(sp_1/(q_1 + k_1 \cdot p_1), sp_2/(q_2 + k_2 \cdot p_2), sp_3/(q_3 + k_3 \cdot p_3)\big)$$

for $s \in \{1, -1\}$ and integers k_1, k_2, k_3 such that $k_1 + k_2 + k_3 = t$.

Before computing the minimum we recall that $d(p/q, \theta^{(0)}) = |p, q| - 1$ and $d(p/q, \theta^{(-1)}) = |p, p + q| - 1$ by Corollary 3.4. Moreover if $0 < q < p$ we have $|p, q - p| = |p, q|$ by Proposition 1.4-(2). So, using the definition of $|\cdot\, , \cdot|$ we have for $h > 0$ that

$$\begin{aligned}
|p, q + hp| &= |p, q| + h, \\
|p, q - hp| &= 1 + |p, hp - q| = 1 + (h - 1) + |p, p - q| = |p, q| + h, \\
&\Rightarrow |p, q + k \cdot p| = |p, q| + |k| \qquad \forall k \in \mathbb{Z}.
\end{aligned}$$

Similarly:

$$\begin{aligned}
|p, -q - hp| &= 1 + |p, q + hp| = 1 + h + |p, q|, \\
|p, -q + hp| &= h - 1 + |p, p - q| = -1 + h + |p, q|, \\
&\Rightarrow |p, -q - k \cdot p| = |p, q| + |k + 1| \qquad \forall k \in \mathbb{Z}.
\end{aligned}$$

Back to our minimum, let us first compute it over realizations with $s = 1$. From the previous discussion we have

$$\begin{aligned}
3 &+ d(\alpha_1, \theta^{(0)}) + d(\alpha_2, \theta^{(0)}) + d(\alpha_3, \theta^{(-1)}) \\
&= \big|p_1, q_1 + k_1 \cdot p_1\big| + \big|p_2, q_2 + k_2 \cdot p_2\big| + \big|p_3, q_3 + (1 + k_3) \cdot p_3\big| \\
&= |k_1| + |k_2| + |1 + k_3| + \sum_{i=1}^{3} |p_i, q_i|
\end{aligned}$$

therefore the minimum is attained when $k_1 = k_2 = 0$ and $k_3 = t$, and its value is $\sum_i |p_i, q_i| + t + 1$. Note that this value is unchanged under reordering of the p_i/q_i's, and that the assumption $t \geqslant -1$ is used here.

Let us now consider the realizations with $s = -1$. From the properties of $|\cdot,\cdot|$ shown above we deduce that

$$\begin{aligned} 3 &+ d(\alpha_1,\theta^{(0)}) + d(\alpha_2,\theta^{(0)}) + d(\alpha_3,\theta^{(-1)}) \\ &= \left|p_1, -q_1 - k_1\cdot p_1\right| + \left|p_2, -q_2 - k_2\cdot p_2\right| + \left|p_3, -q_3 - (k_3-1)\cdot p_3\right| \\ &= |1+k_1| + |1+k_2| + |k_3| + \sum_{i=1}^{3} |p_i, q_i| \end{aligned}$$

so the minimum is attained when $k_1 = k_2 = -1$ and $k_3 = t+2$, and its value is $\sum_i |p_i, q_i| + t + 2$. Again this value is invariant under reordering, and the conclusion of point 3 easily follows.

We are left to consider the three Seifert manifolds which are not genuine and atoroidal because they can also be described as torus bundles. For these manifolds c_1 equals 6 by Proposition 4.13. Now suppose there is an assembling realizing $c_3(T_A)$ in which B_4 appears. Then $c_3(T_A)$ is computed just as above in the atoroidal case, taking a minimum over all fillings of B_4 that give T_A. This minimum is respectively 8, 7, and 6 when A is $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. In the first two cases, we deduce that the triple giving T_A is degenerate. In the last case, the triple is non-degenerate. $\square$

Computation of c_8 and c_9 Recall that there are two bricks B_5 and B_6 having complexity 8, and four $B_7, \ldots, B_{10}$ having complexity 9. The bricks $B_6, \ldots, B_{10}$ are based on hyperbolic manifolds, and it will follow from Propositions 6.3 and 6.4 that they give no contribution to the values of c_8 and c_9 for manifolds that are not negatively curved (see Proposition 7.10). Therefore we have $c_8 = c_9$ on Seifert manifolds, and we conclude this section by studying the contribution of the brick B_5 to $c_8 = c_9$ on the atoroidal ones. Let us first describe B_5. We take $D_2 \times S^1$ with the basis on the boundary defined above for B_4, and again we use the rationals to denote slopes. Then $B_5 = (D_2 \times S^1)_{2,3,\theta}$ where θ is the triod containing the slopes $-1, -6/5, -5/4$.

Proposition 4.16. *1. A slope α on B_5 is non-degenerate if and only if $-5/4 < \alpha < -1$ and $\alpha \notin \{-6/5, -7/6, -8/7\}$;*

2. The fillings of B_5 along non-degenerate slopes are precisely the Seifert manifolds of the form $\big(S^2, (2,1), (3,1), (p,q), -1\big)$ with $p/q > 5,\ p/q \notin \{6,7,8\}$.

3. For $p/q > 5, p/q \notin \{6,7,8\}$ we have $c_8\big(S^2, (2,1), (3,1), (p,q), -1\big) = |p,q| + 2$.

Remark 4.17.

- The formula for c_8 in point 3 indeed does not apply for $p/q \in \{6,7,8\}$, because it would give $p/q+1$, but we can actually show that c_8 equals p/q in these three cases: for $p/q=6$ the corresponding manifold is $T_{\left(\begin{smallmatrix}0 & 1\\ -1 & 1\end{smallmatrix}\right)}$, whose complexity is 6, so c_8 and c_9 also are 6 (and we have seen in Proposition 4.13 that c_1 already is 6). For $p/q=7,8$ we get the closed bricks $E_2, E_3 \in \mathcal{M}^*$, having complexity 7 and 8 respectively.
- Comparison with Proposition 4.14 shows that $c_8 = c_3 - 1$ on manifolds of the form $\big(S^2,(2,1),(3,1),(p,q),-1\big)$ for $p/q>5,\ p/q \notin \{6,7,8\}$.

Proof of 4.16. We first show that the slopes outside $(-5/4,-1)$ are degenerate. Later we will prove that the slopes inside $(-5/4,-1)$, except $-6/5,-7/6,-8/7$, are non-degenerate, and at the same we will describe the fillings they determine and the complexity c_8 of these fillings.

Let us then fix $\alpha \in \partial_{\mathbb{Q}}\mathbb{H}^2$ and define $M=(B_5)_\alpha$, *i.e.* $M=(D_2\times S^1)_{2,3,\alpha}$. We recall that $B_4=(D_2\times S^1)_{\theta^{(0)},\theta^{(0)},\theta^{(-1)}}$ and $c(B_4)=3$. Hence, by Proposition 4.7,

$$c_8(M) \leqslant 3+0+1+d(\alpha,\theta^{(-1)}).$$

A sufficient condition for α to be degenerate is then that

$$4+d(\alpha,\theta^{(-1)}) < 8+d(\alpha,\theta) \tag{11}$$

(where θ is the triod on ∂B_5). Figure 10, which uses an alternative embedding of the Farey tessellation, easily proves that (11) holds if $\alpha \notin [-5/4,-1)$. So all the slopes outside $[-5/4,-1)$ are degenerate, but $-5/4$ also is, because it is contained in θ (see Proposition 4.8), and our first task is accomplished.

We can now concentrate on the slopes inside $(-5/4,-1)$, which are precisely those of the form $p/(q-p)$ for $0<q<p$ and $p/q>5$. Note that M is then given by

$$M=\big(S^2,(2,1),(3,1),(p,q-p)\big)=\big(S^2,(2,1),(3,1),(p,q),-1\big)$$

and the latter expression is the unique normalized one. Now $\alpha=-6/5$ is degenerate because it is contained in θ. For $\alpha=-7/6$ we get $M=E_2$ and $c(M)=7<8=8+d(\alpha,\theta)$, so α is again degenerate. For $\alpha=-8/7$ we get $M=E_3$ and $c(M)=8$, and α is degenerate for the same reason.

Now we list certain facts whose combined effect is to show that the slopes α contained in $(-5/4,-1)$, but distinct from $-6/5,-7/6,-8/7$, are non-degenerate, and to prove points 2 and 3. Fix such an α and note that M is then a genuine atoroidal Seifert manifold by Proposition 1.1. Recall also from Proposition 4.7 that $c_8(M)$ is then the minimum of $c(N,Y)+d(\beta,Y)$ over all realizations of M as N_β, with $(N,Y)\in\mathcal{B}_{\leqslant 8}\setminus\{B_0,B_1,B_3\}$.

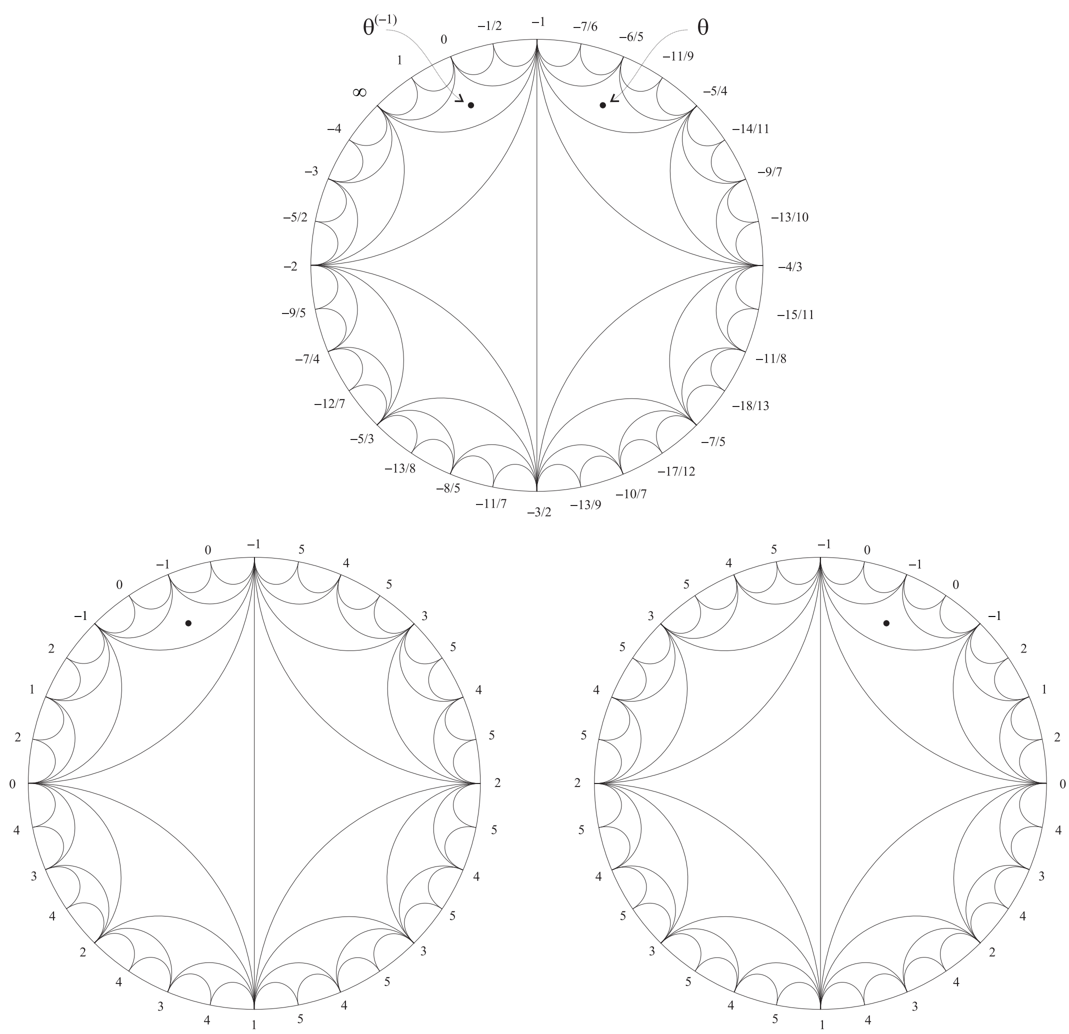

Figure 10: Position of $\theta^{(-1)}$ and θ, and computation of $d(\alpha, \theta^{(-1)})$ and $d(\alpha, \theta)$.

- $\mathcal{B}_{\leqslant 8} \setminus \{B_0, B_1, B_3\}$ consists of the closed bricks of complexity up to 8 and of B_2, B_4, B_5, B_6.

- M is neither a closed brick of complexity up to 8 or a filling of B_2;

- The minimum restricted to $B = B_4$ equals $c_3(M)$ and its value is $3 + |p, q|$ by Proposition 4.14 (which applies here), where $\alpha = p/(q-p)$;

- M can be obtained from B_5 as $(B_5)_\alpha$ only, so only $8 + d(\alpha, \theta)$ appears in the minimum with $(N, Y) = B_5$;

- $3 + |p, q| = 4 + d(\alpha, \theta^{(-1)})$ by Corollary 3.4, and
$$8 + d(\alpha, \theta) = (4 + d(\alpha, \theta^{(-1)})) - 1$$
by direct inspection of Fig. 10.

- We will independently prove below in Proposition 7.10 that if $M = (B_6)_\beta$ then β is degenerate with respect to B_6, so we do not need to consider $B = B_6$ in the minimum.

Our argument is complete. □

5 Dehn surgery on the chain-link

In this section we prove Theorem 1.13. Throughout the section we will denote by N the complement of the three-component chain-link 6^3_1 in S^3, and we will equip the components of ∂N with the usual meridian-longitude homology bases canonically defined for link complements, so we will represent the boundary slopes by rational numbers or ∞. We also recall that for $i \in \mathbb{Z}$ we denote by $\theta^{(i)}$ the triod containing the slopes i, $i+1$, and ∞.

Hyperbolic fillings with symmetries One of the ingredients of the proof of Theorem 1.13 is the list of repetitions among Dehn fillings of N that was already given in Proposition 1.17. We briefly explain here how this proposition is proved, omitting all the details because we regard this result mostly as an experimental discovery. One can see using SnapPea (and check by hand) that the partial Dehn fillings
$$N_{-3/2} \qquad N_{-4} \qquad N_{-5/2} \qquad N_{-1/2} \qquad N_{1,2} \qquad N_{1,-4}$$
are all hyperbolic, that the first two of them are isometric to each other, and that in all cases the symmetry group contains a non-trivial cyclic subgroup that leaves

invariant each boundary component. Using the precise form of these non-trivial isometries and their action on the homology bases on the boundary, one can with some patience deduce the identities listed in equations (2) to (7).

A list of non-hyperbolic fillings Recall now that we are denoting by D_i the disc with i holes, that we have set up a convention to determine a section-fibre homology basis on the boundary components of $D_i \times S^1$, and that we use these bases to express filling slopes by rational numbers and gluings by matrices.

Proposition 5.1. *The following homeomorphisms hold:*

$$\begin{aligned}
N_{\infty,\frac{r}{s},\frac{t}{u}} &= \left(D_1 \times S^1\right)_{\frac{r}{s},-\frac{u}{t}} && (12)\\
N_{-3,\frac{r}{s},\frac{t}{u}} &= \left(D_2 \times S^1\right)_{2,\frac{r+s}{r+2s}} \bigcup\nolimits_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)} \left(D_2 \times S^1\right)_{2,\frac{t+u}{t+2u}} && (13)\\
N_{-2,\frac{r}{s},\frac{t}{u}} &= \left(D_2 \times S^1\right)_{\frac{3}{2},-2-\frac{r}{s},-2-\frac{t}{u}} && (14)\\
N_{-1,\frac{r}{s},\frac{t}{u}} &= \left(D_2 \times S^1\right)_{2,-3-\frac{r}{s},-3-\frac{t}{u}} && (15)\\
N_{0,\frac{r}{s},\frac{t}{u}} &= \left(D_2 \times S^1\right)_{\frac{s}{r+2s},\frac{u}{t+2u}} \bigcup\nolimits_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)} \left(D_2 \times S^1\right)_{2,3} && (16)\\
N_{1,1,\frac{t}{u}} &= \left(D_2 \times S^1\right)_{-\frac{u}{t+2u}} \Big/ {\left(\begin{smallmatrix} 1 & -1 \\ -1 & 0 \end{smallmatrix}\right)} && (17)\\
N_{1,3,3} &= \left(D_2 \times S^1\right)_{2,3} \bigcup\nolimits_{\left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)} \left(D_2 \times S^1\right)_{2,2} && (18)\\
N_{2,2,2} &= \left(D_2 \times S^1\right)_{2,3} \bigcup\nolimits_{\left(\begin{smallmatrix} 2 & 3 \\ -1 & -2 \end{smallmatrix}\right)} \left(D_2 \times S^1\right)_{2,2}. && (19)
\end{aligned}$$

Proof. The first equality is easy: doing an ∞-surgery on a component of a link means canceling this component. So N_∞ is the complement of the Hopf link, which is homeomorphic to $D_1 \times S^1$, and it is easy to see that the actual homeomorphism leads to relation (12). The other relations would probably be hard to show using the topology of N directly. We have established them using skeleta instead. We will now outline the general method and then show how to apply this method to prove (13) to (19).

Our method consists of three steps. We describe them now in general terms, referring to relations (13) to (16) for steps 1 and 2, and to relation (13) only for step 3. The other cases require obvious variations that the reader will easily guess. Later we will provide examples of how these steps are carried out in practice.

Step 1: Construction of standard skeleta. Recall first that, using notation as in Section 4, there is a brick $B_9 = N_{\theta^{(-2)},\theta^{(-2)},\theta^{(-2)}}$ based on N, and we explicitly

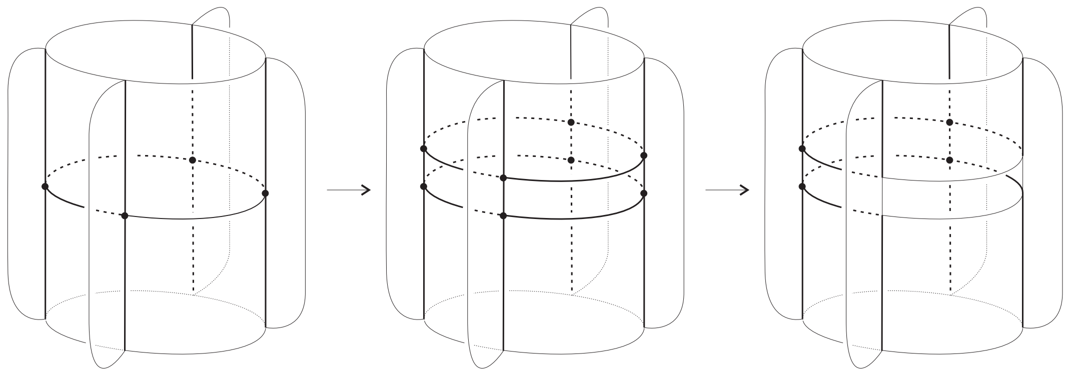

Figure 11: Replacement of an embedded disc with four vertices.

know [11] a minimal skeleton P_9 of B_9. Next, we use the fact stated in Proposition 4.1 that if α is any slope then $N_{\alpha,\theta^{(-2)},\theta^{(-2)}}$ can be realized as an assembling of B_9 with either B_1, when $d(\alpha,\theta^{(-2)}) = -1$, or with some copies of B_3 and one of B_2. Using skeleta P_1, P_2, P_3 of B_1, B_2, B_3 we can then construct a skeleton Q_α of $N_{\alpha,\theta^{(-2)},\theta^{(-2)}}$. This skeleton is always standard unless $d(\alpha,\theta^{(-2)}) = -1$, but in this case we can easily collapse it onto a standard subpolyhedron, so we can reduce Q_α to be standard anyway. In particular, Q_α determines the pair it is the skeleton of.

Step 2: Modification and recognition of skeleta. Now we remind the reader that there exist certain moves, called disc-replacements by Matveev [13] and shown in Fig. 11, which are known to transform a skeleton of a pair into another skeleton of the same pair. Using these moves on Q_α for $\alpha = -3, -2, -1, 0$ we construct another skeleton Q'_α of $N_{\alpha,\theta^{(-2)},\theta^{(-2)}}$ which can be expressed as a gluing of the minimal skeleta of B_2, B_3, B_4.

Step 3: Recognition of fillings. Now we specialize our explanation to the case $\alpha = -3$. Having expressed Q'_{-3} as a gluing of the skeleta of B_2, B_3, B_4, we can correspondingly find an explicit homeomorphism between $N_{-3,\theta^{(-2)},\theta^{(-2)}}$ and an assembling of B_2, B_3, B_4, which is recognized to be

$$(D_2 \times S^1)_2 \bigcup_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)} (D_2 \times S^1)_2.$$

Considering the image under this homeomorphism of the preferred homology bases on the boundary components of $N_{-3,\theta^{(-2)},\theta^{(-2)}}$, and expressing this image in terms

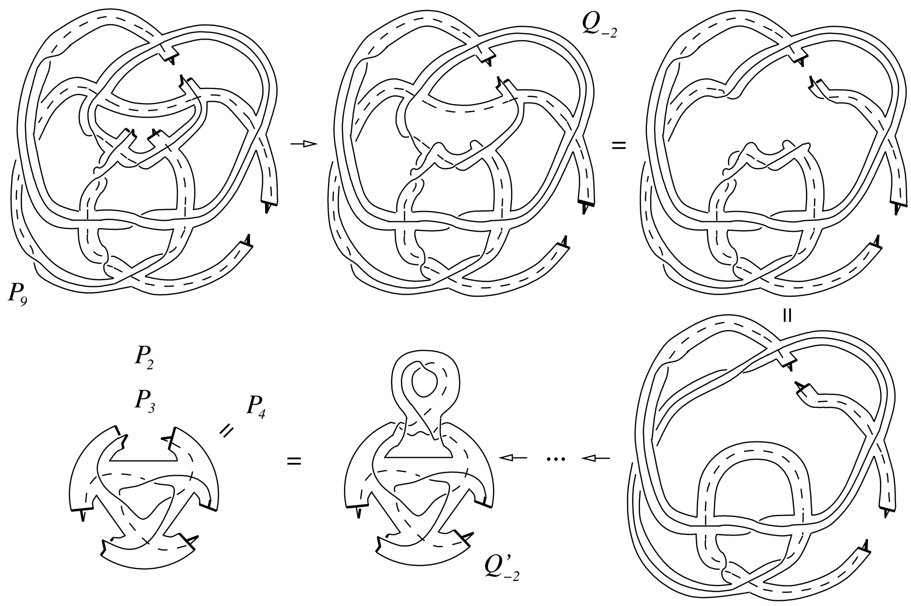

Figure 12: Proof of relation (14)

of the preferred bases for $D_2 \times S^1$, we then get the desired relation (13).

Examples. Having explained the general scheme, we provide some examples of how we have applied it, omitting many passages. We start in Fig. 12 with the easiest case of relation (14), where only one P_1 must be attached. We first show a regular neighbourhood of the singular set of P_9, which allows to reconstruct P_9 completely, and then (using the same method) Q_{-2}, with three different pictures that allow to understand its shape better and better. Then we show Q'_{-2} and its decomposition into one copy of P_4, one of P_3, and one of P_2. For P_2 and P_3 we cannot show a neighbourhood of the singular set only, because this neighbourhood does not allow to reconstruct them. We do not describe explicitly the moves that allow to pass from Q_{-2} to Q'_{-2} (there are two of them, the first one on a disc having four vertices and the second one on a disc having three), but we will show some moves in other cases.

Figure 13 starts with Q_{-1}, first as it appears by attaching P_1 to P_9 and collapsing, and then with an alternative picture where an embedded disc having three vertices is easily visible. We then show the result of the replacement of this disc, and two more moves, eventually giving Q'_{-1} and its decomposition into P_i's. Figure 14 follows a very similar scheme for $N_{-3,\theta(-2),\theta(-2)}$.

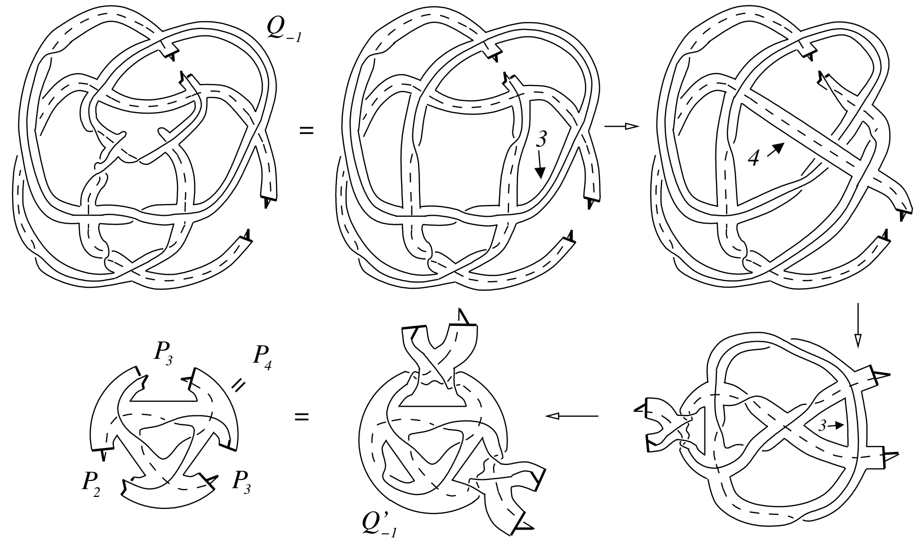

Figure 13: Proof of relation (15)

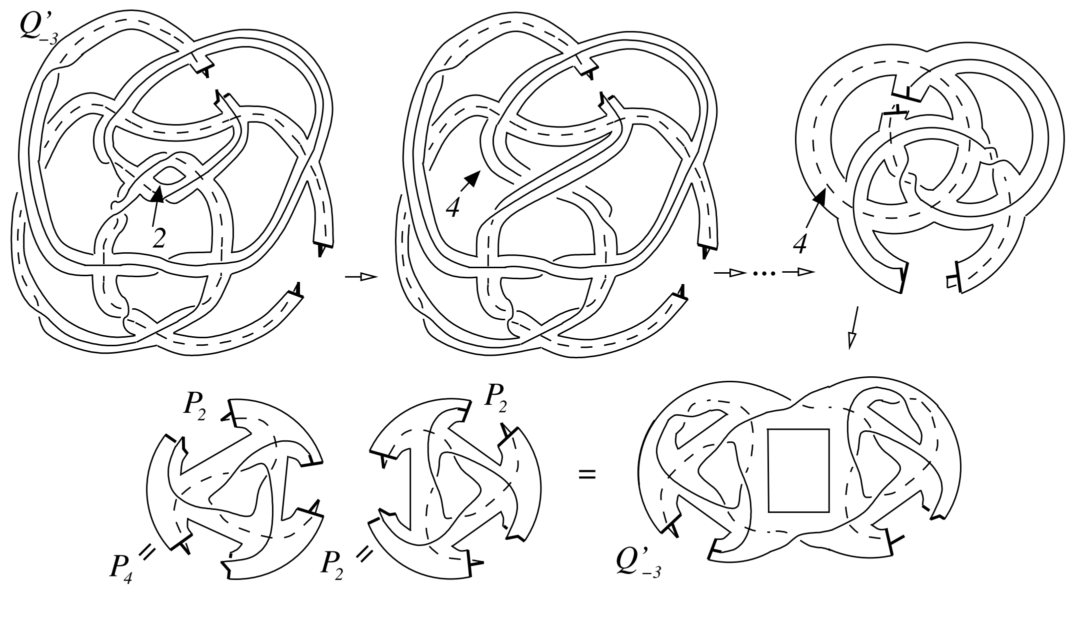

Figure 14: Proof of relation (13)

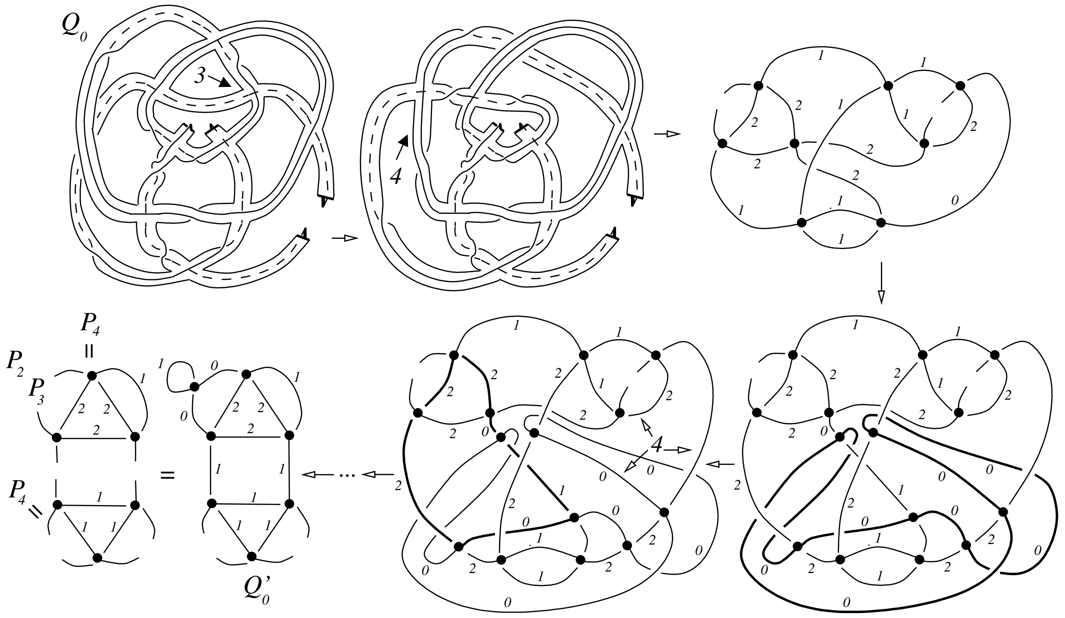

Figure 15: Proof of relation (16)

In Fig. 15 we start with Q_0 and we apply two disc-replacements to it, describing the result by an o-graph (a combinatorial encoding of a standard polyhedron introduced in [3]). We then perform the replacement of a disc having six vertices, showing first an o-graph of the skeleton with the extra disc added (and highlighted in the picture), then the same o-graph with the disc that has to be removed (again highlighted in the picture). We perform more disc-replacements on the resulting polyhedron (starting from a disc having four vertices, as suggested in the picture), eventually getting to Q_0' and its decomposition.

Figure 16 is our most complicated one. We start with $Q_{1,1}$ and then we show $Q'_{1,1}$, omitting the long sequence of moves that relates them. We also mark by solid dots four of the regions of $Q'_{1,1}$. The union of these four regions is a torus embedded in $Q'_{1,1}$, and the figure shows a regular neighbourhood in $Q'_{1,1}$ of this torus, and the complement of this neighbourhood. The figure implies that $N_{1,1,\theta(-2)}$ is a self-assembling of B_4, which (together with some work on the homology bases) implies relation (17)

A proof of relations (18) and (19) would involve skeleta with so many vertices that we have decided to omit the figures altogether. $\square$

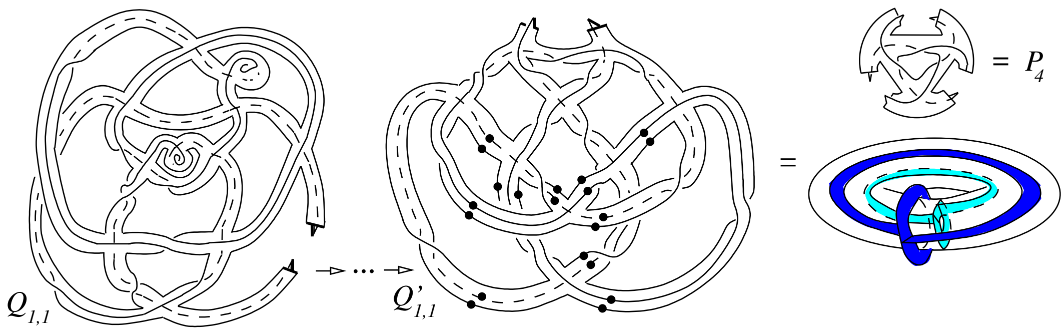

Figure 16: Proof of relation (17)

Classification of fillings We are now ready to prove our main result.

Proof of 1.13. This (very long) proof is organized in the following steps:

1. **Topological preliminaries**: We single out various equalities concerning Seifert fibred spaces which are repeatedly used in the sequel;

2. **Recognition of non-hyperbolic fillings**: We apply Proposition 5.1 to describe various infinite families of triples $(p/q, r/s, t/u)$ that lead to an M which is a graph manifold, so it cannot be negatively curved. In each case we carefully examine the JSJ decomposition of M;

3. **Recognition of more non-hyperbolic fillings**: We apply the symmetries (2) to (7) to the triples found in Step 2, finding more triples that lead to a non negatively curved M. Again we always determine the JSJ decomposition of M;

4. **Application of the 2π-theorem**: we use the Gromov-Thurston 2π-theorem to prove that M is negatively curved except in the cases listed (up to permutation) in Steps 2 and 3, and in a finite number of other cases.

5. **Finite enumeration**: We examine by hand all the cases left uncovered by Steps 2 to 4 and we check that the corresponding M is always hyperbolic.

Along our proof we will sometimes exclude from our discussion triples whose permutations have been considered already. The fact that the list of cases obtained in steps 2 and 3 can be organized as in Tables 1 to 3 is easy and left to the reader.

Step 1: Topological preliminaries Recall that before the statement of Theorem 1.13 we have fixed a homology basis (a_j, b_j) on the j-th boundary component of $D_i \times S^1$, for $j = 0, \ldots, i$. We first note that

$$\big(D_0 \times S^1\big)_{\frac{k}{h}} = L(h,k). \tag{20}$$

Moreover there exists a homeomorphism

$$\begin{array}{c} f : \big(D_2 \times S^1\big)_0 \to (D_0 \times S^1)\#(D_0 \times S^1) \\ \text{such that } f(a_1) = b'_0, \quad f(b_1) = -a'_0, \quad f(a_2) = b''_0, \quad f(b_2) = -a''_0 \end{array} \tag{21}$$

(here we are assuming that the filled component of $D_2 \times S^1$ is the 0-th one, and we are using a'_0, b'_0 for the first copy of $D_0 \times S^1$ and a''_0, b''_0 for the second copy). Equations (20) and (21) readily imply the following:

$$\big(D_2 \times S^1\big)_{\frac{i}{j},\frac{k}{h},0} = L(i,j)\#L(k,h). \tag{22}$$

Next, we note that for $l \in \mathbb{Z}$ there exists a homeomorphism

$$\begin{array}{c} f : D_2 \times S^1 \to D_2 \times S^1 \quad \text{such that } f(b_0) = b_0,\ f(b_1) = b_1,\ f(b_2) = b_2, \\ f(a_0) = a_0 - lb_0,\ f(a_1) = a_1 + lb_1,\ f(a_2) = a_2 \end{array} \tag{23}$$

which gives the relation

$$\big(D_2 \times S^1\big)_{\frac{i}{j},\frac{k}{h}} = \big(D_2 \times S^1\big)_{\frac{i}{j+li},\frac{k}{h-lk}}. \tag{24}$$

Here, and always in the sequel, we use the "=" sign to mean that there exists a homeomorphism which respects the homology bases fixed on the boundary. A direct consequence of this relation is the following one:

$$\big(D_2 \times S^1\big)_{\frac{1}{k},\frac{i}{j}} = \big(D_1 \times S^1\big)_{\frac{i}{j+ki}}. \tag{25}$$

Again (23) easily implies that if X is any manifold with fixed homology bases on the boundary and $\left(\begin{smallmatrix} h & k \\ m & n \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ we have for all $l \in \mathbb{Z}$

$$\big(D_2 \times S^1\big)_{\frac{i}{j}} \bigcup_{\left(\begin{smallmatrix} h & k \\ m & n \end{smallmatrix}\right)} X \;=\; \big(D_2 \times S^1\big)_{\frac{i}{j+li}} \bigcup_{\left(\begin{smallmatrix} h+lk & k \\ m+ln & n \end{smallmatrix}\right)} X, \tag{26}$$

$$X \bigcup_{\left(\begin{smallmatrix} h & k \\ m & n \end{smallmatrix}\right)} \big(D_2 \times S^1\big)_{\frac{i}{j}} \;=\; X \bigcup_{\left(\begin{smallmatrix} h & k \\ m-lh & n-lk \end{smallmatrix}\right)} \big(D_2 \times S^1\big)_{\frac{i}{j+li}}. \tag{27}$$

Now we note that $D_1 \times S^1$ is homeomorphic to the product $T \times [0,1]$ and we let a matrix $\begin{pmatrix} i & i' \\ j & j' \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ act on the T factor, getting an automorphism f of $D_1 \times S^1$ such that

$$f(a_0) = ia_0 + jb_0, \quad f(b_0) = i'a_0 + j'b_0, \quad f(a_1) = ia_1 - jb_1, \quad f(b_1) = -i'a_1 + j'b_1$$

(recall that b_1 is parallel to b_0 in $D_1 \times S^1$, but a_1 is parallel to $-a_0$). So an $(\frac{i}{j}, \frac{k}{h})$-filling of $D_1 \times S^1$ corresponds to a filling of $D_0 \times S^1$ that kills the slope

$$kf^{-1}(a_1) + hf^{-1}(b_1) = (kj' + hi')a_1 + (kj + hi)b_1.$$

Employing also (20) we deduce the following:

$$\left(D_1 \times S^1\right)_{\frac{i}{j}, \frac{k}{h}} = L\left(kj + hi, kj' + hi'\right) \qquad \text{if} \;\; ij' - ji' = \pm 1. \tag{28}$$

Using again the fact that a_1 and b_1 are parallel to $-a_0$ and b_0 on $D_1 \times S^1$ we now easily get:

$$\left(D_1 \times S^1\right)_{\frac{i}{j}} \bigcup\nolimits_{\left(\begin{smallmatrix} h & k \\ m & n \end{smallmatrix}\right)} \left(D_2 \times S^1\right) = \left(D_2 \times S^1\right)_{\frac{hi-kj}{mi-nj}} \quad \text{if} \; \left(\begin{smallmatrix} h & k \\ m & n \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}). \tag{29}$$

The next relations we need refer to the only genuine Seifert manifold with boundary having non-isomorphic fibrations. This manifold can be viewed on one hand as $(D, (2,1), (2,1)) = (D_2 \times S^1)_{2,2}$, so on its boundary we have a fixed section-fibre homology basis (a, b). On the other hand the same manifold can be fibred as the orientable S^1-bundle $S \tilde{\times} S^1$ over the Möbius strip S. And it is a well-known fact that if we select the "obvious" section-fibre positive homology basis (a', b') on $\partial(S \tilde{\times} S^1)$ there exists a homeomorphism

$$f : \left(D_2 \times S^1\right)_{2,2} \to S \tilde{\times} S^1 \text{ such that } f(a) = a' - b', \quad f(b) = a' \tag{30}$$

(this fact will be used again in Lemma 7.7 below). Hence:

$$\left(D_2 \times S^1\right)_{2,2} \bigcup\nolimits_{\left(\begin{smallmatrix} i & j \\ h & k \end{smallmatrix}\right)} X \;\; = \;\; \left(S \tilde{\times} S^1\right) \bigcup\nolimits_{\left(\begin{smallmatrix} j & j-i \\ k & k-h \end{smallmatrix}\right)} X \tag{31}$$

$$X \bigcup\nolimits_{\left(\begin{smallmatrix} i & j \\ h & k \end{smallmatrix}\right)} \left(D_2 \times S^1\right)_{2,2} \;\; = \;\; X \bigcup\nolimits_{\left(\begin{smallmatrix} i+h & j+k \\ -i & -j \end{smallmatrix}\right)} \left(S \tilde{\times} S^1\right). \tag{32}$$

Step 2: Recognition of non-hyperbolic fillings When the triple of filling coefficients is $(1,3,3)$ or $(2,2,2)$, relation (18) or (19) already allows to recognize M. If $p/q = \infty$ we use (12) to show that $M = (D_1 \times S^1)_{\frac{r}{s}, -\frac{u}{t}}$ and then (28) to conclude that $M = L(tr - us, tr' - us')$ where $rs' - sr' = \pm 1$.

We assume from now on that all the filling coefficients are different from ∞, and we employ relations (13) to (17). These relations will typically give the JSJ decomposition of M, but we will have two exceptions to take into account. First, the blocks appearing may be non-genuine Seifert ones. Second, when two blocks are glued together or a block is glued to itself, the gluing may match the fibres (maybe not of the original fibration but of an alternative one, if any). This leads to a comparatively long discussion of cases, during which we will always freely switch the orientation of M without notice. As explicitly mentioned in the statement, we do not discuss the cases where a lens space is S^3, $\mathbb{RP}^3$, or $S^2 \times S^1$. And, when M is a Seifert manifold over S^2 with three exceptional fibres, we do not discuss genuinity, because the JSJ decomposition of M is trivial even in the sporadic cases where M is actually a torus bundle.

The slope -3 We start by considering the case where $p/q = -3$. Relation (13) then shows that M is homeomorphic to

$$\left(D_2 \times S^1\right)_{2,\frac{r+s}{r+2s}} \bigcup\nolimits_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)} \left(D_2 \times S^1\right)_{2,\frac{t+u}{t+2u}}. \tag{33}$$

The first case where the block on the left-hand side is non-genuine occurs when $r/s = -1$, in which case the block becomes $(D_2 \times S^1)_{2,0}$, and by (20) and (21) there exists a homeomorphism

$$f : \left(D_2 \times S^1\right)_{2,0} \to L(2,1)\#(D_0 \times S^1) = \mathbb{RP}^3\#\boldsymbol{T}$$

such that the inverse image under f of a meridian of the solid torus $\boldsymbol{T}$ has slope 0 on the boundary of $(D_2 \times S^1)_{2,0}$. This slope is glued to the slope -1 of the block on right-hand side of (33), so

$$M = \mathbb{RP}^3\#(D_2 \times S^1)_{2,\frac{t+u}{t+2u},-1}.$$

Using (25) with $k = -1$ and $i/j = 2/1$ we see that the second summand is homeomorphic to $(D_1 \times S^1)_{-2,\frac{t+u}{t+2u}}$. Now we employ (28) with $i = 2$, $j = -1$, $i' = 1$, $j' = 0$ to see that this manifold is $L(2(t+2u)-(t+u), t+2u) = L(t+3u, t+2u) = L(t+3u, u)$. Summing up, we have $M = \mathbb{RP}^3\#L(t+3u, u)$ when $r/s = -1$ (and $p/q = -3$).

Another case where the block on left-hand side of (33) is non-genuine occurs when $|r+s| = 1$, namely when $r/s = -1 + 1/n$ for some $n \in \mathbb{Z}$, $n \neq 0$. In this case the block is $(D_2 \times S^1)_{2,\frac{1}{n+1}}$, and we employ (25) with $k = n+1$ and $i/j = 2/1$ to see that the same block can also be described as $(D_1 \times S^1)_{\frac{2}{2n+3}}$. We then use (29) and (24) with $l = -1$ to deduce that

$$M = \left(D_2 \times S^1\right)_{2,\frac{t+u}{t+2u},-\frac{2n+1}{2n+3}} = \left(D_2 \times S^1\right)_{2,\frac{t+u}{u},-\frac{2n+1}{2}}$$

in this case. Now this manifold is the Seifert one

$$\left(S^2, (2,1), (2n+1,-2), (t+u,u)\right)$$

and all three fibres are exceptional unless $|2n+1| \leqslant 1$ or $|t+u| \leqslant 1$. The first inequality is only possible for $n=-1$ (recall that $n \neq 0$), *i.e.* for $r/s=-2$, and $M=(D_2 \times S^1)_{2,\frac{1}{2},\frac{t+u}{u}}$ in this case. Using (25) with $k=2$ and $i/j=2/1$ we see that $M=(D_1 \times S^1)_{\frac{2}{5},\frac{t+u}{u}}$. Now we apply (28) with $i=2, j=5, i'=1, j'=2$ to see that

$$M=L(2u+5(t+u), u+2(t+u))=L(5t+7u, 2t+3u).$$

Let us assume instead that $|t+u| \leqslant 1$. We dismiss the case $t/u=-1$ which was already considered up to permutation, so we have $t/u=-1+1/m$ for $m \in \mathbb{Z}, m \neq 0$, and $M=\left(D_2 \times S^1\right)_{2,\frac{1}{m},-\frac{2n+1}{2}}$. Using (25) with $k=m$ and $i/j=2/1$ we see that

$$M=\left(D_1 \times S^1\right)_{\frac{2}{2m+1},-\frac{2n+1}{2}}.$$

We now apply (28) with $i=2$, $j=2m+1$, $i'=1$, $j'=m$, eventually getting

$$M=L((2n+1)(2m+1)-4,(2n+1)m-2)=L(4nm+2n+2m-3, 2nm+m-2).$$

We also note that the lack of symmetry in n and m is only apparent, because

$$(2nm+m-2)(2nm+n-2) \equiv 1 \pmod{(4nm+2n+2m-3)}.$$

We have now recognized M whenever r/s equals -1 or $-1+1/n$ (and $p/q=-3$). So we assume neither r/s nor t/u have this form, *i.e.* $|r+s|>1$ and $|t+u|>1$. So (33) describes the gluing of two genuine Seifert blocks, and the gluing matrix does not match the fibres of the fibrations implicit in the expressions of the blocks, because these fibres have slope 0, but 0 is matched to -1 and conversely. Therefore (33) gives the JSJ decomposition of M except in the following cases (up to permutation):

(A) the block on left-hand side has an alternative fibration which is matched to the original fibration of the block on the right-hand side;

(B) both the blocks admit alternative fibrations, and these fibrations are matched.

Now the left-hand side has an alternative fibration precisely when $|r+s|=2$, namely when $r/s=-1+\frac{2}{2n+1}$ for some $n \in \mathbb{Z}$, and in this case we deduce from (26) and (31) that

$$\begin{aligned} M &= \left(D_2 \times S^1\right)_{2,\frac{2}{2n+3}} \bigcup_{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}} \left(D_2 \times S^1\right)_{2,\frac{t+u}{t+2u}} \\ &= \left(D_2 \times S^1\right)_{2,2} \bigcup_{\begin{pmatrix} -n & 1 \\ n+1 & -1 \end{pmatrix}} \left(D_2 \times S^1\right)_{2,\frac{t+u}{t+2u}} \\ &= \left(S \widetilde{\times} S^1\right) \bigcup_{\begin{pmatrix} 1 & n+1 \\ -1 & -n-2 \end{pmatrix}} \left(D_2 \times S^1\right)_{2,\frac{t+u}{t+2u}}. \end{aligned}$$

Now case (A) occurs precisely when $n+1=0$, whence $r/s=-3$, and using (27) we see that

$$\begin{aligned} M &= (S\widetilde{\times}S^1)\bigcup_{\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}} (D_2\times S^1)_{2,\frac{t+u}{t+2u}} \\ &= (S\widetilde{\times}S^1)\bigcup_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} (D_2\times S^1)_{2,\frac{t+u}{u}}. \end{aligned}$$

Up to changing sign the gluing matrix is now properly matching both the sections and the fibres, so M is Seifert fibred over the gluing of S and D, which gives

$$M=\big(\mathbb{RP}^2,(2,1),(t+u,u)\big)$$

for $r/s=-3$ (and $p/q=-3$).

Assume case (B) occurs. Then t/u also has the form $-1+\frac{2}{2m+1}$, and by (27) and (32) we have

$$\begin{aligned} M &= (S\widetilde{\times}S^1)\bigcup_{\begin{pmatrix} 1 & n+1 \\ -1 & -n-2 \end{pmatrix}} (D_2\times S^1)_{2,\frac{2}{2m+3}} \\ &= (S\widetilde{\times}S^1)\bigcup_{\begin{pmatrix} 1 & n+1 \\ m & mn+m-1 \end{pmatrix}} (D_2\times S^1)_{2,2} \\ &= (S\widetilde{\times}S^1)\bigcup_{\begin{pmatrix} m+1 & m+n+mn \\ -1 & -n-1 \end{pmatrix}} (S\widetilde{\times}S^1). \end{aligned}$$

Moreover the fibers should match, so $m+n+mn=0$. Therefore we have either $n=m=0$ or $n=m=-2$, namely either $r/s=t/u=1$ or $r/s=t/u=-5/3$. The two manifolds we get are

$$M=(S\widetilde{\times}S^1)\bigcup_{\begin{pmatrix} -1 & 0 \\ \pm1 & 1 \end{pmatrix}} (S\widetilde{\times}S^1)$$

which are easily seen to be $(K,\pm1)$, where K is the Klein bottle. Finally, we have $(K,+1)=(K,-1)=T_{\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}}$.

The slope -2 Let us consider now the case $p/q=-2$, where equation (14) shows that M is

$$(D_2\times S^1)_{\frac{3}{2},-\frac{r+2s}{s},-\frac{t+2u}{u}}.$$

This manifold is Seifert fibred over S^2, and there are three exceptional fibres unless $|r+2s|\leqslant 1$ (up to permutation). If $r/s=-2$, we have $M=(D_2\times S^1)_{\frac{3}{2},0,-\frac{t+2u}{u}}$, which is equal to $L(3,1)\#L(t+2u,u)$ by (22). If $|r+2s|=1$, *i.e.* $r/s=-2+1/n$, then $M = (D_2 \times S^1)_{\frac{3}{2},-\frac{1}{n},-\frac{t+2u}{u}}$, and we can apply (25) with $k = -n$ and $i/j = 3/2$, deducing that $M = (D_1 \times S^1)_{\frac{3}{2-3n},-\frac{t+2u}{u}}$. We further apply (28) with $i = 3$, $j = 2 - 3n$, $i' = 1$, $j' = 1 - n$, getting

$$M = L\big(3u-(2-3n)(t+2u), u-(1-n)(t+2u)\big) = L\big(3n(t+2u)-2t-u, n(t+2u)-t-u\big).$$

If $|r + 2s| > 1$ and $|t + 2u| > 1$, then M is Seifert with three exceptional fibres and equal to

$$\big(S^2, (3, 2), (r + 2s, -s), (t + 2u, -u)\big).$$

The slope -1 The case $p/q = -1$ is similar to the previous one. Relation (15) proves that M is

$$M = (D_2 \times S^1)_{2,-\frac{r+3s}{s},-\frac{t+3u}{u}}.$$

This manifold is Seifert fibred over S^2, and there are three exceptional fibres unless $|r + 3s| \leqslant 1$ up to permutation. If $r/s = -3$ then $M = (D_2 \times S^1)_{2,0,-\frac{t+3u}{u}}$, which is equal to $\mathbb{RP}^3 \# L(t + 3u, u)$ by (22). If $|r + 3s| = 1$, *i.e.* $r/s = -3 + 1/n$, we have $M = (D_2 \times S^1)_{2,-\frac{1}{n},-\frac{t+3u}{u}}$, and we employ (25) with $k = -n$ and $i/j = 2/1$, getting $M = (D_1 \times S^1)_{\frac{2}{1-2n},-\frac{t+3u}{u}}$. Now we use (28) with $i = 2$, $j = 1 - 2n$, $i' = -1$, $j' = n$, deducing that

$$M = L\big(-2u+(1-2n)(t+3u), u+n(t+3u)\big) = L\big(2n(t+3u)-t-u, n(t+3u)+u\big).$$

If $|r + 3s| > 1$ and $|t + 3u| > 1$, then M is Seifert with three exceptional fibres and equal to

$$\big(S^2, (2, 1), (r + 3s, -s), (t + 3u, -u)\big).$$

The slope 0 Let us turn now to the case $p/q = 0$, where (16) shows that M is

$$\big(D_2 \times S^1\big)_{\frac{s}{r+2s},\frac{u}{t+2u}} \bigcup\nolimits_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)} \big(D_2 \times S^1\big)_{2,3}. \tag{34}$$

The right-hand side block is now always genuine, but the left-hand side block is not if $|s| = 1$ (recall that $s \neq 0$), namely when $r/s = n \in \mathbb{Z}$. In this case the block becomes $(D_2 \times S^1)_{\frac{1}{n+2},\frac{u}{t+2u}}$, which is seen to be equal to $(D_1 \times S^1)_{\frac{u}{t+(n+4)u}}$ using (25) with $k = n + 2$. Now we use (29) to see that

$$M = \big(D_2 \times S^1\big)_{2,3,-\frac{t+(4+n)u}{t+(3+n)u}}.$$

So M is the Seifert manifold with three exceptional fibres

$$\begin{aligned} M &= \big(S^2,(2,1),(3,1),(t+(4+n)u,-t-(3+n)u)\big) \\ &= \big(S^2,(2,-1),(3,1),(t+(4+n)u,u)\big) \end{aligned}$$

unless $|t+(4+n)u| \leqslant 1$. If $t+(4+n)u=0$, namely $t/u=-4-n$, we use (22) and see that $M=\mathbb{RP}^3\#L(3,1)$. If $|t+(4+n)u|=1$, *i.e.* $t/u=-4-n+1/m$, we have $M=(D_2\times S^1)_{2,3,\frac{1}{m-1}}$, and we can apply (25) with $k=m-1$ and $i/j=2/1$, getting $M=(D_1\times S^1)_{3,\frac{2}{2m-1}}$. We eventually use (28) with $i=3$, $j=1$, $i'=1$, $j'=0$ and see that $M=L(6m-1,2m-1)$.

We consider now the case where $r/s,t/u\notin\mathbb{Z}$ in (34), so both the blocks involved are genuine Seifert, and the gluing matrix does not match the fibrations. In addition, the right-hand side block has a unique fibration, so the JSJ decomposition of M is given by

$$\big(D,(s,r+2s),(u,t+2u)\big)\bigcup\nolimits_{\left(\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big) \tag{35}$$

unless the left-hand side block has an alternative fibration whose fibre is matched to the fibre of $(D,(2,1),(3,1))$. Assuming this happens, we must have $|s|=|u|=2$, so $r/s=n+1/2$ and $t/u=m+1/2$. The left-hand side block is then $(D_2\times S^1)_{\frac{2}{2n+5},\frac{2}{2m+5}}$. By (26) and (31) we have

$$\begin{aligned} M &= \big(D_2\times S^1\big)_{\frac{2}{2n+5},\frac{2}{2m+5}}\bigcup\nolimits_{\left(\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right)}\big(D_2\times S^1\big)_{2,3} \\ &= \big(D_2\times S^1\big)_{2,2}\bigcup\nolimits_{\left(\begin{smallmatrix}m+n+4&-1\\-m-n-3&1\end{smallmatrix}\right)}\big(D_2\times S^1\big)_{2,3} \\ &= \big(S\widetilde{\times}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}-1&-m-n-5\\1&m+n+4\end{smallmatrix}\right)}\big(D_2\times S^1\big)_{2,3}. \end{aligned}$$

Fibers match for $n+m+5=0$, *i.e.* $m=-n-5$. Summing up, we have shown that for $r/s,t/u\notin\mathbb{Z}$ the JSJ decomposition of M is given by (35) unless $r/s=n+1/2$ and $t/u=-n-9/2$. In this case we have, using (27) with $l=-1$ and switching the orientation of $S\widetilde{\times}S^1$:

$$\begin{aligned} M &= \big(S\widetilde{\times}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}-1&0\\1&-1\end{smallmatrix}\right)}\big(D_2\times S^1\big)_{2,3} \\ &= \big(S\widetilde{\times}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)}\big(D_2\times S^1\big)_{-2,3} \\ &= \big(S\widetilde{\times}S^1\big)\bigcup\nolimits_{\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)}\big(D_2\times S^1\big)_{-2,3} \\ &= \big(\mathbb{RP}^2,(2,1),(3,1),-1\big). \end{aligned}$$

The slopes (1,1) We now study the case $p/q = r/s = 1$, where equation (17) shows that

$$M = \left(D_2 \times S^1\right)_{-\frac{u}{t+2u}} \Big/ \left(\begin{smallmatrix} 1 & -1 \\ -1 & 0 \end{smallmatrix}\right).$$

The block which is being glued to itself is non-genuine if $t/u = n \in \mathbb{Z}$. If this is the case the block becomes $(D_2 \times S^1)_{-\frac{1}{n+2}}$ and we can employ (26) with $l = n+2$, easily deducing that

$$M = \left(D_1 \times S^1\right) \Big/ \left(\begin{smallmatrix} -1-n & -1 \\ -1 & 0 \end{smallmatrix}\right) = T_{\left(\begin{smallmatrix} n+1 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$$

(recall again that a_1 and b_1 on $\partial(D_1 \times S^1)$ are parallel to $-a_0$ and b_0 respectively, whereas the monodromy of a torus bundle is expressed using parallel bases).

If $t/u \notin \mathbb{Z}$ then $(D_2 \times S^1)_{-\frac{u}{t+2u}}$ is genuine Seifert with a unique fibration and the self-gluing does not match the fibres, so the JSJ decomposition of M is given by

$$M = \left(D_1, (u, -t-2u)\right) \Big/ \left(\begin{smallmatrix} 1 & -1 \\ -1 & 0 \end{smallmatrix}\right).$$

Step 3: Recognition of more non-hyperbolic fillings As announced, we use now relations (2) to (7) to find triples which were not listed in Step 2 but give the same fillings as triples listed in Step 2. For the sake of conciseness and clarity, we have organized our constructions in Table 6.

Let us first comment on the first nine lines of this table. Here we find on the first column a triple not found in Step 2 (except for $t/u \in \{-3, -2, -1, 0\}$ in the first two lines). In the second column there is a reference to one of the symmetry relations listed at the beginning of the section. We view this relation as a move acting on the triple to its left, and in the third column we give the triple resulting from this action; this triple is readily seen to be one of those of Step 2. The last two lines are similar, except that in the last column the triple is *not* one of those of Step 2. Instead, it appears in the first column of the ninth and second line respectively.

Let us now recognize M in all the lines listed in the first column of Table 6. Of course this is trivial for all the lines except the first two, because for the other cases we only have to use Step 2 for the triple on the last column (but in Table 3 we have often simplified the expression using formulae (23), (26), and (27), and changes of orientation).

Then, let us start from the case $-\frac{3}{2}, -\frac{5}{2}, \frac{t}{u}$, so M is the same as the manifold obtained with the filling coefficients $0, -\frac{3}{2}, -\frac{2t+5u}{t+2u}$. First, if $t+2u = 0$, *i.e.* $t/u = -2$, we easily get from the case $p/q = \infty$ that $M = L(2,1) = \mathbb{RP}^3$. Otherwise we have to refer to the case $p/q = 0$. Now $-\frac{2t+5u}{t+2u}$ is an integer when $|t + 2u| = 1$,

Original triple	Move	New triple
$-\frac{3}{2}, -\frac{5}{2}, \frac{t}{u}$	(3)	$-\frac{3}{2}, 0, -\frac{u}{t+2u} - 2$
$-4, -\frac{1}{2}, \frac{t}{u}$	inverse of (2)	$-\frac{3}{2}, 0, -\frac{t}{u} - 3$
$-\frac{3}{2}, -\frac{7}{3}, -\frac{7}{3}$	(3)	$-\frac{3}{2}, 1, 1$
$-4, -4, -\frac{2}{3}$	inverse of (2)	$-\frac{3}{2}, 1, 1$
$-\frac{3}{2}, -\frac{3}{2}, -4$	(2)	$-4, 1, 1$
$-\frac{5}{2}, -\frac{5}{2}, -\frac{4}{3}$	(4)	$-\frac{5}{2}, 1, 1$
$-\frac{1}{2}, -5, -5$	(5)	$-\frac{1}{2}, 1, 1$
$1, 2, n$ $n = 2, 3, 4, 5$	(6)	$1, 2, 2 - n$
$1, -4, -\frac{1}{3}$	(7)	$1, -4, -3$
$-\frac{3}{2}, -\frac{3}{2}, -\frac{8}{3}$	(2)	$-4, -\frac{1}{3}, 1$
$-\frac{5}{2}, -\frac{5}{3}, -\frac{5}{3}$	(4)	$-\frac{5}{2}, -4, -\frac{1}{2}$

Table 6: Sporadic triples giving non-hyperbolic fillings

i.e. $t/u = -2 + 1/n$, and in this case $-\frac{2t+5u}{t+2u} = -n - 2$. We further need to make distinctions depending on whether

$$-3 + (4 - (n+2))2 = -2n + 1$$

has value 0 or ± 1. Of course 0 cannot be attained, and $+1$ also cannot (because $n \neq 0$), so the only special case occurs for $n = 1$, namely for $t/u = -1$, and $M = L(13, 5)$, as we knew already from the discussion of slope -1. We then deduce that for $t/u = -2 + 1/n$ but $n \neq 1$ we have

$$M = \big(S^2, (2, -1), (3, 1), (3 - (4 - n - 2)2, -2)\big) = \big(S^2, (2, -1), (3, 1), (2n - 1, -2)\big).$$

When $-\frac{2t+5u}{t+2u}$ is neither an integer nor infinity, *i.e.* when t/u is not -2 or $-2 + 1/n$, we will have, after easy calculations,

$$M = \big(D, (t + 2u, -u), (2, 1)\big) \bigcup_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)} \big(D, (2, 1), (3, 1)\big)$$

except in the special case where $-\frac{2t+5u}{t+2u} = -(-\frac{3}{2}) - 4$, which occurs for $t/u = 0$, and

$$M = \big(\mathbb{RP}^2, (2, 1), (3, 1), -1\big)$$

in this case, as we knew already.

The discussion of the cases $-4, -1/2, t/u$ is easier, because it reduces to that of $0, -3/2, -t/u - 3$. Namely, for $t/u = n$ we have to consider the special cases $n = 0, -1$, leading to $L(13, 5)$ and $L(11, 3)$ respectively (as we knew already), otherwise

$$M = \big(S^2, (2, -1), (3, 1), (2n + 1, -2)\big).$$

Moreover we have the exceptional case where $-t/u - 3 = -(-3/2) - 4$, which occurs for $t/u = -1/2$ and leads to

$$M = \big(\mathbb{RP}^2, (2, 1), (3, 1), -1\big).$$

In all other cases we have

$$M = \big(D, (u, -t - u), (2, 1)\big) \bigcup_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)} \big(D, (2, 1), (3, 1)\big).$$

Now that we have recognized the M's arising from the triples on the first column of Table 6, we could show, with a little patience, that even a repeated application of the symmetries (2) to (7) to any of the triples found so far does not lead to new triples. This will actually be deduced from the rest of our argument, because we will see that all the triples, except those listed so far (up to permutation), give a negatively curved M.

Step 4: Application of the 2π-theorem As announced, we will now use the Gromov-Thurston 2π-theorem [5] to show that a Dehn filling of N is negatively curved for all the triples of filling slopes except those listed in Steps 2 and 3 and a finite number of other ones. More precisely, let us define $\mathcal{G}$ as the set of all slopes, pairs of slopes, and triples of slopes that were proved in Steps 2 and 3 to give a non negatively curved filling of N. The aim of Step 4 is to construct three finite sets of slopes S_1, S_2, S_3 such that *if α, β, γ are slopes on ∂N and $N_{\alpha,\beta,\gamma}$ is not negatively curved then either $(\alpha, \beta, \gamma) \in \mathcal{G}$ or, up to permutation, $(\alpha, \beta, \gamma) \in S_1 \times S_2 \times S_3$.*

We begin by recalling the statement of the 2π-theorem, so we consider a cusped hyperbolic manifold M and a horospherical cusp section H of M, that is an embedded surface parallel to ∂M such that the metric of M restricts to a Euclidean metric on H. Then $H = \sqcup_{i=1}^k T_i'$, where T_i' is a Euclidean torus isotopic to the i-th component T_i of ∂M. If α_i is a slope on T_i we now define the length of α_i with respect to H (or H-length for short) as the shortest length of a curve in T_i' parallel to α_i. The Gromov-Thurston theorem now asserts that $M_{\alpha_{i_1},\ldots,\alpha_{i_h}}$ is negatively curved whenever all filling slopes α_{i_j} have H-length larger than 2π.

Before proceeding we note that if a homology basis (a, b) is fixed on a torus T, the Euclidean structures on T up to isotopy are determined by two parameters $x + iy \in \mathbb{C}$ with $y > 0$ and $A \in \mathbb{R}$ with $A > 0$. Here $x + iy$ gives the *shape* of T (*i.e.* the Euclidean structure up to scaling) while A gives the area. The relation with the fixed basis (a, b) comes from the fact that there should exist a locally isometric covering $\mathbb{C} \to T$ on which a and b act as the translations $z \mapsto z + \sqrt{A/y}$ and $z \mapsto z + (x + iy)\sqrt{A/y}$ respectively. It is now an easy exercise to show that the minimal length $\ell(p/q)$ of a slope $pa + qb$ on T satisfies

$$\ell(p/q)^2 = \frac{A}{y}\big((p + xq)^2 + (yq)^2\big). \tag{36}$$

We now specialize to our manifold $N = M6_1^3$ and note that it has symmetries realizing any permutation of the boundary components, so the shapes of its three cusps are actually the same. We describe this common shape using the homology bases (a_i, b_i) chosen by SnapPea, where $a_i = \mu_i$ and $b_i = 2\mu_i - \lambda_i$ for $i = 1, 2, 3$, with (μ_i, λ_i) being the meridian-longitude basis used above. The shape is then described by the parameter $1/2 + i\sqrt{7}/2$. Moreover, if we choose a maximal horospherical cusp section H consisting of three tori having the same area, this area turns out to be $\sqrt{7}/2$. Using this information and (36) it is now easy to prove that the slopes on any component of ∂N having H-length less that 2π are those in the following set:

$$\begin{aligned}\tilde{S}_1 \;\; = \;\; & \Big\{\infty, -6, -5, -4, -3, -\frac{5}{2}, -2, -\frac{5}{3}, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, -1, \\ & -\frac{3}{4}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3, 4, 5\Big\}.\end{aligned}$$

Now we remark that $\tilde{S}_1 \cap \mathcal{G} = \{\infty, -2, -1, 0, 1\}$ (recall that the homology bases are changed: a slope previously corresponding to p/q is now given by $-2 - p/q$, and conversely). And we set $S_1 = \tilde{S}_1 \setminus \mathcal{G}$, noting that S_1 has 23 elements. Using SnapPea it is now easy to check that N_α is actually hyperbolic for all $\alpha \in S_1$. Moreover, since there is a symmetry of N fixing any boundary component and exchanging the other two, the two cusps of N_α have the same shape given by a certain $z_\alpha \in \mathbb{C}$. Let us consider as above the maximal cusp section H_α in N_α consisting of two tori of the same area, and let us denote by A_α this area. Again using SnapPea one can determine z_α and A_α for all $\alpha \in S_1$, and correspondingly compute, via (36), the set $\tilde{S}_2(\alpha)$ of slopes on (any component of) ∂N_α having H_α-length less than 2π. We now define

$$S_2(\alpha) = \{\beta \in \tilde{S}_2(\alpha) :\ (\alpha, \beta) \notin \mathcal{G}\}, \qquad S_2 = \bigcup\nolimits_{\alpha \in S_1} S_2(\alpha).$$

This set S_2 consists of 55 elements. Again using SnapPea one can now check that $N_{\alpha,\beta}$ is actually hyperbolic for all $(\alpha, \beta) \in (S_1 \times S_2) \setminus \mathcal{G}$, and compute the shape $z_{\alpha,\beta} = x_{\alpha,\beta} + iy_{\alpha,\beta}$ and the area $A_{\alpha,\beta}$ of the maximal cusp $H_{\alpha,\beta}$ of $N_{\alpha,\beta}$. With the list of actual values at hand one can check that

$$-1 \leqslant x_{\alpha,\beta} \leqslant 4, \quad 0.5 \leqslant y_{\alpha,\beta} \leqslant 4, \quad A_{\alpha,\beta} \geqslant 1 \qquad \forall (\alpha, \beta) \in (S_1 \times S_2) \setminus \mathcal{G}$$

(the last inequality is actually known from the theory, but one can also check it experimentally). Now let p/q be a slope on $\partial N_{\alpha,\beta}$. Up to switching signs we can assume $q \geqslant 0$, whence, by (36) and the previous inequalities,

$$\ell(p/q)^2 = \frac{A_{\alpha,\beta}}{y_{\alpha,\beta}}\big((p + q \cdot x_{\alpha,\beta})^2 + (q \cdot y_{\alpha,\beta})^2\big) \geqslant \frac{1}{4}\big(f(p,q)^2 + q^2 \cdot (0.5)^2\big) =: g(p,q)$$

where $f(p,q)$ is 0 if $0 \in (p-q, p+4q)$, and $\min\{|p-q|, |p+4q|\}$ otherwise. We then define $S_3 = \{p/q :\ g(p,q) \leqslant 4\pi^2\}$, and note that S_3 has 200 elements.

Having constructed S_1, S_2, S_3, let us check that they work. If $N_{\alpha,\beta,\gamma}$ is not negatively curved then, up to permutation, we have $\alpha \in \tilde{S}_1$ by the 2π-theorem. So either $\alpha \in S_1$ or $\alpha \in \mathcal{G}$. And $(\alpha, \beta, \gamma) \in \mathcal{G}$ in the latter case. Assuming $\alpha \in S_1$, we know N_α is hyperbolic, so $\beta \in \tilde{S}_2(\alpha)$ up to permutation by the 2π-theorem. So either $\beta \in S_2$, or $(\alpha, \beta) \in \mathcal{G}$, or both. And $(\alpha, \beta, \gamma) \in \mathcal{G}$ if $(\alpha, \beta) \in \mathcal{G}$. Assuming $(\alpha, \beta) \notin \mathcal{G}$ we know that $N_{\alpha,\beta}$ is hyperbolic, so $\gamma \in S_3$ by the 2π-theorem.

Step 5: Finite enumeration To conclude we are left to investigate all the $23 \cdot 55 \cdot 200 = 253000$ triples of slopes in $S_1 \times S_2 \times S_3$, and check that those not lying in $\mathcal{G}$ define hyperbolic fillings. We have done this using the scriptable Pyhthon version of SnapPea. Many of the triples in $(S_1 \times S_2 \times S_3) \setminus \mathcal{G}$ turned out to give solutions of the hyperbolic Dehn filling equations with some inverted tetrahedra. For these triples we have tried with several alternative triangulations of N and of some partial fillings of N, getting a genuine solution after some attempts. The only triple resisting to this method was $\{1, -5, -1/2\}$, which is known [10] to give the closed hyperbolic manifold with volume $1.0149416\ldots$ (and complexity 9). $\square$

6 Complexity of negatively curved manifolds

In this section we pursue our analysis of the Dehn fillings of $M6_1^3$. In particular, we compute c_8 and c_9 on the negatively curved fillings (proving Theorem 1.16) and we show that all the other fillings are degenerate. The latter result will be used in Section 7 to conclude the computation of c_n on Seifert manifolds. We keep using the same notation fixed in the previous section. In particular we denote $M6_1^3$ by N and we use the homology bases fixed on its boundary to encode slopes and triods.

Computation of c_8 The non-closed bricks of complexity 8 are the Seifert brick B_5 already discussed above and $B_6 = N_{1,-4,\theta^{(-1)}}$, based on the "figure-8 knot sister" $N_{1,-4} = M2_2^1$. Even if we will not need this information, it is worth mentioning that the slopes $-1, 0, \infty$ contained in $\theta^{(-1)}$ are geometrically remarkable with respect to the hyperbolic metric of $M2_2^1$, because -1 is the shortest slope, and $0, \infty$ are the unique slopes having the second shortest length. We also note that there is an orientation-reversing isometry f of $N_{1,-4}$ which acts on $H_1(\partial N_{1,-4})$ as $(p, q) \mapsto (q, p)$ —this is the isometry which gives rise to the symmetry (7) of Proposition 1.17. The action of f on slopes and triods, when translated into an action on $\partial_{\mathbb{Q}}\mathbb{H}^2$ and $\mathcal{T}^{(0)}$, is induced by the reflection with respect to the geodesic with ends at -1 and 1. In particular, f maps $\theta^{(-1)}$ to itself.

We start with the following easy result, which implies the first half of Theorem 1.16:

Proposition 6.1. *Suppose M is closed, irreducible, atoroidal, and not a Seifert manifold. Then $c_8(M)$ equals the minimum of $7 + |p, -q|$ over all realizations of M as $N_{1,-4,p/q}$.*

Proof. By Proposition 4.7-(1), $c_8(M)$ is the minimum of $c(M', Y) + d(\alpha, Y)$ with $(M', Y) \in \mathcal{B}_{\leqslant 8}$ and $M = M'_\alpha$. Since no filling of a Seifert manifold can be irreducible, atoroidal, and non-Seifert, we deduce that (M', Y) must be B_6. So $c_8(M)$ is the minimum of $8 + d(p/q, \theta^{(-1)})$ over all realizations of M as $N_{1,-4,p/q}$. Now $d(p/q, \theta^{(-1)}) = |p + q, q| - 1$ by Corollary 3.4, and it is easy to see that $|p + q, q| = |p, -q|$, using Proposition 3.3 and the matrix S_1 of Section 1, which fixes $\theta^{(0)}$ and maps $(p + q)/q$ to $-p/q$. $\square$

Now we turn to the discussion of non negatively curved fillings of $N_{1,-4}$. Before doing this we prove certain complexity estimates repeatedly used below. Our statement involves the norm $|A|$ defined in Section 1 for $A \in \mathrm{GL}_2(\mathbb{Z})$. It also uses the homology bases fixed on $\partial(D_2 \times S^1)$ before Theorem 1.13 to encode slopes and triods.

Lemma 6.2. *1. If α_1 and α_2 are slopes or triods on distinct boundary components of $D_2 \times S^1$ then*

$$c_3\big((D_2 \times S^1)_{\alpha_1,\alpha_2,\theta^{(-1)}}\big) \leqslant 3 + d(\theta^{(0)}, \alpha_1) + d(\theta^{(0)}, \alpha_2).$$

2. Let $\left(\begin{smallmatrix} i & j \\ h & k \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$. Assume (α_1, α_2) and (α_3, α_4) are pairs as in the previous point. Then:

$$c_3\left((D_2 \times S^1)_{\alpha_1,\alpha_2} \bigcup_{\left(\begin{smallmatrix} i & j \\ h & k \end{smallmatrix}\right)} (D_2 \times S^1)_{\alpha_3,\alpha_4}\right) \leqslant 6 + \sum_{l=1}^{4} d(\alpha_l, \theta^{(0)}) + \left|\left(\begin{smallmatrix} i & -j \\ -h & k \end{smallmatrix}\right)\right|.$$

3. If α is a slope or a triod on $\partial(D_2 \times S^1)$ then

$$c_3\left((D_2 \times S^1)_\alpha \Big/ \left(\begin{smallmatrix} 1 & -1 \\ -1 & 0 \end{smallmatrix}\right)\right) \leqslant 9 + d(\alpha, \theta^{(-1)}).$$

Proof. Point 1 readily follows from Proposition 4.1, using $B_4 = (D_2 \times S^1)_{\theta^{(0)},\theta^{(0)},\theta^{(-1)}}$, because $c(B_4) = 3$.

For point 2 we apply the inequality of point 1 to both the blocks involved, and we change the homology bases on the boundary of both blocks, performing a change of type $(a, b) \mapsto (-a, b)$. In the new basis the old triod $\theta^{(-1)}$ is transformed into $\theta^{(0)}$, and the gluing matrix becomes $A = \left(\begin{smallmatrix} i & -j \\ -h & k \end{smallmatrix}\right)$. Now Proposition 4.1 implies that the gluing matching $\theta^{(0)}$ to $A\theta^{(0)}$ can be realized by insertion of $d(\theta^{(0)}, A\theta^{(0)})$ copies of B_3 and assembling. But $d(\theta^{(0)}, A\theta^{(0)}) = |A|$ by Corollary 3.5 and the conclusion follows.

Turning to point 3, we note that $c_3(D_2 \times S^1)_{\alpha,\theta^{(0)},\theta^{(0)}} \leqslant 3 + d(\alpha, \theta^{(-1)})$, again from Proposition 4.1 applied to B_4. Now the gluing map with matrix $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ matches $\theta^{(0)}$ to $\theta^{(-1)}$, and $d(\theta^{(0)}, \theta^{(-1)}) = 1$, so the map can be realized via a self-assembling. We deduce the upper bound $6 + 3 + d(\alpha, \theta^{(-1)})$ on c_3, and the proof is complete. □

The following result will be used in Propositions 7.10 and 7.11 to show that B_6 gives no contribution to $c_8(M)$ if M is a Seifert manifold or a torus bundle.

Proposition 6.3. *Every slope on $B_6 = N_{1,-4,\theta^{(-1)}}$ producing a manifold that is not negatively curved is degenerate.*

Proof. Corollary 1.14 shows that the slopes that me must prove to be degenerate are precisely $\infty, -3, -2, -1, -1/2, -1/3, 0, 1$. The slopes $\infty, -1, 0$ are degenerate by Proposition 4.8-(1). It follows from Theorem 1.13 that

$$\begin{aligned} N_{1,-4,-3} = N_{1,-4,-1/3} &= \big(D,(2,1),(2,1)\big)\bigcup\nolimits_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big) \\ &= \big(D_2\times S^1\big)_{2,2}\bigcup\nolimits_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)}\big(D_2\times S^1\big)_{2,3}. \end{aligned}$$

Lemma 6.2-(2) then implies that

$$\begin{array}{c} c_3(N_{1,-4,-3}) = c_3(N_{1,-4,-1/3}) \leqslant 6+0+0+0+1+0 = 7 \\ < 9 = c(B_6) + d(-3,\theta^{(-1)}) = c(B_6) + d(-1/3,\theta^{(-1)}) \end{array} \tag{37}$$

so -3 and $-1/3$ are degenerate. For the slopes -2 and $-1/2$ we use the following equality, again taken from Theorem 1.13:

$$N_{1,-4,-2} = N_{1,-4,-1/2} = \big(S^2,(3,2),(3,2),(2,1),-1\big).$$

We can then apply Proposition 4.14-(4) to see that

$$c_3(N_{1,-4,-2}) = c_3(N_{1,-4,-1/2}) = 2+2+1+(-1)+1 = 5 < 8 = c(B_6).$$

Therefore -2 and $-1/2$ are also degenerate. We are left to deal with the slope 1. By Theorem 1.13 we have $N_{1,-4,1} = T_A$ with $A = \left(\begin{smallmatrix} -3 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Now $\|A\| \leqslant d(\theta^{(-1)}, A\theta^{(-1)})$ by Corollary 3.5, but $A\theta^{(-1)} = \theta^{(-3)}$, so $\|A\| \leqslant 2$, and $c_1(T_A) \leqslant 7$ by Proposition 4.13. Since $7 < 8 = c(B_6) + d(1,\theta^{(-1)})$, we deduce that 1 is also degenerate, and the proof is complete. $\square$

Computation of c_9 The bricks B_6, B_7, B_8, B_9, and B_{10} are based respectively on $M2_2^1$, $M3_4^1$, $M4_1^2$, $N = M6_1^3$, and N again (as above we use the notation of [7] for cusped hyperbolic manifolds). Each of these manifolds can be obtained as an appropriate Dehn filling of N. So for $i = 6,\ldots,10$ we can describe B_i as $N_{\alpha^{(i)}}$ where $\alpha^{(i)}$ is a triple of slopes and/or triods. Due to the symmetries of N the triples $\alpha^{(i)}$ are not unique, but we choose the following definite ones:

$$\alpha^{(6)} = \big(1,-4,\theta^{(-1)}\big), \qquad \alpha^{(7)} = \big(1,-5,\theta^{(-1)}\big), \qquad \alpha^{(8)} = \big(1,\theta^{(-2)},\theta^{(-2)}\big),$$

$$\alpha^{(9)} = \big(\theta^{(-2)},\theta^{(-2)},\theta^{(-2)}\big), \qquad \alpha^{(10)} = \big(\theta^{(-2)},\theta^{(-2)},\theta^{(-3)}\big).$$

The proof that indeed $B_i = N_{\alpha^{(i)}}$ for $i = 6, \ldots, 10$ uses the explicit description of the B_i's given in [11].

As opposed to what we did for c_8, we start for c_9 with the non negatively curved fillings of N, because we have not proved yet well-behavedness on tori of $n = 9$. This property is readily implied by the next result, using Remark 4.10 and the information already shown on B_4 in Proposition 4.14. The next result will also be used in Propositions 7.10 and 7.11 to show that c_9 equals c_8 on Seifert manifolds and torus bundles.

Proposition 6.4. *If $B \in \{B_7, \ldots, B_{10}\}$ and α is a set of slopes on ∂B such that B_α is not negatively curved then α is degenerate.*

Proof. A by-product of our proof of Theorem 1.13 is that N_α is not negatively curved precisely if one of the following happens:

(A) α contains one of the slopes -3, -2, -1, 0, or ∞;

(B) α contains one of the pairs of slopes $(1, 1)$, or $(-3/2, -5/2)$, or $(-4, -1/2)$;

(C) α is one of the 14 triples listed in Table 3.

(One may also deduce this fact using only the statement of Theorem 1.13, rather than its proof, but one would need to use the hyperbolization theorem for negatively curved manifolds with non-empty boundary, the hyperbolic Dehn filling theorem, and the fact that a filling of a manifold which is not negatively curved is also not negatively curved.)

Since $c(B_i) > 2$ for $i = 7, \ldots, 10$ we can now apply Proposition 4.8-(2). According to this result, to prove our proposition it is sufficient to show degeneracy with respect to B_i of the α's as follows:

(a) α is a triple of slopes and triods obtained from $\alpha^{(i)}$ by removing some triods and inserting slopes instead of them;

(b) α satisfies one of the properties (A), (B), or (C);

(c) it is impossible to attain one of the properties (A), (B), or (C) with a smaller number of triod-to-slope replacements on $\alpha^{(i)}$.

To prove that all such α's are degenerate, we first establish upper bounds on $c_9(N_\alpha)$ for all α's obtained as in (a)-(c) from $\alpha^{(9)}$. Our inequalities use many facts already established, together with the easy relation

$$\left\| \left(\begin{smallmatrix} n & -1 \\ 1 & 0 \end{smallmatrix} \right) \right\| \leqslant d\left(\theta^{(-1)}, \left(\begin{smallmatrix} n & -1 \\ 1 & 0 \end{smallmatrix} \right) \theta^{(-1)}\right) = d(\theta^{(-1)}, \theta^{(n)}) = |n+1| \qquad (38)$$

(already used with $n=-3$ in the proof of Proposition 6.3). To save space while helping the reader follow our arguments, we will write under each equality or inequality sign the equation or statement it is implied by:

$$\begin{aligned} c_9(N_{\infty,\theta^{(-2)},\theta^{(-2)}}) &\underset{(12)}{=} c_9\big(T\times[0,1],\theta^{(-2)}\times\{0\},\theta(-1,-1/2,0)\times\{1\}\big) \\ &\underset{\text{Prop. 4.1}}{\leqslant} d(\theta^{(-2)},\theta(-1,-1/2,0))=2 \end{aligned} \tag{39}$$

(here $\theta(-1,-1/2,0)$ denotes the triod containing the slopes -1, $-1/2$, and 0).

$$\begin{aligned} c_9(N_{-3,\theta^{(-2)},\theta^{(-2)}}) &\underset{(13)}{=} c_9\left((D_2\times S^1)_{2,\theta^{(0)}}\bigcup_{\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right)}(D_2\times S^1)_{2,\theta^{(0)}}\right) \\ &\underset{\text{Lem. 6.2(2)}}{\leqslant} 6+0+0+0+0+0=6. \end{aligned} \tag{40}$$

$$c_9(N_{-2,\theta^{(-2)},\theta^{(-2)}}) \underset{(14)}{=} c_9\big((D_2\times S^1)_{3/2,\theta^{(-1)},\theta^{(-1)}}\big) \underset{\text{Lem. 6.2(1)}}{\leqslant} 3+1+1=5. \tag{41}$$

$$\begin{aligned} c_9(N_{-1,\theta^{(-2)},\theta^{(-2)}}) &\underset{(15)}{=} c_9\big((D_2\times S^1)_{2,\theta^{(-2)},\theta^{(-2)}}\big) \\ &\underset{\text{Prop. 4.1}}{\leqslant} c_9\big((D_2\times S^1)_{2,\theta^{(-2)},\theta^{(-1)}}\big)+d(\theta^{(-2)},\theta^{(-1)}) \\ &\underset{\text{Lem. 6.2(1)}}{\leqslant} 3+0+2+1=6. \end{aligned}$$

$$\begin{aligned} c_9(N_{0,\theta^{(-2)},\theta^{(-2)}}) &\underset{(16)}{=} c_9\left((D_2\times S^1)_{\theta^{(0)},\theta^{(0)}}\bigcup_{\left(\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right)}(D_2\times S^1)_{2,3}\right) \\ &\underset{\text{Lem. 6.2(2)}}{\leqslant} 6+0+0+0+1+0=7. \end{aligned} \tag{42}$$

$$c_9(N_{1,1,\theta^{(-2)}}) \underset{(17)}{=} c_9\left((D_2\times S^1)_{\theta^{(-1)}}\Big/\left(\begin{smallmatrix}1&-1\\-1&0\end{smallmatrix}\right)\right) \underset{\text{Lem. 6.2(3)}}{\leqslant} 9+0=9. \tag{43}$$

$$\begin{aligned} c_9(N_{-4,-1/2,\theta^{(-2)}}) &\underset{(2)^{-1}}{=} c_9(N_{-3/2,0,\theta^{(-2)}}) \\ &\underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{0,\theta^{(-2)},\theta^{(-2)}})+d(-3/2,\theta^{(-2)}) \underset{(42)}{\leqslant} 7+0=7. \end{aligned}$$

$$c_9(N_{-5/2,-3/2,\theta^{(-2)}}) \underset{(4)}{=} c_9(N_{-5/2,-3,\theta^{(-2)}})$$
$$\underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{\theta^{(-2)},-3,\theta^{(-2)}}) + d(-5/2,\theta^{(-2)}) \underset{(40)}{\leqslant} 6+1=7.$$

$$c_9(N_{-5,-5,-1/2}) \underset{(5)}{=} c_9(N_{1,1,-1/2}) \underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{1,1,\theta^{(-2)}}) + d(-1/2,\theta^{(-2)}) \underset{(43)}{\leqslant} 9+1=10.$$

$$c_9(N_{-4,-4,-2/3}) \underset{(2)}{=} c_9(N_{-3/2,1,1})$$
$$\underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{1,1,\theta^{(-2)}}) + d(-3/2,\theta^{(-2)}) \underset{(43)}{\leqslant} 9+0=9. \qquad (44)$$

$$c_9(N_{-3/2,-7/3,-7/3}) \underset{(3)}{=} c_9(N_{-3/2,1,1}) \underset{(44)}{\leqslant} 9.$$

$$c_9(N_{-4,-3/2,-3/2}) \underset{\text{Thm. 1.13}}{=} c_9\left(T_{\left(\begin{smallmatrix} -3 & -1 \\ 1 & 0 \end{smallmatrix}\right)}\right) \underset{\text{Prop. 4.13}}{\leqslant} 5 + \left\|\left(\begin{smallmatrix} -3 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right\| \underset{(38)}{\leqslant} 5+2=7.$$

$$c_9(N_{-8/3,-3/2,-3/2}) \underset{(2)}{=} c_9(N_{-4,-1/3,1}) \underset{(7)}{=} c_9(N_{1,-4,-3}) \underset{(37)}{\leqslant} 7.$$

$$c_9(N_{-5/2,-5/2,-4/3}) \underset{(4)}{=} c_9(N_{-5/2,1,1}) \underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{1,1,\theta^{(-2)}}) + d(-5/2,\theta^{(-2)}) \underset{(43)}{\leqslant} 9+1=10.$$

$$c_9(N_{-5/2,-5/3,-5/3}) \underset{\text{Thm. 1.13}}{=} c_9\left((D_2\times S^1)_{2,2}\bigcup\nolimits_{\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)}(D_2\times S^1)_{2,3}\right)$$
$$\underset{\text{Lem. 6.2(2)}}{\leqslant} 6+0+0+0+1+1=8.$$

$$c_9(N_{1,2,2}) \underset{\text{Thm. 1.13}}{=} c_9\big(S^2,(2,1),(3,1),(7,1),-1\big) = 7$$

(the last equality holds because $\big(S^2,(2,1),(3,1),(7,1),-1\big)$ is the closed brick E_2 of $\mathcal{M}^*$, and its complexity is 7).

$$c_9(N_{1,2,3}) \underset{\text{Thm. 1.13}}{=} c_9\big(S^2,(2,1),(4,1),(5,1),-1\big) \underset{\text{Prop. 4.14-(4)}}{\leqslant} 1+3+4+(-1)+1=8.$$

$$c_9(N_{1,2,4}) \underset{\text{Thm. 1.13}}{=} c_9\big(S^2,(3,1),(3,1),(4,1),-1\big) \underset{\text{Prop. 4.14-(4)}}{\leqslant} 2+2+3+(-1)+1=7.$$

$$\begin{aligned} c_9(N_{1,2,5}) &\underset{\text{Thm. 1.13}}{=} c_9\left((D_2\times S^1)_{2,2}\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}(D_2\times S^1)_{2,3}\right)\\ &\underset{\text{Lem. 6.2(2)}}{\leqslant} 6+0+0+0+1+0=7.\end{aligned}$$

$$\begin{aligned} c_9(N_{1,3,3}) &\underset{\text{Thm. 1.13}}{=} c_9\left((D_2\times S^1)_{2,3}\bigcup_{\left(\begin{smallmatrix}1&2\\0&-1\end{smallmatrix}\right)}(D_2\times S^1)_{2,2}\right)\\ &\underset{\text{Lem. 6.2(2)}}{\leqslant} 6+0+1+0+0+1=8.\end{aligned}$$

$$\begin{aligned} c_9(N_{2,2,2}) &\underset{\text{Thm. 1.13}}{=} c_9\left((D_2\times S^1)_{2,3}\bigcup_{\left(\begin{smallmatrix}2&3\\-1&-2\end{smallmatrix}\right)}(D_2\times S^1)_{2,2}\right)\\ &\underset{\text{Lem. 6.2(2)}}{\leqslant} 6+0+1+0+0+2=9.\end{aligned} \tag{45}$$

Using these inequalities, we prove now that the triples α obtained as explained in (a)-(c) from $\alpha^{(i)}$ are degenerate with respect to B_i for $i=7,\ldots,10$. We start with $i=7$. Since $\alpha^{(7)}=(1,-5,\theta^{(-1)})$ the only triples we have to consider are $(1,-5,n)$ for $n=-3,-2,-1,0,1,\infty$. Now $-1,0,\infty$ are degenerate because they are contained in $\theta^{(-1)}$. And the slopes -3, -2, and 1 are degenerate because

$$\begin{aligned} c_9(N_{1,-5,-3}) &\underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{\theta^{(-2)},\theta^{(-2)},-3})+d(\theta^{(-2)},1)+d(\theta^{(-2)},-5)\\ &\underset{(40)}{\leqslant} 6+1+2=9<10=9+1=c(B_7)+d(-3,\theta^{(-1)})\end{aligned}$$

$$\begin{aligned} c_9(N_{1,-5,-2}) &\underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{\theta^{(-2)},\theta^{(-2)},-2})+d(\theta^{(-2)},1)+d(\theta^{(-2)},-5)\\ &\underset{(41)}{\leqslant} 5+1+2=8<9=9+0=c(B_7)+d(-2,\theta^{(-1)})\end{aligned}$$

$$\begin{aligned} c_9(N_{1,1,-5}) &\underset{\text{Thm. 1.13}}{=} c_9\left(T_{\left(\begin{smallmatrix}4&1\\-1&0\end{smallmatrix}\right)}\right) \underset{\text{Prop. 4.13}}{\leqslant} 5+\left\|\left(\begin{smallmatrix}4&1\\-1&0\end{smallmatrix}\right)\right\|\\ &\underset{(38)}{\leqslant} 5+3=8<9=9+0=c(B_7)+d(1,\theta^{(-1)}).\end{aligned}$$

To conclude for $i = 8, 9, 10$ we now claim that for all the triples α obtained as explained in (a)-(c) from $\alpha^{(9)}$ the following inequality holds:

$$c_9(N_\alpha) < 8 + d(\alpha, \alpha^{(9)}). \tag{46}$$

Before proving (46), we show that it is sufficient to conclude not only for $i = 9$, which is obvious since $c(B_9) = 9$, but also for $i = 8$ and $i = 10$. We start with $i = 10$ and assume that α is obtained as in (a)-(c) from $\alpha^{(10)}$ by some triod-to-slope replacement. Let $\tilde{\alpha}$ be obtained by performing the same replacements on $\alpha^{(9)}$. Then (46) applies to $\tilde{\alpha}$. Using the fact that $d(\alpha^{(9)}, \alpha^{(10)}) = d(\theta^{(-2)}, \theta^{(-3)}) = 1$, it is also easy to see that either $d(\alpha, \tilde{\alpha}) = 0$ and $d(\tilde{\alpha}, \alpha^{(9)}) \leqslant d(\alpha, \alpha^{(10)}) + 1$, or $d(\tilde{\alpha}, \alpha) = 1$ and $d(\tilde{\alpha}, \alpha^{(9)}) = d(\alpha, \alpha^{(10)})$. In particular, the following holds anyway:

$$d(\tilde{\alpha}, \alpha^{(9)}) + d(\tilde{\alpha}, \alpha) \leqslant d(\alpha, \alpha^{(10)}) + 1. \tag{47}$$

Hence

$$\begin{aligned} c_9(N_\alpha) & \underset{\text{Prop. 4.1}}{\leqslant} c_9(N_{\tilde{\alpha}}) + d(\tilde{\alpha}, \alpha) \underset{(46)}{<} 8 + d(\tilde{\alpha}, \alpha^{(9)}) + d(\tilde{\alpha}, \alpha) \\ & \underset{(47)}{\leqslant} 9 + d(\alpha, \alpha^{(10)}) = c(B_{10}) + d(\alpha, \alpha^{(10)}). \end{aligned}$$

For $i = 8$ a very similar argument applies, except that to define $\tilde{\alpha}$ it is necessary in some cases to replace $\theta^{(-2)}$ by 1, to ensure that one of the properties (A), (B), or (C) is fulfilled. The key point is in any case that $d(\alpha^{(8)}, \alpha^{(9)}) = d(1, \theta^{(-2)}) = 1$.

Having shown that (46) allows to conclude the proof, we then establish it. Recall that the α's to consider are precisely those appearing in the left-hand side of the inequalities (39) to (45), and there is nothing to prove whenever $d(\alpha, \alpha^{(9)}) \geqslant 0$ and the right-hand side is 7 or less. In the other cases we get the result by computing $d(\alpha, \alpha^{(9)})$ and showing that it is greater than -8 plus the number on the right-hand side:

$$
\begin{array}{rcl}
d((\infty,\theta^{(-2)},\theta^{(-2)}),\alpha^{(9)}) & = & -1>2-8 \\
d((-2,\theta^{(-2)},\theta^{(-2)}),\alpha^{(9)}) & = & -1>5-8 \\
d((-1,\theta^{(-2)},\theta^{(-2)}),\alpha^{(9)}) & = & -1>6-8 \\
d((1,1,\theta^{(-2)}),\alpha^{(9)}) & = & 2>9-8 \\
d((-5,-5,-1/2),\alpha^{(9)}) & = & 5>10-8 \\
d((-4,-4,-2/3),\alpha^{(9)}) & = & 4>9-8 \\
d((-7/3,-7/3-3/2,),\alpha^{(9)}) & = & 4>9-8 \\
d((-5/2,-5/2,-4/3),\alpha^{(9)}) & = & 3>10-8 \\
d((-5/2,-5/3,-5/3),\alpha^{(9)}) & = & 3>8-8 \\
d((1,2,2),\alpha^{(9)}) & = & 5>9-8 \\
d((1,2,3),\alpha^{(9)}) & = & 6>8-8 \\
d((1,3,3),\alpha^{(9)}) & = & 7>8-8 \\
d((2,2,2),\alpha^{(9)}) & = & 6>9-8.
\end{array}
$$

Our argument is now complete □

Having proved eventually that 9 is well-behaved on tori, we are now ready to apply Proposition 4.7-(1) to compute c_9 on the negatively curved fillings of N. To do this we need to slightly extend the "distance" function d introduced in Section 3. Recall that $d(\alpha,\alpha')$ was originally defined when α is either a triod or a slope on a given torus, and α' is a triod. Now we also allow the case where both α and α' are slopes, defining $d(\alpha,\alpha')$ to be 0 if they coincide and $+\infty$ if they do not.

Lemma 6.5. *Let (M,X) be a pair. If M is irreducible, atoroidal, and not Seifert, then $c_9(M)$ equals the minimum of $c(B_i)+d(\alpha,\alpha^{(i)})$ over all realizations of (M,X) as N_α and all $6\leqslant i\leqslant 10$.*

Proof. Since $B_1,\ldots,B_5$ are Seifert, M is not a Dehn filling of $B_1,\ldots,B_5$. Denoting B_i by $(M^{(i)},Y^{(i)})$, Proposition 4.7-(1) then implies that $c_9(M,X)$ equals the minimum of $c(B_i)+d(\alpha,Y^{(i)})$ over all realizations of (M,X) as $M^{(i)}_\alpha$ for $i\in\{6,\ldots,10\}$.

Let $k(M,X)$ be the minimum described in the statement, that we want to show to be equal to $c_9(M,X)$. We denote the components of ∂N by T_1,T_2,T_3 and note that $B_i=N_{\alpha^{(i)}}$, so $\partial M^{(i)}$ is a union of some T_j's.

To prove that $c_9(M,X)\geqslant k(M,X)$ it is sufficient to show that if $(M,X)=M^{(i)}_\alpha$ then there exists $\tilde\alpha$ such that $(M,X)=N_{\tilde\alpha}$ and $d(\alpha,Y^{(i)})=d(\tilde\alpha,\alpha^{(i)})$. The required

$\tilde{\alpha}$ is given as follows:

$$\tilde{\alpha}(T_j) = \begin{cases} \alpha(T_j) & \text{if } \alpha^{(i)}(T_j) \in \Theta(T_j), \\ \alpha^{(i)}(T_j) & \text{if } \alpha^{(i)}(T_j) \in \mathcal{S}(T_j). \end{cases}$$

Inequality $c_9(M,X) \leqslant k(M,X)$ is a consequence of the following fact: if $(M,X) = N_\alpha$ and $d(\alpha, \alpha^{(i)}) < \infty$ then there exists $\tilde{\alpha}$ such that $(M,X) = M^{(i)}_{\tilde{\alpha}}$ and $d(\alpha, Y^{(i)}) = d(\tilde{\alpha}, \alpha^{(i)})$. We simply set $\tilde{\alpha}(T_j) = \alpha(T_j)$ for $T_j \subset \partial M^{(i)}$. $\square$

Recall that at the end of Section 1 we have defined a certain function h on triples of rational numbers. The next result implies the second part of Theorem 1.16.

Proposition 6.6. *Assume M is irreducible, closed, atoroidal, and not a Seifert manifold. Then $c_9(M)$ equals the minimum of $h(p/q, r/s, t/u)$ over all realizations of M as $N_{p/q,r/s,t/u}$.*

Proof. By Lemma 6.5, it is sufficient to fix $p/q, r/s, t/u$ and prove that $h(p/q, r/s, t/u)$ is $\min\{k_6, \ldots, k_{10}\}$ where $k_i = c(B_i) + d\big((p/q, r/s, t/u), \alpha^{(i)}\big)$.

We first note that $d(p/q, \theta^{(-2)}) = |p+2q, q| - 1$, whence $k_9 = 6 + |p+2q, q| + |r+2s, s| + |t+2u, u|$. We will compute our minimum by systematically analyzing for $i \neq 9$ the triples on which k_i is smaller than k_9. We start with k_{10} and note that it is smaller than k_9 precisely when $p/q < -2$ up to permutation, and $k_{10} = -1 + k_9$ in this case.

Concerning k_8, we note that it is finite precisely when $p/q = 1$ up to permutation. Moreover $d(1, \theta^{(-2)}) = 1$, so k_8 is smaller than k_9 for $p/q = 1$, and again the difference is 1.

Now k_7 is finite precisely when $p/q = 1$ and $r/s = -5$ up to permutation. Moreover $d(1, \theta^{(-2)}) = 1$ and $d(-5, \theta^{(-2)}) = 2$, so there is a -3 gain on the first two coordinates. And $d(\theta^{(-1)}, \theta^{(-2)}) = 1$, so there is a global gain whatever the last coordinate. More precisely, there is a further -1 gain with respect to k_9 when $t/u > -1$, and a $+1$ loss if $t/u < -1$. Summing up we have $k_7 = -4 + k_9$ if $p/q = 1, r/s = -5, t/u > -1$ up to permutation, and $k_7 = -2 + k_9$ if $p/q = 1, r/s = -5, t/u < -1$ up to permutation.

We are left to analyze k_6, which is finite when $p/q = 1$ and $r/s = -4$ up to permutation. Since $d(1, \theta^{(-2)}) = d(-4, \theta^{(-2)}) = 1$ we have a -2 gain on the first two coordinates, together with a -1 gain coming from the fact that $c(B_6) = 8$. As above there is a further -1 gain if $t/u > -1$, and a $+1$ loss $t/u < -1$. Therefore $k_7 = -4 + k_9$ if $p/q = 1, r/s = -4, t/u > -1$ up to permutation, and $k_7 = -2 + k_9$ if $p/q = 1, r/s = -4, t/u < -1$ up to permutation.

This analysis easily implies that indeed $\min\{k_6, \ldots, k_{10}\}$ is $h(p/q, r/s, t/u)$. $\square$

7 Complexity of Seifert manifolds and torus bundles

The key tool used for the computation of c_n in the atoroidal case, namely Proposition 4.6, is typically not valid for Seifert manifolds and torus bundles, so a different technique has to be employed.

Compatible Seifert pairs We develop here a general machinery to treat Seifert pairs and their assemblings.

Definition 7.1. Let (M, X) be a pair where M is an *oriented* Seifert manifold. We will say that $X = \{\theta_i\}_{i=1}^k$ is *compatible* with the Seifert fibration if each $\theta_i \in X$ contains a fibre (up to isotopy).

We show now that the set of X's compatible with a given M is parameterized by an integer $e(M, X) \in \mathbb{Z}$ called the generalized Euler number (which reduces to the customary $e(M)$ when M is closed). As we did for closed manifolds, instead of $e(M, X)$ itself we will define and use a certain "generalized twisting number" $t(M, X)$. This $t(M, X)$ is an integer, and $e(M, X)$ is obtained from it by adding a contribution q/p for each exceptional fibre of filling type (p, q) with $p > q > 0$. We first give a concise definition of $t(M, X)$ and state the classification theorem. Later we will prove the theorem using a more detailed description of how $t(M, X)$ is computed. We start with the following result, whose proof is easy and left to the reader.

Lemma 7.2. *Let (M, X) be an oriented compatible Seifert pair with $\partial M = \sqcup_{i=1}^k T_i$ and $X = \{\theta_i, \}_{i=1}^k$.*

- *If b_i is an arbitrarily oriented fibre on T_i then there exists a unique oriented curve a_i on T_i such that (a_i, b_i) is a positive basis of $H_1(T)$ and a_i, b_i, $a_i + b_i$ are contained in θ_i up to isotopy;*
- *The equivalence class of a_i as an unoriented loop is independent of the initial orientation on b_i;*
- *The Seifert fibration of M extends to a unique Seifert fibration of the closed manifold $\widehat{M}_X$ obtained from M by Dehn filling each T_i so to kill a_i.*

According to this result we can define $t(M, X) = t(\widehat{M}_X)$, assuming the definition is known for closed manifolds (see Section 1).

Proposition 7.3. *Let the following be given:*

- *A compact and connected surface F;*

- *$h \in \mathbb{N}$ and coprime pairs $(p_1, q_1), \ldots, (p_h, q_h)$ with $0 < q_i < p_i$;*
- *$t \in \mathbb{Z}$.*

Then there exists and is unique up to fibrewise orientation-preserving homeomorphism a compatible Seifert pair (M, X) such that M is fibred over F with h exceptional fibres of types $(p_1, q_1), \ldots, (p_h, q_h)$, and $t(M, X) = t$. Moreover $(-M, X)$ is also a compatible Seifert pair, and, if F has k boundary components, then

$$t(-M, X) = -t(M, X) - h - k.$$

To prove this result we will need to spell out the definition of $t(M, X)$. We start with the following known fact (see [8]):

Lemma 7.4. *Let M be an oriented Seifert manifold without exceptional fibres and with non-empty boundary $T_1 \sqcup \ldots \sqcup T_k$. Choose an arbitrarily oriented fibre b_i on each T_i. Let s be a section of the fibration and choose a loop $a_i \subset T_i$ contained in s and oriented so that (a_i, b_i) is a positive basis of $H_1(T_i)$. Then for a set $\{a'_i\}_{i=1}^k$ with $a'_i \in H_1(T_i)$ the following are equivalent:*

- *$\{a'_i\}_{i=1}^k$ is induced by another section s' of the fibration;*
- *there exist $n_1, \ldots, n_k \in \mathbb{Z}$ such that $a'_i = a_i + n_i \cdot b_i$ and $\sum_{i=1}^k n_i = 0$.*

Moreover, given s and s', the associated set $\{n_i\}_{i=1}^k$ depends only on s and s', not on the orientation of the b_i's, and there exists an orientation preserving self-homeomorphism f of M that maps s to s'.

To compute the generalized twisting number $t(M, X)$ of a compatible Seifert pair (M, X) we will take a section s of the fibration on the complement of the exceptional fibres and compare s with a system of curves determined by X and by the types of the exceptional fibres. To make this explicit we need some notation. We include the case $\partial M = \emptyset$ to remind the reader of the classical definition of $t(M)$.

Let M have $h \geqslant 0$ exceptional fibres and $k \geqslant 0$ boundary components. We construct a new Seifert manifold N with non-empty boundary $T_1 \sqcup \ldots \sqcup T_\ell$ by removing from M a fibred solid torus around each exceptional fibre, except when $h = k = 0$: in this case we remove an arbitrary fibred solid torus. So ℓ is either $h + k$ or 1. We assume that $T_1, \ldots, T_{h'}$ are the boundary components produced by the drilling of the solid tori, so h' is either h or 1. Now we choose an arbitrarily oriented fibre b_i on T_i for $i = 1, \ldots, \ell$. For $i \leqslant h'$ we denote by μ_i a meridian of the drilled solid torus, oriented so that the geometric intersection p_i of μ_i and b_i is positive, and we choose another oriented loop a_i on T_i such that (a_i, b_i) is a positive basis of $H_1(T_i)$ and $\mu_i = p_i \cdot a_i + q_i \cdot b_i$ with $0 \leqslant q_i < p_i$ (of course a_i exists and is unique).

Note that if $h > 0$ then $p_i \geqslant 2$, $q_i > 0$, and the type of the i-th fibre is (p_i, q_i). On the contrary, when $h = k = 0$, we have $q_1 = 0$ and $p_1 = 1$.

For $i = h' + 1, \ldots, \ell$ we now choose a_i as in Lemma 7.2. Then we pick a section s of N and for $i = 1, \ldots, \ell$ we take $a_i' \subset T_i$ contained in s such that (a_i', b_i) is a positive basis of $H_1(T_i)$. So $a_i' = a_i - m_i \cdot b_i$, and Lemma 7.4 shows that $\sum_{i=1}^{\ell} m_i$ is actually independent of s and of the arbitrary orientation chosen on $b_1, \ldots, b_\ell$. For $k = 0$ the twisting number $t(M)$ is defined precisely as $\sum_{i=1}^{\ell} m_i$. For $k > 0$, comparing our construction with the definition of $t(\widehat{M_X})$ just recalled, one easily sees that $\sum_{i=1}^{\ell} m_i = t(M, X)$.

Proof of 7.3. Existence and uniqueness in the closed case are well-known [8], so we only deal with the case $k > 0$. Then F and $(p_1, q_1), \ldots, (p_h, q_h)$ already determine M completely. The existence of X now easily follows from the method of computation of $t(M, X)$ just described, and uniqueness also does, using Lemma 7.4.

Now we consider orientation reversal, supposing first that $h + k > 0$ and using the same b_i's as in the above computation of $t(M, X)$. Since $k + h > 0$, all drilled tori are exceptional, and one easily sees that a_i gets replaced by $\overline{a}_i = -a_i - b_i$ for all i. Using the same section s it follows that $\overline{m}_i = -m_i - 1$, whence the conclusion. When $k = h = 0$ we simply have $\overline{a}_1 = -a_1$, and the conclusion follows again. $\square$

Small bricks are compatible Seifert pairs Take $\chi, t \in \mathbb{Z}$, $h, k \in \mathbb{N}$, $o \in \{0, 1\}$, and integers p_i, q_i for $i = 1, \ldots, h$, so that $\chi + k \leqslant 2$, if $o = 1$ then $\chi + k$ is even, p_i, q_i are coprime, and $0 < q_i < p_i$ for all i. We will denote from now on by

$$\mathrm{Seif}\big(\chi, k, o, \{(p_1, q_1), \ldots, (p_h, q_h)\}, t\big)$$

the oriented compatible Seifert pair such that:

- The base surface has Euler characteristic χ and k boundary components, it is orientable if $o = 1$, and it is not if $o = 0$;
- There are h exceptional fibres of types (p_i, q_i) for $i = 1, \ldots, h$;
- The generalized twisting number of the compatible pair is t.

As an example we show how the first bricks $B_0, \ldots, B_4$ can be presented as compatible Seifert pairs, and we introduce another pair Z used below. For all pairs except B_0 there are two choices of t because pairs are only defined up to homeomorphism, whereas compatible Seifert pairs are oriented. In all cases there are no singular fibres:

$$\begin{array}{rcll} B_0 & = & \mathrm{Seif}\big(0,2,1,\emptyset,-1\big), & \\ B_1 & = & \mathrm{Seif}\big(1,1,1,\emptyset,t\big), & t\in\{-1,0\} \\ B_2 & = & \mathrm{Seif}\big(1,1,1,\emptyset,t\big), & t\in\{-2,1\}, \\ B_3 & = & \mathrm{Seif}\big(0,2,1,\emptyset,t\big), & t\in\{-2,0\}, \\ B_4 & = & \mathrm{Seif}\big(-1,3,1,\emptyset,t\big), & t\in\{-2,-1\}, \\ Z & = & \mathrm{Seif}\big(0,1,0,\emptyset,t\big), & t\in\{-1,0\}. \end{array} \tag{48}$$

Given two oriented compatible Seifert pairs, we define now an assembling of them to be *compatible* if it is orientation-reversing and fibre-preserving. The same definition applies to a self-assembling of one oriented compatible Seifert pair.

Lemma 7.5. *Let the following oriented compatible Seifert pairs be given:*

$$\begin{array}{rcl} (M,X) & = & \mathrm{Seif}\big(\chi,k,o,\{(p_1,q_1),\ldots,(p_h,q_h)\},t\big), \\ (M',X') & = & \mathrm{Seif}\big(\chi',k',o',\{(p'_1,q'_1),\ldots,(p'_{h'},q'_{h'})\},t'\big). \end{array}$$

Assume $k,k'\geqslant 1$. Then any compatible assembling of (M,X) and (M',X') produces the compatible Seifert pair

$$\mathrm{Seif}\big(\chi+\chi',k+k'-2,o\cdot o',\{(p_1,q_1),\ldots,(p_h,q_h),(p'_1,q'_1),\ldots,(p'_{h'},q'_{h'})\},t+t'+1\big).$$

Proof. The only non-trivial thing to prove is that the twisting number t'' of the compatible pair we obtain is $t+t'+1$. Fix homology bases $\{(a_i,b_i)\}_{i=1}^{k}$ and $\{(a'_i,b'_i)\}_{i=1}^{k'}$ on the components of ∂M and $\partial M'$ as in Lemma 7.2. Assume the assembling glues $T_1\subset\partial M$ to $T'_1\subset\partial M'$ along a map ψ. Since ψ is orientation-reversing and fibre-preserving, and it maps $\theta_1\subset T_1$ to $\theta'_1\subset T'_1$, we can assume (up to reversing b'_1 and hence a'_1) that ψ maps b_1 to b'_1 and a_1 to $-a'_1-b'_1$.

Now let N be obtained from M by removing a regular neighborhood of the singular fibres, or a non-singular fibre when $k=1$ and $h=0$. Then N has at least two boundary components, one of which is T_1, so there is a section s containing a_1. In a similar way we can construct $N'\subset M'$ and a section s' of N' containing $a'_1+b'_1$. Therefore $s\cup s'$ is a section of $N\cup N'$. With obvious meaning of symbols we then have:

$$\partial s = a_1+\sum\nolimits_{i>1}(a_i-m_i\cdot b_i),\qquad \partial s' = a'_1+b'_1+\sum\nolimits_{i>1}(a'_i-m'_i\cdot b'_i),$$

$$\partial(s\cup s') = \sum\nolimits_{i>1}(a_i-m_i\cdot b_i)+\sum\nolimits_{i>1}(a'_i-m'_i\cdot b'_i).$$

By definition of twisting number we get:

$$t=\sum\nolimits_{i>1}m_i,\qquad t'=-1+\sum\nolimits_{i>1}m'_i,\qquad t''=\sum\nolimits_{i>1}m_i+\sum\nolimits_{i>1}m'_i$$

whence the conclusion at once. □

Lemma 7.6. *Let the following oriented compatible Seifert pair be given:*

$$(M,X) = \mathrm{Seif}\big(\chi, k, o, \{(p_1,q_1),\ldots,(p_h,q_h)\}, t\big).$$

Assume $k \geqslant 2$. Then a compatible self-assembling of (M,X) produces a compatible Seifert pair of the form

$$\mathrm{Seif}\big(\chi, k-2, o', \{(p_1,q_1),\ldots,(p_h,q_h)\}, t+1\big).$$

where $o' = 0$ when $o = 0$, and o' attains both values 0 and 1 when $o = 1$, depending on the choice of the self-assembling map.

Proof. The only non-trivial aspects to discuss are orientability of the new base surface and the proof that the new twisting number t' equals $t+1$. Of course the new base surface is non-orientable if the original one is. If the original surface is orientable we can choose a coherent orientation for the fibres, and we can realize the self-assembling both with a map that preserves the orientation of fibres and with one that reverses this orientation. In the former case the new base surface is orientable, in the latter it is not.

Turning to the computation of t', we choose homology bases (a_i, b_i) on ∂M as in Lemma 7.2, and we assume the self-assembling is performed along a $\psi : T_1 \to T_2$. As in the previous lemma we can assume that ψ maps (a_1, b_1) to $(-a_2 - b_2, b_2)$. Now we choose away from the singular fibres a section s that contains a_1 and $a_2 + b_2$ (of course when $k = 2$ and $h = 0$ we must first remove a trivially fibred torus). So s gives rise to a new section s'. With obvious meaning of symbols we then have

$$\partial s = a_1 + (a_2 + b_2) + \sum\nolimits_{i>2}(a_i - m_i \cdot b_i), \qquad \partial s' = \sum\nolimits_{i>2}(a_i - m_i \cdot b_i)$$

whence $t = -1 + \sum_{i>2} m_i$ and $t' = \sum_{i>2} m_i$, and the conclusion readily follows. □

We recall now that the Seifert parameters for a manifold with non-empty boundary are almost always unique. In the setting of compatible Seifert pairs, the next lemma describes two non-uniqueness instances needed below:

Lemma 7.7. *1. B_2 is homeomorphic to* $\mathrm{Seif}(1,1,1,\{(2,1)\},-1)$*;*

2. The following are the only genuine compatible Seifert pairs with non-empty boundary which are homeomorphic as oriented pairs but not as Seifert pairs:

$$\begin{aligned}\mathrm{Seif}\big(0,1,0,\emptyset,0\big) &\cong \mathrm{Seif}\big(1,1,1,\{(2,1),(2,1)\},-2\big)\\ \mathrm{Seif}\big(0,1,0,\emptyset,-1\big) &\cong \mathrm{Seif}\big(1,1,1,\{(2,1),(2,1)\},-1\big).\end{aligned}$$

Proof. The first assertion is easy, because any Seifert fibred solid torus is homeomorphic to the trivial one. Using the explicit map between $\big(D,(2,1)\big)$ and $D\times S^1$ one easily finds the desired homeomorphism.

Let us turn to the second assertion. It is well-known [8] that for genuine orientable Seifert manifolds with non-empty boundary the only case of non-uniqueness of the fibration is due to the homeomorphism between the twisted circle bundle over the Möbius strip and the manifold fibred over the disc with two exceptional fibres of type $(2,1)$. We give this manifold any orientation and denote it by M. The two fibrations of M give distinct fibres on ∂M. If a triod is compatible with both fibrations of M then it contains both these fibres, and there are two such triods θ and θ'. Proposition 7.3 shows that for some integers i,i',j,j' we have

$$\begin{aligned}(M,\theta) &= \mathrm{Seif}\big(0,1,0,\emptyset,i\big)=\mathrm{Seif}\big(1,1,1,\{(2,1),(2,1)\},j\big)\\ (M,\theta') &= \mathrm{Seif}\big(0,1,0,\emptyset,i'\big)=\mathrm{Seif}\big(1,1,1,\{(2,1),(2,1)\},j'\big).\end{aligned}$$

Reversing orientation i gets replaced by $-i-1$, but the property that a triod contains both fibres is insensitive to orientation, so $-i-1\in\{i,i'\}$ whence $i'=-i-1$. Moreover θ and θ' share two curves, so i and i' must be consecutive, which implies that $\{i,i'\}=\{-1,0\}$. A similar argument proves that $j'=-j-3$, whence $\{j,j'\}=\{-2,-1\}$. We have shown that

$$\{\mathrm{Seif}(0,1,0,\emptyset,i):\ i=-1,0\}=\{\mathrm{Seif}(1,1,1,\{(2,1),(2,1)\},j):\ j=-2,-1\}.$$

The proof that indeed $i=0$ corresponds to $j=-2$ and $i=-1$ corresponds to $j=-1$ now follows from (30), with a little patience. $\square$

Lemmas 7.5 and 7.7 readily imply the following:

Corollary 7.8. *The pair Z can be realized as $B_4\oplus B_2\oplus B_2$.*

We can now compute c_3 on Seifert manifolds.

Theorem 7.9. *Let $M=\big(F,(p_1,q_1),\ldots,(p_k,q_k),t\big)$ be a closed genuine Seifert manifold with parameters normalized so that $k-\chi(F)>0$, $p_i>q_i>0$, $t\geqslant -k/2$. If M is not $\big(S^2,(2,1),(2,1),(j,1),-1\big)$ for $j\in\{2,3\}$ then:*

$$c_3(M)=\max\{0,t-1+\chi(F)\}+6(1-\chi(F))+\sum_{i=1}^{k}(|p_i,q_i|+2).$$

Proof. Our proof is organized as a sequence of claims and constructions, and it uses the compatible Seifert pair Z introduced above, fibred as $\mathrm{Seif}(0,1,0,\emptyset,0)$. Switching orientation the same fibration can also be described as $\mathrm{Seif}(0,1,0,\emptyset,-1)$. Recall that Z admits an alternative fibration.

The assumption that M is not $\big(S^2,(2,1),(2,1),(j,1),-1\big)$ for $j\in\{2,3\}$ ensures that M is not a closed brick of complexity up to 3, so every realization of $c_3(M)$ will consist of blocks B_0,B_1,B_2,B_3,B_4.

Claim 1. There exists an assembling of bricks realizing $c_3(M)$ and such that:

- *it involves only copies of B_2, B_3, and B_4;*
- *each copy of B_3 is saturated (*i.e. *it is a union of fibres);*
- *each copy of B_4 is either saturated or assembled to two copies of B_2 so that $B_4\oplus B_2\oplus B_2$ is saturated and isomorphic to Z as a fibred unoriented pair;*
- *each copy of B_2 not contained in the Z's just described is a regular neighbourhood of an exceptional fibre of M, and conversely each exceptional fibre is contained in one such B_2.*

To prove the claim we fix an assembling realizing $c_3(M)$ with a minimal number of copies of B_4. The only case where B_0's cannot be dismissed is when M is a torus bundle, and this is not our case. Lemma 4.5 easily implies that our assembling cannot involve B_1's, so there are only B_2's, B_3's, and B_4's. Moreover, there is at least one B_4, otherwise M is a lens space.

Let us concentrate on one B_4 and note that, by irreducibility of the blocks involved, each component of ∂B_4 is either incompressible in M or the boundary of a solid torus $\boldsymbol{T}$. Since $n=3$ is well-behaved on tori, minimality of the number of B_4's shows that such a $\boldsymbol{T}$ must be an assembling of some B_3's and one B_2. In particular, $\boldsymbol{T}$ lies outside B_4, and hence defines a Dehn filling of B_4. Moreover, if the filled slope is r/s with respect to the basis fixed on $H_1(\partial B_4)$, we have $r\geqslant 2$ (up to switching signs) by Proposition 4.7-(2). Let $i\in\{0,1,2,3\}$ be the number of incompressible components of ∂B_4. Recall that incompressible tori are always fibred (because M is genuine, see [8]). The next discussion shows that we can modify the assembling so that all B_4's are as in the claim:

- If $i=3$ then B_4 is saturated and compatible (because its fibration is unique);
- If $i=2$, the one solid torus incident to B_4 gives a saturated Dehn filling N of B_4. Assume the slope killed by the filling is r/s. Then $r\geqslant 2$, so $N=(D_1,(r,s))$ has a unique Seifert fibration, which restricts to the fibration of B_4. It follows that B_4 is saturated;
- If $i=1$ the two $\boldsymbol{T}$'s incident to B_4 give some saturated Dehn filling of B_4, again denoted by N. Let r_1/s_1 and r_2/s_2 be the filled slopes, with $r_1,r_2\geqslant 2$.

If $r_1 > 2$ or $r_2 > 2$ then $N \cong (D, (r_1, s_1), (r_2, s_2))$ has a unique fibration, so B_4 is saturated. If $r_1 = r_2 = 2$ and B_4 is not saturated then N is fibred as the twisted S^1-bundle over the Möbius strip. As noted above, each of the two $\boldsymbol{T}$'s incident to B_4 is an assembling of some B_3's and one B_2, and there are $b = |2, s_1| + |2, s_2| - 2$ copies of B_3 in all, by Proposition 4.1. Moreover, denoting by θ the triod on ∂N and using the definition of twisting number, we see that $(N, \{\theta\}) = \mathrm{Seif}(1, 1, 1, \{(2, 1), (2, 1)\}, t + b)$ for some $t \in \{-2, -1\}$. But we cannot have $t = -2$ (unless $b = 0$, in which case $t = -2$ and $t = -1$ give the same pair), for otherwise we could switch the orientation of the original B_4 and realize $(N, \{\theta\})$ with $b - 1$ copies of B_3. We have shown that $(N, \{\theta\}) = \mathrm{Seif}(1, 1, 1, \{(2, 1), (2, 1)\}, -1 + b)$. Lemma 7.5 and Corollary 7.8 now imply that the sub-assembling giving $(N, \{\theta\})$ can be replaced, without increasing complexity, by an assembling $B_4 \oplus B_2 \oplus B_2$ fibred as Z together with b compatible B_3's.

- If $i = 0$ then there is only one B_4, and the conclusion follows from the same argument.

Turning to the B_2's not contained in the Z's and to the B_3's, we note that they are saturated, because their boundary is (up to isotopy). The correspondence between exceptional fibres and B_2's not contained in the Z's is a byproduct of the above construction.

A modified realization and its graph. By Remark 2.3, a self-assembling which identifies two tori T and T' matches $\theta \subset T$ either with $\theta' \subset T'$ or with another triod on T' having distance 1 from θ'. In the second case we can insert one copy of B_3 and get a self-assembling which matches the triods. Doing this to an assembling giving $c_3(M)$ as in Claim 1, we get a realization of M as an assembling of blocks of type B_2, B_3, B_4, Z such that triods are always matched. We call *fake* the B_3's we have inserted, and we define a graph $\mathcal{G}$ with the blocks of our assembling as vertices and the gluings as edges. So the B_4's are trivalent, the B_3's are bivalent, the B_2's and Z's are univalent.

Claim 2. Let $V_0, \ldots, V_{n+1}$ be the vertices of a simple (possibly closed) path in $\mathcal{G}$. Suppose V_0, V_{n+1} are (possibly equal and) of type B_4 or Z and the $V_1, \ldots, V_n$ are of type B_3. Then $V_1, \ldots, V_n$ are compatible. Let T_i be the torus along which V_{i-1} is glued to V_i and let θ_i be the triod on T_i. Since the T_i's are parallel in M, we can identify them to a single fibred torus T. Now $\theta_1, \ldots, \theta_{n+1}$ can be viewed as the vertices of a path in $\Theta(T)$, because $d(\theta_i, \theta_{i+1}) = 1$. Minimality shows that this path must be simple, so it is determined by θ_1 and θ_{n+1}. Knowing that θ_1 and θ_{n+1} contain a fibre (because the blocks B_4 and Z are compatible), it is now easy to show that the other θ_i's do, and the claim is proved.

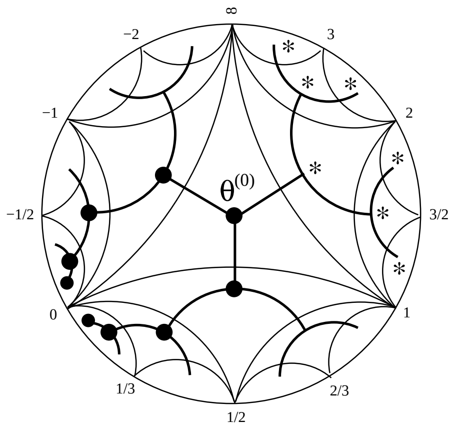

Figure 17: The triod θ_1 is one of the stars, and θ_{n+1} is one of the solid dots.

Claim 3. Let $V_0,\ldots,V_{n+1}$ be the vertices of a simple path in $\mathcal{G}$. Suppose V_0 is a B_2 block containing an exceptional fibre of type (p,q). Suppose V_{n+1} is of type B_4 or Z and $V_1,\ldots,V_n$ are of type B_3. Then V_i is compatible for $i=|p,q|,\ldots,n$ and it is not for $i=1,\ldots,|p,q|-1$. With notation as for the previous claim, we note that there is a preferred positive basis (a,b) for $H_1(T)$, such that b is a fibre and $p\cdot a+q\cdot b$ is a meridian of V_0. We use (a,b) to identify $\mathcal{S}(T)$ to $\partial_{\mathbb{Q}}(\mathbb{H}^2)$ and note that θ_{n+1} contains the slope 0, so it is one of the vertices marked by solid dots in Fig. 17. In addition θ_1 contains p/q and $1<p/q<\infty$, so it is one of the vertices marked by stars. Using Proposition 3.3 and recalling that $\theta^{(0)}$ is the centre of the disc, we deduce that

$$n=d(\theta_1,\theta_{n+1})=d(\theta_1,\theta^{(0)})+d(\theta^{(0)},\theta_{n+1})=|p,q|+d(\theta^{(0)},\theta_{n+1}).$$

The figure also shows that precisely the last $1+d(\theta^{(0)},\theta_{n+1})$ triods among $\theta_1,\ldots,\theta_{n+1}$ contain 0, whence the conclusion at once.

Claim 4. If b_3^c is the number of non-fake compatible B_3's in our assembling then

$$c_3(M)=b_3^c+6(1-\chi(F))+\sum_{i=1}^{k}(|p_i,q_i|+2).$$

Let us denote by b_3,b_4,z,s the number of non-fake B_3's, of B_4's, of Z's, and of self-assemblings. By assumption we have $c_3(M)=b_3+3b_4+3z+6s$, and Claims 2 and 3 show that

$$b_3=b_3^c+\sum_{i=1}^{k}(|p_i,q_i|-1)=b_3^c-3k+\sum_{i=1}^{k}(|p_i,q_i|+2).$$

We now claim that $b_4 = k - \chi(F)$ and $z = 2 - 2s - \chi(F)$. These relations easily imply the desired conclusion. Relation $\chi(F) = -b_4 + k$ readily follows from Lemmas 7.5 and 7.6, and the identities (48). To show the other relation consider the fibred space arising from the assembling of the blocks of type B_2, B_3, B_4 only, without self-assemblings. The base surface is then a sphere with some holes. Filling z of these holes and identifying the boundary circles of s pairs of them we get the closed surface F. So there are $z + 2s$ holes in all, and $\chi(F) = 2 - (z + 2s)$, whence the desired relation.

Claim 5. Let (M', X) be obtained from our assembling by dismissing the B_2's and the non-compatible B_3's. Then $t(M', X) = t$. Consider a B_2 block and the neighbouring non-compatible B_3 blocks, as in Claim 3. Their union is a solid torus $\boldsymbol{T}$, and we can identify $\partial \boldsymbol{T}$ with a component of $\partial M'$. Recall that to compute t one has to express a section σ of M' in terms of a basis of $H_1(\partial \boldsymbol{T})$ determined by the filled slope p/q. Similarly, to compute $t(M', X)$ one must express σ in terms of a basis determined by the triod of X lying on $\partial \boldsymbol{T}$. The argument proving Claim 3 shows that these bases are actually the same, whence the conclusion at once.

Claim 6. Conclusion: $b_3^c = \max\{0, t - 1 + \chi(F)\}$. To show this claim we first note that $z + b_4 > 0$, otherwise M would be a lens space. Then we slightly change our perspective and ask ourselves to what extent do the numbers b_3^c, b_4, z, k actually determine the unoriented compatible Seifert pair (M', X). Namely we discuss what pairs (N, Y) can be realized as follows:

- N is connected and has k boundary components;
- (N, Y) is obtained via fibre-preserving gluings from b_4 copies of B_4, z copies of Z, b_3^c copies of B_3, and some number $f \geqslant 0$ of additional copies of B_3, with $z + b_4 > 0$;
- the graph of this gluing remains connected after removing the f additional copies of B_3.

These conditions easily imply that there are $n = (z + 2(b_3^c + f) + 3b_4 - k)/2$ gluings, precisely $s = (3b_4 - z - k)/2 - (b_4 - 1)$ of which are self-assemblings. (Of course our notation for s is consistent with above.) Moreover (N, Y) is a compatible Seifert pair fibred over a surface with Euler characteristic $-b_4$ and no exceptional fibres. For the other Seifert parameters we have some flexibility, because the gluing maps are not uniquely determined. The flexibility reduces if we fix orientations on all the blocks, because then the twisting number of each block is determined, so $t(N, Y)$ is computed via Lemmas 7.5 and 7.6. If $s = 0$ then the base surface is orientable, otherwise we can realize both an orientable and a non-orientable one without changing the other parameters.

Let us now discuss exactly what possible $t(N,Y)$'s can be realized once b_3^c, b_4, z, k are given. Recall that, depending on orientation, $t(B_4)$ is -2 or -1. Similarly $t(Z)$ is -1 or 0 and $t(B_3)$ is -2 or 0. So:

$$-2b_4 - z - 2(b_3^c + f) + n \leqslant t(N,Y) \leqslant -b_4 + n$$

and all possibilities can be realized by appropriate choices of orientations (because $z + b_4 > 0$). Noting that f can take any value between 0 and s and plugging in the values of n and s these inequalities easily imply that

$$-b_4 - b_3^c - 1 \leqslant t(N,Y) \leqslant b_4 + b_3^c - k + 1 \tag{49}$$

and all these values can be realized.

We are eventually ready to prove the desired equality, so we assume that the parameters b_3^c, b_4, z, k and f arise as above from an assembling realizing $c_3(M)$. Since $b_4 - k = -\chi(F)$, the right-hand side of equation (49) readily implies that $b_3^c \geqslant \max\{0, t - 1 + \chi(F)\}$. We prove equality by contradiction, so we assume that $b_3^c > 0$ and $e < b_4 + b_3^c - k + 1$. Since $t \geqslant -k/2$ by assumption, we also have $t > -b_4 - b_3^c - 1$. Now equation (49) and the discussion leading to it prove that (M', X) could be realized with $b_3^c - 1$ copies of B_3 and all other parameters unchanged: a contradiction to minimality. □

Proposition 7.10. *Let M be a closed Seifert manifold. Then:*

- *the bricks $B_6, \ldots, B_{10}$ give no contribution to $c_8(M)$ and $c_9(M)$;*
- *if M is toroidal, the brick B_5 gives no contribution to $c_8(M)$.*

Proof. The first point is easy. Assume there is a hyperbolic brick $B \in \{B_6, \ldots, B_{10}\}$ in an assembling that realizes $c_8(M)$ or $c_9(M)$. Each boundary component of B either bounds a solid torus outside B or is incompressible in M. Let M' be the filling of B given by the union of B and the solid tori incident to it. Then, by Propositions 6.3 and 6.4, either the filling is degenerate or M' is negatively curved, so it is one of the blocks of the JSJ decomposition of M, and the conclusion easily follows.

Turning to the second point, we first recall that, using the preferred homology bases on $\partial(D_2 \times S^1)$ introduced before Theorem 1.13 and the notation N_α defined before Proposition 4.7, we have

$$B_4 = (D_2 \times S^1)_{\theta^{(0)}, \theta^{(0)}, \theta^{(-1)}} \quad \text{and} \quad B_5 = (D_2 \times S^1)_{2,3,\theta} \quad \text{with } \theta \supset \{-5/4, -6/5, -1\}.$$

Let us then assume by contradiction that there is a copy of B_5 in an assembling realizing $c_8(M)$. If the single boundary component of B_5 is compressible in M then

M is either atoroidal or non-irreducible, which is impossible. Therefore B_5 is a union of fibres, and its induced Seifert structure, being unique, is precisely that of $(D_2 \times S^1)_{2,3,\theta}$. Moreover, glued to ∂B_5, we have some B_3's (possibly none of them) and then a brick B equal to either B_4 or B_5 (not B_1 or B_2, otherwise ∂B_5 would be compressible).

Consider first the case where $B = B_4$. Precisely as in the proof of Theorem 7.9 we can see that either this B_4 is compatible or its two free boundary components are glued to two B_2's so to give a compatible Z. In both cases we see that the original θ on ∂B_5 is modified along a simple path in $\mathcal{T}$ until a compatible triod θ' is reached, and the $d(\theta, \theta')$ edges in this path correspond to the copies of B_3. The union of B_5 with the neighbouring B_3's, viewed as a portion of the assembling giving $c_8(M)$, gives $(B_5)_{\theta'}$ as a result, and its contribution to $c_8(M)$ is

$$c(B_5) + d(\theta, \theta') = 8 + d(\theta, \theta^{(-1)}) + d(\theta^{(-1)}, \theta') = 8 + 5 + d(\theta^{(-1)}, \theta') = 13 + d(\theta^{(-1)}, \theta')$$

(identity $d(\theta, \theta') = d(\theta, \theta^{(-1)}) + d(\theta^{(-1)}, \theta')$ is easily deduced from Fig. 10, because θ' contains the slope 0). Now the same $(B_5)_{\theta'}$ can also be realized as $(B_4)_{2,3,\theta'}$ with complexity

$$c(B_4) + d(\theta^{(0)}, 2) + d(\theta^{(0)}, 3) + d(\theta^{(-1)}, \theta') = 3 + 0 + 1 + d(\theta^{(-1)}, \theta') = 4 + d(\theta^{(-1)}, \theta').$$

This proves that B_5 can be dismissed with a gain of 9 on complexity, so the original assembling was not a realization of $c_8(M)$, a contradiction.

Next, consider the case where $B = B_5$. Then M is given by

$$(D_2 \times S^1)_{2,3} \bigcup\nolimits_A (D_2 \times S^1)_{2,3}.$$

But M is Seifert, so $A = \left(\begin{smallmatrix} \pm 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ up to switching sign to the basis of one block. Since the two manifolds glued along A both have θ on the boundary, the number of B_3's required to assemble them is $d(\theta, A\theta)$, so $c_8(M) = 2 \cdot c(B_5) + d(\theta, A\theta) = 16 + d(\theta, A\theta)$. As above we can replace B_5 with $(B_4)_{2,3,\theta^{(-1)}}$ on both sides, deducing that

$$c_3(M) \leqslant 2 \cdot (3 + 0 + 1) + d(\theta^{(-1)}, A\theta^{(-1)}) = 8 + d(\theta^{(-1)}, A\theta^{(-1)}).$$

The conclusion now follows as soon as we show that $d(\theta^{(-1)}, A\theta^{(-1)}) < 8 + d(\theta, A\theta)$ for $A = \left(\begin{smallmatrix} \varepsilon & 0 \\ n & 1 \end{smallmatrix}\right)$, $\varepsilon = \pm 1$, $n \in \mathbb{Z}$. We begin with $\varepsilon = +1$ and note that $A = B^n$, where $B = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We consider the alternative picture in the half-space model of $\mathbb{H}^2$ of the Farey tessellation and its dual graph, as shown in Fig. 18. (As above, the picture is only combinatorially, not geometrically, correct. Note also that we have used a smaller and smaller character for p/q as $|p, q|$ grows.) It is very easy to see that B acts on the tessellation of Fig. 18 as the horizontal translation that

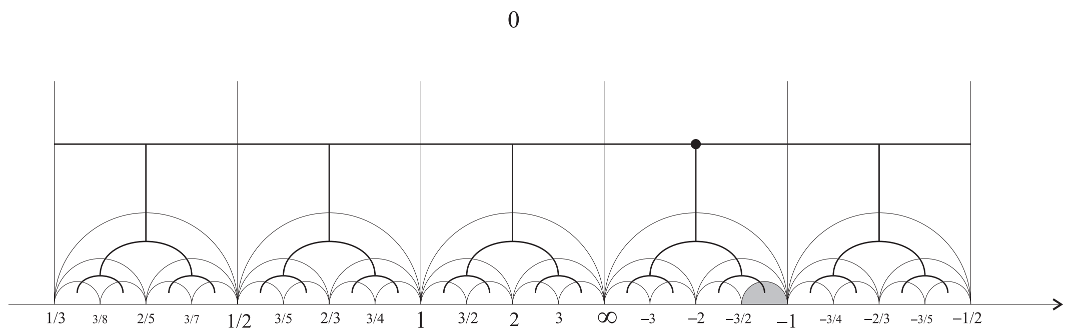

Figure 18: An alternative picture of the Farey tessellation.

moves each vertical half-line to the previous one. Since $\theta^{(-1)}$ is the dot marked in the picture whereas θ lies somewhere in the small shadowed region, we deduce that

$$d(\theta^{(-1)}, A\theta^{(-1)}) = |n|, \qquad d(\theta, A\theta) = 2d(\theta, \theta^{(-1)}) + |n| = 10 + |n|$$

whence the desired inequality. When $A = \left(\begin{smallmatrix} -1 & 0 \\ n & 1 \end{smallmatrix}\right)$ the action of A on the tessellation is a horizontal translation followed by the reflection in the vertical half-line through 0 and ∞. It is then easy to see that $d(\theta^{(-1)}, A\theta^{(-1)}) = |n-1|$ and $d(\theta, A\theta) = 10 + |n-1|$, whence the conclusion. $\square$

The two previous results conclude the proof of Theorem 1.11 for genuine Seifert manifolds, because $c_n(M)$ was already computed in Propositions 4.12, 4.14, and 4.16 for $n \leqslant 8$ when M is atoroidal and Seifert but not a member of $\mathcal{M}^*$. For torus bundles, having already computed c_1 in Proposition 4.13, we are left to show the following:

Proposition 7.11. *If $A \in \mathrm{GL}_2(\mathbb{Z})$ and T_A is the torus bundle over the circle with monodromy A then $c_n(T_A) = c_1(T_A)$ for $n = 2, \ldots, 9$.*

Proof. We begin by noting the following topological fact: if H is an incompressible torus embedded in T_A then, cutting T_A along H, we get either $T \times [0,1]$ or two copies of $S \widetilde{\times} S^1$ (the twisted S^1-bundle over the Möbius strip). And, in the latter case, we have $A = \left(\begin{smallmatrix} -1 & 0 \\ m & -1 \end{smallmatrix}\right)$ up to conjugation, and $T_A = (K, +m) = (K, -m)$, where K denotes the Klein bottle.

Let us then consider an assembling realizing $c_n(T_A)$, and let $\mathcal{H} \subset T_A$ be the family of tori along which the gluings of the assembling are performed. If all tori in $\mathcal{H}$ are compressible, well-behavedness of $n \leqslant 9$ easily implies that the assembling has a certain centre B_i with $i \geqslant 4$. However Propositions 4.14, 4.16, 6.3, and 6.4 imply that a filling of B_i giving T_A is always degenerate, except for $i=4$ and $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, but in this case we have $c_1(T_A) = c_3(T_A) = c(T_A) = 6$, so $c_n(T_A) = 6$ for $1 \leqslant n \leqslant 9$.

Assume one $H \in \mathcal{H}$ cuts T_A into $T \times [0,1]$. Then a self-assembling is performed along H of a certain pair based on $T \times [0,1]$. Well-behavedness of $n \leqslant 9$ implies that we can realize this pair as either B_0 or an assembling of B_3's, so we have $c_n(T_A) = c_1(T_A)$.

Assume now one $H \in \mathcal{H}$ cuts T_A into two copies of $S \widetilde{\times} S^1 = \big(D, (2,1), (2,1)\big)$. By Proposition 7.10 (stated for the closed case but true, with the same proof, also in the case with boundary), both copies of $S \widetilde{\times} S^1$ are assemblings of copies of B_2, B_3, B_4. Precisely as in the first step of the proof of Theorem 7.9 we then see that each $S \widetilde{\times} S^1$ is an assembling of one B_4, two B_2's, and some B_3's. More precisely, the whole assembling giving T_A consists of two compatible Seifert pairs $\big(D_2 \times S^1\big)_{2,2,\theta^{(-1)}}$, both realized as an assembling of one B_4 and two B_2's, with some B_3's in between. So T_A is given by

$$(D_2 \times S^1)_{2,2} \bigcup\nolimits_B (D_2 \times S^1)_{2,2}$$

for some $B \in \mathrm{GL}_2(\mathbb{Z})$, and there are $d(\theta^{(-1)}, B\theta^{(-1)})$ copies of B_3 (recall that we always use the fixed homology bases on the boundary components of $D_2 \times S^1$). We must now discuss when such a gluing actually gives $(K, \pm m)$ as a result. Since the blocks glued are genuine Seifert and the result is also Seifert, B must match a fibration of the first block with a fibration of the second block (there are two on each).

If the original fibrations $(D_2 \times S^1)_{2,2}$ are matched, then the result is some $\big(S^2, (2,1), (2,1), (2,1), (2,1), t\big)$. This manifold equals $(K, \pm m)$ only for $t = -2$ and $m = 0$, whence B is the identity, and the alternative fibrations $S \widetilde{\times} S^1$ are also matched. We then postpone this case to discuss it with the more general one where the fibrations $S \widetilde{\times} S^1$ are matched.

If one original fibration $(D_2 \times S^1)_{2,2}$ is matched with a fibration $S \widetilde{\times} S^1$, the result is some $\big(\mathbb{RP}^2, (2,1), (2,1), t\big)$, which is never equal to $(K, \pm m)$.

Assume now the two fibrations $S \widetilde{\times} S^1$ are matched, and recall from (30) that the section and fibre of $S \widetilde{\times} S^1$, when expressed with respect to the homology basis fixed on $(D_2 \times S^1)_{2,2}$, are the slopes 0 and -1 respectively. So B should map -1 to -1. The result in this case is always $(K, \pm m)$, with $|m|$ being the geometric intersection between the slope 0 and its image under B. Using the fact that both 0 and -1 are contained $\theta^{(-1)}$ we easily see that $d(\theta^{(-1)}, B\theta^{(-1)}) \geqslant \max\{|m|-1, 0\}$. Moreover $A = \left(\begin{smallmatrix} -1 & 0 \\ \pm m & -1 \end{smallmatrix}\right)$ and $\|A\| = |m|$, whence

$$\begin{aligned} c_n(T_A) &\geqslant 2c(B_4) + 4c(B_2) + c(B_3) \cdot \max\{|m|-1, 0\} \\ &= 6 + \max\{|m|-1, 0\} = c_1(T_A) \end{aligned}$$

and the proof is eventually complete. □

We conclude the paper noting that a proof of Theorem 2.8-(3) is already implicit in our results. In fact, Proposition 4.14-(4) proves that $c_3(C_{i,j}) = 1 + i + j$ if $c^*(C_{i,j}) > 9$, whereas $c^*(C_{i,j}) = i + j$, Proposition 4.16-(3) proves that $c_8(E_k) = k + 6$ if $c^*(E_k) > 9$, whereas $c^*(E_k) = k + 5$, and Proposition 7.10 proves that $c_9(C_{i,j}) = c_3(C_{i,j})$ and $c_9(E_k) = c_8(E_k)$ under the same constraints.

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